An Extension Model of Financially-Balanced Bonus-Malus System

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1. Introduction

1.1 Pricing the automobile third-party liability insurance

- A priori rate making
- A posteriori evaluation

Premium = A base premium x Bonus-Malus factors :

- A base premium (BP)： a function of the current rating factors, resorting on generalized linear model
- Bonus-Malus factors BMF(k,t)： depending on the number of years t the policy is in force and on the number of claims k reported during this period

1.2 Optimal BMS by Bayes theorem

- Assume that the distribution of the number of claims of each individual conforms to a Poisson(λ)
- Each policyholder is characterized by the value of his parameter λ, and λ is the observed value of a random variable Λ whose density function is called structure function $u(.)$.
- Assume the distribution of Λ is a Gamma with parameters α and β , that is

$$u(x) = \frac{\beta^{\alpha} e^{-\lambda\beta} x^{\alpha-1}}{\Gamma(\alpha)} \qquad \alpha, \beta > 0, and \quad E\Lambda = \frac