# AN ALTERNATIVE METHOD FOR CALCULATING THE PROBABILITY OF RUIN IN RISK THEORY[1]. THEORETICAL APPROACH AND APPLICATIONS

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ABSTRACT

We propose a new method for calculating the risk of ruin with reference to both life and damages insurance portfolios in a finite period of time and in particular for calculating:
a) the total claims probability distribution in multivariate situations, according to the Collective Risk Theory model;
b) the time of ruin in addition to the probability distribution of the ruin amount and of the insurer's capital before ruin, according to the Individual Risk Theory model.
The method moves from an original idea of Amsler (1992), i.e. the idea of defining directly the random variables representing the risks incumbent on a given portfolio during a given period of time.
After making a brief discussion of the existing literature, we illustrate in detail the alternative method proposed and some numerical results obtained by applying it to the above mentioned problems.

KEYWORDS

Distribution. Risk. Ruin.

## 1. INTRODUCTION

In the present paper we illustrate a new method for calculating the risk of ruin with reference to both life and damages insurance portfolios in a finite period of time[3].
This method is useful in particular for calculating:
a) the total claims probability distribution in multivariate situations like the ones recently discussed in literature by Hesselager (1996) and Sundt (1999);
b) the time of ruin in addition to the probability distribution of the ruin amount and of the insurer's capital before ruin in a more practical and economic way than the one recently proposed in literature by Gerber and Shiu (1998).
In the first case, the calculation is based on two different types of random variables, the claims number and the individual claim amount, according to the Collective Risk Theory model. In this case the proposed method, unlike the others proposed in literature, allows to calculate the total claims distribution whatever the distribution of the claims number may be[4].

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[1] In the present paper we use the terms Collective Risk Theory and Individual Risk Theory according to Bowers et al. (1986, 1997) and Tomassetti, Manna, Pucci (1995).
[2] The author thanks prof. Alvaro Tomassetti and dr. Emanuela Camerini for the contribution they gave on the computational level.
[3] Usually, a period of maximum forty years in the life insurance.
[4] See Ferrara, Manna, Tomassetti (1996) for an example in the univariate case with a distribution of the claims number given by a mixture of a Poisson distribution and a generalized Pareto one.

In the latter case, the calculation is based on a single type of random variables, the one describing for each policy included in the portfolio the insurer's annual financial flow, according to the Individual Risk Theory model. In this case the proposed method allows to calculate the time of ruin, the amount of ruin and the capital before ruin by allowing for the services due and the accrued capital.
In both cases, by moving from an original idea of Amsler (1992), the proposed method is based on the definition of the random variables representing the risks (i.e. the individual claim amount in the first case, the annual financial flow in the second case) and on the direct calculation of their sum. The method does not require any particular hypothesis about the above mentioned variables and therefore it can be applied in most of the practical circumstances.
In this paper, after making a brief discussion of the existing literature, we illustrate in detail the alternative method proposed and some numerical results obtained by applying it to the problems mentioned sub a) and b)[5].

## 2. OUTLINE OF THE RECENT LITERATURE

### 2.1. About the calculation of the total claims distribution by recursion

The entire literature on this subject starts with a paper by Panjer (1981) where the author proposed a recursive formula for the calculation of the total claims distribution of insurance policies by assuming that:
a) the probability distribution of the claims number satisfies a given recursive relation;
b) the random variables defining the individual claim amount are independent and identically distributed.
During almost twenty years of contributions on this topic, all the authors have tried to generalize Panjer's recursive procedure by keeping the assumption b) and by extending the assumption a) to various distributions. Besides, several authors have tried to overcome the underflow and overflow problems that Panjer's formula presents when practically applied[6].
Recently, Hesselager (1996) and Sundt (1999) have derived some multivariate extensions of Panjer's recursion using different multivariate generalizations of the assumptions a) and b).

#### 2.1.1. *The bivariate case studied by Hesselager*

Hesselager considers a situation with two portfolios of policies. The problem can be formalized in the following way.
Let $N_1$ denote the random variable representing the claims number for the first portfolio with generical realization $n_1$ ($n_1$=0,1,...) and $N_2$ the random variable representing the claims number for the second portfolio with generical realization $n_2$ ($n_2$=0,1,...). The random variables $N_1$ and $N_2$ are assumed to be dependent with joint probability function $p(n_1,n_2)$.
Let $U_{i1}$ i=(1,2,...,$N_1$) and $U_{i2}$ i=(1,2,...,$N_2$) denote the random variables representing, respectively for the first and the second portfolio, the amount of the i-th claim and besides let us make the following assumptions:
1. the random variables $U_{i1}$ are mutually independent and identically distributed with non-negative integer generical realization $u_1$ and corresponding probability $\varphi_1(u_1)$, with $\varphi_1(0)=0$;

[5] On this subject, it is important to notice that the authors mentioned sub a) and b) formally illustrate their respective models but they do not develop any numerical application.
[6] See Bruno (1998) for an alternative solution of these problems based on the same logic of the procedure that we propose in this paper with reference to multivariate situations.

2. the random variables $U_{i2}$ are mutually independent and identically distributed with non-negative integer generical realization $u_2$ and corresponding probability, $\varphi_2(u_2)$, with $\varphi_2(0)=0$;
3. the random variables $U_{i1}$ and $U_{i2}$ are mutually independent (for both a fixed and a varying i);
4. the random variables $U_{i1}$ and $U_{i2}$ are independent of both $N_1$ and $N_2$.
In this case, the aggregate claim amount of the first and the second portfolio is respectively given by the following random variables:

$$(1) \qquad (X_1, X_2) = \left( \sum_{i=0}^{N_1} U_{i1}, \sum_{i=0}^{N_2} U_{i2} \right)$$

with $U_{01}=U_{02}=0$. Then, under the stated assumptions, the joint probability function of $X_1$ and $X_2$ is given by the following expression:

$$(2) \qquad g(x_1, x_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} p(n_1, n_2) \varphi_1^{*n_1}(x_1) \varphi_2^{*n_2}(x_2) \qquad \text{for } (x_1, x_2) \geq 0$$

where $x_1$ ($x_1$=0,1,2,...) denotes the generical numerical realization of the first variable and $x_2$ ($x_2$=0,1,2,...) the generical numerical realization of the second one. Besides, $\varphi_1^{*n1}(x_1)$ is the probability that the aggregate amount of $n_1$ claims is $x_1$ for the first portfolio and, in a similar way, $\varphi_2^{*n2}(x_2)$ is the probability that the aggregate amount of $n_2$ claims is $x_2$ for the second portfolio.
Under the additional assumption that the counting distribution $p(n_1,n_2)$ is of a given kind, Hesselager derives a recursive formula to rapidly compute (2). Hesselager's recursive formula is valid in particular in the following three different cases (which are equivalent to three different kinds of counting distribution):
*Case 1*: if, once defined the random variable $N=N_1+N_2$ and called n its generical numerical realization and p(n) the corresponding probability, the random variable ($N_1/N=n$) is binomial with parameters ($n,\rho$), i.e.:

$$(3) \qquad \text{prob}\{(N_1 = n_1) \mid (N = n)\} = \binom{n}{n_1} \rho^n \, (1-\rho)^{n-n_1}$$

and the probability p(n) satisfies the following recursive relation:

$$(4) \qquad p(n) = \left( a + \frac{b}{n} \right) p(n-1) \qquad \text{for } n=1,2,\ldots.$$

In this case, we have:

(5)

$$
\begin{aligned}
p(n_1,n_2) &= \text{prob}\{(N_1=n_1)\cap(N_2=n_2)\}= \\
&= \text{prob}\{(N_1=n_1)\cap(N=n_1+n_2)\}= \\
&= \text{prob}\{N=n_1+n_2\}\cdot\text{prob}\{(N_1=n_1)\,|\,(N=n_1+n_2)\}= \\
&= p(n_1+n_2)\binom{n_1+n_2}{n_1}\rho^{n_1}(1-\rho)^{n_2}
\end{aligned}
$$

*Case 2*: if $N_1=K_0+K_1$ and $N_2=K_0+K_2$ with $K_0$, $K_1$, $K_2$ mutually independent random variables and the following recursive relations are satisfied:

(6) $$q_0(k_0)=\left(a_0+\frac{b_0}{k_0}\right)q_0(k_0-1) \qquad \text{for } k_0=1,2,\ldots.$$

(7) $$q_1(k_1)=\left(a_1+\frac{b_1}{k_1}\right)q_1(k_1-1) \qquad \text{for } k_1=1,2,\ldots.$$

(8) $$q_2(k_2)=\left(a_2+\frac{b_2}{k_2}\right)q_2(k_2-1) \qquad \text{for } k_2=1,2,\ldots$$

where $k_0$, $k_1$, $k_2$ denote the generical numerical realizations of respectively $K_0$, $K_1$, $K_2$ and $q_0(k_0)$, $q_1(k_1)$, $q_2(k_2)$ the corresponding probabilities.
An example is the case where $K_0$, $K_1$ and $K_2$ are Poisson independent random variables with mean value respectively equal to $\lambda_0$, $\lambda_1$ and $\lambda_2$. In this case we have:

(9) $$p(n_1,n_2)=e^{-(\lambda_0+\lambda_1+\lambda_2)}\sum_{k=0}^{\min(n_1,n_2)}\frac{(\lambda_0)^k}{k!}\cdot\frac{(\lambda_1)^{n_1-k}}{(n_1-k)!}\cdot\frac{(\lambda_2)^{n_2-k}}{(n_2-k)!}$$

*Case 3*: if $N_1$ and $N_2$ are compound Poisson independent random variables with parameters $\lambda_1 P$ and $\lambda_2 P$ where P is a random variable with generical realization $\theta\in[\sigma_1,\sigma_2]$, $0\le\sigma_1<\sigma_2\le\infty$, and probability density function $v(\theta)$ satisfying the following relation:

(10) $$\frac{d}{d\theta}\log v(\theta)=\frac{\sum_{h=0}^{k}a_h\theta^h}{\sum_{h=0}^{k}b_h\theta^h}$$

In this case, apart from (10), we have:

(11) $$p(n_1,n_2)=\int_{\sigma_1}^{\sigma_2}\frac{(\theta\lambda_1)^{n_1}}{n_1!}e^{-\theta\lambda_1}\frac{(\theta\lambda_2)^{n_2}}{n_2!}e^{-\theta\lambda_2}v(\theta)d\theta$$

*2.1.2. The multivariate case studied by Sundt*

Sundt considers a multivariate situation where each claim event generates a vector of random variables[7]. The problem can be formalized in the following way.
Let N denote the random variable representing the number of claims occurring in an insurance portfolio within a given period, n (n=0,1,...) its generical numerical realization and p(n) the corresponding probability.
Let $\mathbf{U}_i=(U_{i1},U_{i2},\ldots,U_{im})$ (i=1,...,N) denotes the generical m-dimensional vector generated by the i-th claim event and let us make the following assumptions:
1. the random vectors $\mathbf{U}_i$ are mutually independent (i.e. the random variables $U_{ij}$ (i=1,...,N; j=1,...,m) are mutually independent for a varying i and a fixed j; on the contrary, they are dependent for a varying j and a fixed or a varying i);
2. the random vectors $\mathbf{U}_i$ are identically distributed with generical realization given by the m-dimensional vector of non-negative integers $\mathbf{u}=(u_1,u_2,\ldots,u_m)$ and corresponding joint probability $f(\mathbf{u})=f(u_1,u_2,\ldots,u_m)$, with $f(\mathbf{0})=f(0,0,\ldots,0)=0$;
3. the random variables $U_{ij}$ are independent of N.
In this case, the portfolio aggregate claim amount is given by the following vector of random variables:

$$\mathbf{X}=(X_1,X_2,\ldots,X_m)=\sum_{i=0}^{N}\mathbf{U}_i=\left(\sum_{i=0}^{N}U_{i1},\sum_{i=0}^{N}U_{i2},\ldots,\sum_{i=0}^{N}U_{im}\right) \tag{12}$$

with $U_{01}=U_{02}=\ldots=U_{0m}=0$. Then, under the stated assumptions, the joint probability function of this vector is given by the following expression:

$$g(\mathbf{x})=\sum_{n=0}^{\infty}p(n)f^{*n}(\mathbf{x}) \qquad \text{for } \mathbf{x}\geq 0 \tag{13}$$

that can be also written in the following way:

$$g(x_1,\ldots,x_m)=\sum_{n=0}^{\infty}p(n)f^{*n}(x_1,\ldots,x_m) \qquad \text{for } (x_1,\ldots,x_m)\geq(0,\ldots,0) \tag{14}$$

where the m-dimensional vector of non-negative integers $\mathbf{x}=(x_1,x_2,\ldots,x_m)$ denotes the generical numerical realization of $\mathbf{X}=(X_1,X_2,\ldots,X_m)$ and besides $f^{*n}(\mathbf{x})=f^{*n}(x_1,x_2,\ldots,x_m)$ is the probability that the aggregate claim amount caused by n claim events is $\mathbf{x}=(x_1,x_2,\ldots,x_m)$.
Under the additional assumption that the counting distribution p(n) satisfies the following recursive relation[8]:

$$p(n)=\left(a+\frac{b}{n}\right)p(n-1) \qquad \text{for } n=1,2,\ldots. \tag{15}$$

[7] For instance, according to Sundt, a situation where each claim event can affect more than one policy included in a given portfolio or a situation where each claim event can induce various types of claims, or still a situation where all claims are settled after a given number of years.
[8] Note that this is exactly the same recursive relation to be satisfied for deriving Panjer's recursion in the univariate case but it is also the same relation (4) to be satisfied for deriving Hesselager's recursion in the bivariate case 1.

Sundt derives a multivariate recursive formula to rapidly compute (14) and he discusses some special cases of the general algorithm. All these cases refer to bivariate situations.
Sundt discusses in particular a situation where each claim event can induce only two different types of claim, i.e. a claim of a given type (type 1) with probability c and a claim of a different type (type 2) with opposite probability 1-c. In this case the joint probability distribution of the individual claim amount can be written in the following way:

$$f(u_1,u_2)=\begin{cases} ch(u_1) & \text{for } u_1=1,2,\ldots \text{ and } u_2=0 \\ (1-c)k(u_2) & \text{for } u_1=0 \text{ and } u_2=1,2,\ldots \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

where $h(u_1)$ denotes the conditional probability function of the individual claim amount given that the claim is of type 1 and $k(u_2)$ the conditional probability function of the individual claim amount given that the claim is of type 2. Besides, we have:

$$\text{prob}\{(N_1=n_1)\,|\,(N=n)\}=\binom{n}{n_1}c^{n_1}(1-c)^{n-n_1} \tag{17}$$

where $N_1$ is the random number of claim events inducing a claim of type 1 and $n_1$ its generical numerical realization.
As for the applications of our model in section 4.1, it may be useful to notice here that (14), once explicated according to (15), (16) and (17) with $c=\rho$, $h(u_1)=\varphi_1(u_1)$ and $k(u_2)=\varphi_2(u_2)$ becomes equivalent to (2) explicated according to (3) and (5) of Hesselager's case 1.

### 2.2. On the time value of ruin

In literature, this problem is approached in two different ways:
a) by means of general algorithms, by defining the discrete random variables representing the annual risks for each policy included in the portfolio and by directly performing the convolution of these random variables;
b) by means of analytical methods in a continuous environment, by making some specific assumptions about the time evolution of the insurer's surplus.
Amsler (1992) was the first author stating the problem as described sub a). According to the mentioned approach, Amsler formulated in particular the calculation of the probability distribution of the time and of the amount of ruin.
Only later, however, other authors[9] have removed the non-realistic assumptions and the application restrictions of Amsler's method making it useful only for academic purposes (as said by Amsler itself in his paper).
On the contrary, Gerber and Shiu (1998) have recently approached the problem as described sub b). Their method can be formalized in particular in the following way.
Let $W(t)$ denote the insurer's surplus at time t ($t\geq 0$) and w its generical realization. Let in particular $W(0)=w_0$ denote the insurer's initial surplus. Besides, let $\{N(t)\}$ and $\{X(t)\}$ denote respectively the claims number stochastic process and the total claim amount one and $U_i$ ($i=1,\ldots,N(t)$) the random variables representing the individual claim amount.
Under the assumption that the premiums are received continuously at a constant rate c per unit time, the insurer's surplus can be written in the following way[10]:

[9] See, for example, Tomassetti, Manna, Pucci (1995), Bruno, Camerini, Tomassetti (1998), Bruno (1998) and see also section 3 and 4 of this paper.

(18) $$W(t) = w_0 + ct - X(t)$$

where:

(19) $$X(t) = \sum_{i=1}^{N(t)} U_i$$

Let us make also the following assumptions:
1. the random variables $U_i$ are independent and identically distributed with generical realization $u \geq 0$ and probability density function $f(u)$;
2. the claims number follows a Poisson process with mean per unit time $\lambda$.
Under the stated assumptions, Gerber and Shiu consider three different random variables, the time of ruin T ($T=\inf\{t|W(t)<0\}$, the surplus immediately before ruin W(T-) and the surplus at ruin W(T). They analytically study the joint probability density function of (W(T-),|W(T)|,T) and derive a renewal equation for the following function:

(20) $$\phi(w) = E\left[g(W(T-),|W(T)|)e^{-\delta T} I(T < \infty \mid W(0) = w_0\right]$$

which may be interpreted as an estimation of the expected future loss in case of ruin given the initial surplus[11]. They derive in particular some explicit expressions of (20) under the additional assumption that the individual claim amount distribution is exponential or a mixture of exponentials.

## 3. THEORETICAL CONTRIBUTION

We propose a new method for calculating the risk of ruin and in particular for calculating:
a) the total claims probability distribution in multivariate situations like the ones illustrated in section 2.1;
b) the time of ruin in addition to the probability distribution of the ruin amount and of the insurer's capital before ruin in more general situations than the ones illustrated in section 2.2.
The logic underlying the construction of this method differs from the classical one adopted in literature. As a matter of fact, instead of looking for general solutions subjected to various assumptions apart from the specific application, we have looked for general algorithms self-adactable to the different applications with the aim of providing an useful and economic tool to solve various actual problems.
The method moves from the original idea of Amsler discussed in the previous section, i.e. the idea of defining directly the random variables representing the risks incumbent on a given portfolio during a given period of time. Thus, it allows to remove a lot of the assumptions of the traditional approaches but, obviously, it requires the acknowledgement of a lot of elements of automatic calculus[12] in additional to the fundamentals of mathematics.

### 3.1. Formalization according to the Collective Risk Theory model

[10] Notice that this expression doesn't allow for the investment opportunities of the surplus.
[11] In other words, they do not calculate the probability distribution of the time and of the amount of ruin but only their expected values.
[12] For this aspect, we have availed ourselves of the help given by prof. Alvaro Tomassetti.

According to the Collective Risk Theory model, we assume to be known the random variable representing the claims number and the random variables representing the individual claim amount and we intend to calculate the probability distribution of the total claim amount.
To start, we refer to the bivariate case of two portfolios of policies studied by Hesselager and mentioned in section 2.1.1. We remind that, in this case, the problem consists in the calculation of the following function:

$$(21)\, g(x_1,x_2) = \sum_{n_1=0}^{MN_1}\sum_{n_2=0}^{MN_2} p(n_1,n_2)\varphi_1^{*n_1}(x_1)\varphi_2^{*n_2}(x_2) \text{ for } x_1 = 0,\dots,MX_1; x_2 = 0,\dots,MX_2$$

where, for simplicity but without loss of generality, we have assumed that the maximum number of claims is finite (in particular $MN_1$ for the first portfolio and $MN_2$ for the second one) as well as the maximum total claim amount (in particular $MX_1$ for the first portfolio and $MX_2$ for the second one).
On this subject, we remark that for $n_1=0,\dots,MN_1$ and $n_2=0,\dots,MN_2$:

$$(22) \qquad \varphi_1^{*n_1}(x_1) = \text{prob}\left\{\sum_{i=0}^{n_1} U_{i1} = x_1\right\} \qquad \text{for } x_1 \geq 0$$

$$(23) \qquad \varphi_2^{*n_2}(x_2) = \text{prob}\left\{\sum_{i=0}^{n_2} U_{i2} = x_2\right\} \qquad \text{for } x_2 \geq 0$$

where $U_{i1}$ $(i=1,\dots,MN_1)$ and $U_{i2}$ $(i=1,\dots,MN_2)$ are independent random variables, with $U_{01}=U_{02}=0$. Under a theoretical point of view, it is therefore possible to derive the above mentioned functions (22) and (23) by convolution. This is not easy however in actual facts, especially when the number of claims and the number of possible individual claim values are very large, since $U_{i1}$ $(i=1,\dots,MN_1)$ and $U_{i2}$ $(i=1,\dots,MN_2)$ are also discrete random variables.
So, we propose an approximated procedure to operate convolution[13].
Actually, we first identify the values of $p(n_1,n_2)$ whose sum is equal to one with an error lower than $10^{-6}$ and then we perform the calculation of (22) and (23) only for the values of $n_1$ and $n_2$ corresponding to the above mentioned probabilities. Besides, when calculating (22) and (23), we consider only the values of $x_1$ and $x_2$ for which respectively the sum of the functions $\varphi_1^{*n1}(x_1)$ and $\varphi_2^{*n2}(x_2)$ is equal to 1 with an error lower than $10^{-6}$ and simultaneously the first three moments of the related approximated distributions differ in relative terms from the corresponding exact values with an error lower than $10^{-6}$.
On this subject, it may be useful to remark that, according to (22) and (23), the functions $\varphi_1^{*n1}(x_1)$ and $\varphi_2^{*n2}(x_2)$ are respectively the probability distribution functions of the following random variables:

$$(24) \qquad X_1^{(n_1)} = \sum_{i=0}^{n_1} U_{i1}$$

[13] This is an extension of the univariate procedure proposed by Ferrara, Manna, Tomassetti (1996) and later applied by Bruno (1998) for overcoming the underflow and overflow problems of Panjer's formula.

(25) $$X_2^{(n_2)} = \sum_{i=0}^{n_2} U_{i2}$$

These random variables are sums of independent random variables and it is therefore possible to calculate exactly all their moments. Let in particular $E\left[\left(X_1^{(n_1)}\right)^k\right]$ and $E\left[\left(X_2^{(n_2)}\right)^k\right]$ (k=1,2,3) denote the first three exact moments of respectively (24) and (25). Besides, let $E_a\left[\left(X_1^{(n_1)}\right)^k\right]$ and $E_a\left[\left(X_2^{(n_2)}\right)^k\right]$ (k=1,2,3) denote the first three moments of the approximated distributions of respectively the same random variables.
Then formally, according to our procedure, we first look for the reduced set $\Omega$ of those values of $n_1$ and $n_2$ so that:

(26) $$1 - \sum_{(n_1,n_2)\in\Omega} p(n_1,n_2) < 10^{-6}$$

and afterwards, for each $(n_1,n_2)\in\Omega$, we look for the reduced sets $\Lambda(n_1)$ and $\Lambda(n_2)$ of those values respectively of $x_1$ and $x_2$ so that:

(27) $$1 - \sum_{x_1\in\Lambda(n_1)} \varphi_1^{*n_1}(x_1) < 10^{-6}$$

(28) $$1 - \sum_{x_2\in\Lambda(n_2)} \varphi_2^{*n_2}(x_2) < 10^{-6}$$

and simultaneously, for k=1,2,3:

(29) $$\left|1 - \frac{E_a\left[(X_1^{(n_1)})^k\right]}{E\left[(X_1^{(n_1)})^k\right]}\right| < 10^{-6}$$

(30) $$\left|1 - \frac{E_a\left[(X_2^{(n_2)})^k\right]}{E\left[(X_2^{(n_2)})^k\right]}\right| < 10^{-6}$$

This means that, instead of calculating the function (21) for all the possible values of $x_1$ and $x_2$ so that with certainty:

(31) $$\sum_{x_1=0}^{MX_1}\sum_{x_2=0}^{MX_2} g(x_1,x_2) = \sum_{n_1=0}^{MN_1}\sum_{n_2=0}^{MN_2} p(n_1,n_2) \sum_{x_1=0}^{MX_1}\sum_{x_2=0}^{MX_2} \varphi_1^{*n_1}(x_1)\varphi_2^{*n_2}(x_2) = 1$$

we calculate (21) only for those values of $x_1$ and $x_2$ so that with a given error and without making any assumption about the counting distribution $p(n_1,n_2)$:

(32)

$$\sum_{x_1\in\Lambda(n_1)}\sum_{x_2\in\Lambda(n_2)} g(x_1,x_2)=\sum_{(n_1,n_2)\in\Omega} p(n_1,n_2)\sum_{x_1\in\Lambda(n_1)}\sum_{x_2\in\Lambda(n_2)}\varphi_1^{*n_1}(x_1)\varphi_2^{*n_2}(x_2)\cong 1$$

To conclude, it is important to notice that a similar procedure can be applied also in the multivariate situations studied by Sundt and mentioned in section 2.1.2.

### 3.2. Formalization according to the Individual Risk Theory model

We refer to the same problem studied by Gerber and Shiu mentioned in section 2.2, i.e. the problem of studying the probability of ruin for a portfolio of policies. We refer, in particular, to the case of a portfolio of life insurance policies but we remark that the results obtained in this case can be easily generalized to portfolios of damages insurance policies.
Unlike Gerber and Shiu, we intend to calculate the time of ruin and the amount of ruin by allowing for the investment opportunities of the insurer's surplus during a finite period of time.
According to the Individual Risk Theory model, we start by defining for each policy s (s=1,...,S) included in the portfolio and for each management year r (r=1,...,R) the random variable $X_s^{(r)}$ representing the annual financial flow to be expected at the end of the year by allowing for the accrued capital in various possible interest rate scenarios c (c=1,...,C)[14] and by assuming that this capital forms the reserve for the next year if the insured event occurs. In other words, we identify all the possible numerical realizations of this random variable (whose mean value corresponds to the traditional risk premium) and the corresponding probabilities (which consist in probabilities of death as well as of other elimination causes and, in some contracts, in probabilities of survival - i.e. of non-elimination - at maturity, too).
Let us take, for instance, a generical contract providing for payments on death and invalidity and also for a payment on survival at maturity. Let $i_c(0;r-1,r)$ denote the annual future interest rate structure according to the scenario c and $[1+i_c(0;r-1,r)]$ the corresponding compounding factor. Besides let:
- $n_s$ denotes the maturity of each policy s;
- $B_s^{(r)}$ denotes the benefit payable at time r to each policy s on death during the period (r-1,r] (event occurring with probability ${}_{r-1/}q_x^{(a,d)}$);
- $C_s^{(r)}$ denotes the benefits payable at time r to each policy s on invalidity during the period (r-1,r] (event occurring with probability ${}_{r-1/}q_x^{(a,i)}$);
- $E_s^{(ns)}$ denotes the benefits payable to each policy s on survival at maturity (event occurring with a probability opposite to the elimination one);
- $P_s^{(r-1)}$ denotes the premium collected for each policy s at the beginning of the year r;
- $V_{s,c}^{(r)}$ denotes the reserve available for each policy s at the end of the year r, where $V_{s,c}^{(r)}=(V_{s,c}^{(r-1)}+P_s^{(r-1)})[1+i_c(0;r-1,r)]$ with $V_{s,c}^{(0)}=0$[15].
Thus, at any time before maturity, we define the generical random variable $X_s^{(r)}$ by considering that:
a) if the insured event occurs during the year (r-1,r], at the end of the same year the insurer Company will use the existing reserve $V_{s,c}^{(r)}$ to pay the services contractually provided for, $B_s^{(r)}$ or $C_s^{(r)}$. In this case, the numerical realization of the random variable is $+V_{s,c}^{(r)}-B_s^{(r)}$ on

[14] For simplicity, in the financial evaluation we use an annual future interest rate structure varying in a deterministic way according to the given scenarios. On this subject, see Bowers et al. (1997).
[15] Notice that, even if we always use the same subscript c, the interest rate scenario used for calculating the reserve existing at time r may differ from the ones used for calculating the reserve existing at any previous time.

death or $+V_{s,c}^{(r)}-C_s^{(r)}$ on invalidity (in both cases, it corresponds in absolute value to the capital under risk);
b) if the insured event does not occur, at the end of the year the insurer Company will transfer the existing reserve to form the one for the next year. In this case, the numerical realization of the random variable is $+V_{s,c}^{(r)}-V_{s,c}^{(r)}=0$.
In conclusion, at any time before maturity, we define the generical random variable $X_s^{(r)}$ in the following way:

(33)

| Numerical realizations | Probabilities |
|---|---|
| $+V_{s,c}^{(r)}-B_s^{(r)}$ | ${}_{r-1/}q_x^{(a,d)}$ |
| $+V_{s,c}^{(r)}-C_s^{(r)}$ | ${}_{r-1/}q_x^{(a,i)}$ |
| 0 | $1-{}_{r-1/}q_x^{(a,d)}-{}_{r-1/}q_x^{(a,i)}$ |

And, in a similar way, at maturity:

(34)

| Numerical realizations | Probabilities |
|---|---|
| $+V_{s,c}^{(n_s)}-B_s^{(n_s)}$ | ${}_{n_s-1/}q_x^{(a,d)}$ |
| $+V_{s,c}^{(n_s)}-C_s^{(n_s)}$ | ${}_{n_s-1/}q_x^{(a,i)}$ |
| $+V_{s,c}^{(n_s)}-E_s^{(n_s)}$ | $1-{}_{n_s-1/}q_x^{(a,d)}-{}_{n_s-1/}q_x^{(a,i)}$ |

Under a theoretical point of view, once defined the above mentioned random variables $X_s^{(r)}$, it is possible to solve the problem of ruin in discussion by simply perform their sum for the first m (m=1,...,R) management years[16], i.e. the sum $W^{(m)}$:

(35)
$$W^{(m)}=\sum_{r=1}^{m}\left(\sum_{s=1}^{S}X_s^{(r)}\right)=\sum_{r=1}^{m}X^{(r)}$$

where $X^{(r)}$ denotes the annual financial flow to be expected for the entire portfolio. As a matter of fact, the time of ruin is that value of m so that the expected value of $W^{(m)}$ becomes negative for the first time (i.e. $T=\inf\{m|E[W^{(m)}]<0\}$), the probability distribution of the amount of ruin is the probability distribution of the random variable $W^{(T)}$ and also the probability distribution of the insurer's surplus before ruin is the one of the random variable $W^{(T-1)}$.
Notice, however, that the random variables $X_s^{(r)}$ to be summed up are independent random variables for a fixed r since they refer to different policies but they are dependent random variables for a fixed s since they refer to events concerning each individual policy during

[16] Notice that nothing changes in practice if we want to financially refer the random variables to be summed up to the same evaluation time.

various management years. It is therefore possible to calculate by convolution[17] the probability distribution of $X^{(r)}$ (r=1,...,R) but it is not possible to calculate in the same way the probability distribution of $W^{(m)}$.
We then state the problem in a different way. Actually, we re-write (35) in the following equivalent[18] way:

$$W^{(m)} = \sum_{s=1}^{S} X_s^{(1,2,\dots,m)} \tag{36}$$

where $X_s^{(1,2,\dots,m)}$ denotes, for each policy s, the random variable representing the financial flow to be expected during the first m management years. In order to define this random variable we go on by identifying all its numerical realizations and corresponding probabilities as done for the random variable $X_s^{(r)}$.
Let us take for instance the same contract of the previous example. We define the random variable $X_s^{(1,2)}$ for the first two management years in the following way:

(37)

| Numerical realizations | Probabilities |
|---|---|
| $+V_{s,c}^{(1)} - B_s^{(1)}$ | ${}_{0/}q_x^{(a,d)}$ |
| $+V_{s,c}^{(1)} - C_s^{(1)}$ | ${}_{0/}q_x^{(a,i)}$ |
| $+V_{s,c}^{(2)} - B_s^{(2)}$ | ${}_{1/}q_x^{(a,d)}$ |
| $+V_{s,c}^{(2)} - C_s^{(2)}$ | ${}_{1/}q_x^{(a,i)}$ |
| 0 | $1-{}_{0/}q_x^{(a,d)}-{}_{0/}q_x^{(a,i)}-{}_{1/}q_x^{(a,d)}-{}_{1/}q_x^{(a,i)}$ |

where the last numerical realization is $+V_{s,c}^{(2)}-E_s^{(2)}$ instead of 0, if $n_s=2$.
In the same way, by obviously availing ourselves of a suitable algorithm, we define for all the policies included in the portfolio the random variables $X_s^{(1,2,\dots,m)}$ concerning the first m management years, up to the entire contractual life. Although these random variables are functions of dependent random variables, they are independent from one another since they refer to different policies. It is therefore possible to directly perform their sum[19] and then calculate the probability distribution of (36), for m=1,...,R.

## 4. APPLICATIONS

### 4.1. Calculation of the total claims distribution

We intend to apply the procedure illustrated in section 3.1 for calculating the probability distribution of the total claim amount. We refer in particular to the bivariate case of two

[17] Actually, by means of an approximated procedure similar to the one discussed in section 3.1 and illustrated in detail in Bruno, Camerini, Tomassetti (1998).
[18] It is possible to verify that (36) and (35) have the same mean value and the same variance.
[19] See Bruno, Camerini, Tomassetti (1998) for a detailed description of the general running of the used algorithm performing the sum of independent discrete random variables.

portfolios of policies by making the assumptions of the case 1 studied by Hesselager and discussed in section 2.1.1. We remind that in this case, as shown in section 2.1.2, the solution of the problem in discussion is identical according to the approach of both Hesselager and Sundt but we remind also that our procedure can be applied in more general cases.
Actually, we have to compute (21) under the assumption that the bivariate counting distribution $p(n_1,n_2)$ is (5) with $p(n_1+n_2)$ satisfying (4). We assume in particular that $\rho=0.39$ and $p(n_1+n_2)$ is a Poisson distribution with parameter $\lambda=4.841423259$, i.e.:

$$p(n_1+n_2)=\frac{e^{-\lambda}\lambda^{n_1+n_2}}{(n_1+n_2)!} \tag{38}$$

Besides, we assume that $MN_1$ and $MN_2$ are both equal to 18. As a matter of fact it is easy to show that in this case:

$$\sum_{n_1=0}^{18}\sum_{n_2=0}^{18}\frac{e^{-\lambda}\lambda^{n_1+n_2}}{(n_1+n_2)!}\binom{n_1+n_2}{n_1}\rho^{n_1}(1-\rho)^{n_2}\cong 1 \tag{39}$$

with an error lower than $10^{-7}$.
Let us take for instance a collective of 1,440 policies distributed in two different portfolios with an equal number of policies[20] and, finally, let us assume that the probability distribution of the individual claim amount $\varphi_1(u_1)$ for the first portfolio is[21]:

(40)

| Realizations | Probabilities |
|---|---|
| 14 | 0.0103301 |
| 15 | 0.0307990 |
| 16 | 0.0293511 |
| 17 | 0.0103301 |
| 18 | 0.0730414 |
| 19 | 0.0111568 |
| 20 | 0.0264554 |
| 24 | 0.1002133 |
| 26 | 0.0815418 |
| 28 | 0.0252126 |
| 30 | 0.0212857 |
| 31 | 0.0254214 |
| 55 | 0.0991756 |
| 60 | 0.4556837 |
| otherwise | 0 |

and the probability distribution of the individual claim amount $\varphi_2(u_2)$ for the second portfolio is[22]:

[20] See Panjer (1981) and Ferrara, Manna, Tomassetti (1996) for numerical exemplifications in the univariate case of an individual portfolio with 1,440 policies.
[21] This is the same probability distribution used in Panjer, Willmot (1992, pp. 178,229).
[22] This is a probability distribution thought for numerical exemplification purposes.

(41)

| Realizations | Probabilities |
|---|---|
| 10 | 0.010 |
| 11 | 0.025 |
| 13 | 0.030 |
| 15 | 0.035 |
| 16 | 0.050 |
| 17 | 0.060 |
| 19 | 0.180 |
| 22 | 0.125 |
| 24 | 0.149 |
| 30 | 0.336 |
| otherwise | 0 |

The total claims distribution obtained in this case by applying the proposed procedure is reported in the following table in the form of a cumulative distribution function.

**Table 1: Total claim cumulative distribution**

| Total claim amount | Cumulative distribution |
|---|---|
| =0 | 0.007896 |
| ≤50 | 0.079882 |
| ≤60 | 0.105357 |
| ≤70 | 0.143682 |
| ≤80 | 0.190220 |
| ≤90 | 0.230474 |
| ≤100 | 0.280772 |
| ≤120 | 0.385218 |
| ≤140 | 0.496784 |
| ≤148.546 | 0.539984 |
| ≤160 | 0.594165 |
| ≤180 | 0.685002 |
| ≤200 | 0.766004 |
| ≤220 | 0.828510 |
| ≤250 | 0.898497 |
| ≤300 | 0.962527 |
| ≤400 | 0.996606 |
| ≤1000 | 1.000000 |

This distribution has a mean value of 148.546 and a standard deviation of 75.929.

### 4.2. Calculation of the time of ruin and of the ruin amount distribution

We apply the approach illustrated in section 3.2 for calculating the time of ruin and the probability distribution of the amount of ruin for a portfolio of life insurance policies. We refer in particular to a portfolio of 10,000 policies of term life insurance and of endowment

insurance which is composed as described in the following table[23]. Note that the benefits are expressed in thousands of Euro and so are the other monetary values presented in this section.

**Table 2. Portfolio composition**

| Type | Number | Benefits on death | Benefits on survival | Term | Age | Sex |
|---|---|---|---|---|---|---|
| 1 | 2500 | 80 | 30 | 20 | 35 | Female |
| 2 | 2500 | 100 | 30 | 20 | 45 | Male |
| 3 | 1500 | 130 | 0 | 15 | 50 | Female |
| 4 | 3500 | 200 | 0 | 10 | 60 | Male |

We use for simplicity a deterministic and constant annual interest rate of 2% but we remind that, as shown in section 3.2, the general approach can allow for various different interest rates scenarios.
In the following table, we report some characteristic parameters of the probability distributions of the random variables $W^{(m)}$ for m=1,...,5, given an initial capital of 40,000.

**Table 3. Characteristic parameters of $W^{(m)}$ for m=1,2...,5**

| m | Mean | Standard deviation | Skewness |
|---|---|---|---|
| 1 | 32,794 | 1,206 | -0.154 |
| 2 | 25,087 | 1,709 | -0.105 |
| 3 | 16,818 | 2,132 | -0.083 |
| 4 | 8,235 | 2,478 | -0.070 |
| 5 | -924 | 2,792 | -0.060 |

According to the proposed approach, we also know the probability distribution of the random variables $W^{(m)}$ for m=6,...,20 but we do not present the corresponding characteristic parameters since, as evident from table 3, the ruin occurs during the fifth year.
As for the capital to be expected at ruin $W^{(5)}$, we obtain a distribution with 52,083 numerical realizations and corresponding probabilities that we summarize in the following table.

**Table 4. Cumulative distribution function of $W^{(5)}$**

| Classes of realizations | Cumulative function |
|---|---|
| $\mu-4\sigma$ | 0.000056 |
| $\mu-3\sigma$ | 0.001720 |
| $\mu-2.5\sigma$ | 0.007136 |
| $\mu-2\sigma$ | 0.024336 |
| $\mu-1.9\sigma$ | 0.030408 |
| $\mu-1.8\sigma$ | 0.037671 |
| $\mu-1.7\sigma$ | 0.046310 |
| $\mu-1.6\sigma$ | 0.056491 |
| $\mu-1.5\sigma$ | 0.068383 |

[23] It is important to notice that this application is more general and realistic than the ones presented by Amsler (1992).

| | |
|---|---|
| $\mu-\sigma$ | 0.158667 |
| $\mu-0.75\sigma$ | 0.225357 |
| $\mu-0.5\sigma$ | 0.305965 |
| $\mu$ | 0.496063 |
| $\mu+0.25\sigma$ | 0.594976 |
| $\mu+0.5\sigma$ | 0.688724 |
| $\mu+0.75\sigma$ | 0.771989 |
| $\mu+\sigma$ | 0.841326 |
| $\mu+1.5\sigma$ | 0.934868 |
| $\mu+2\sigma$ | 0.978918 |
| $\mu+2.5\sigma$ | 0.994710 |
| $\mu+3\sigma$ | 0.998990 |
| $\mu+4\sigma$ | 0.999985 |
| $\mu+5\sigma$ | 1.000000 |

This distribution has the following characteristic parameters:
Mean ($\mu$): -924. This value consists of two component: a positive component due to the expected capital at the beginning of the year equal to 44,163 and a negative one due to the total risks equal to -45,087;
Standard deviation ($\sigma$): 2,791.577;
Skewness: -0.060;
Kurtosis: 3.003.

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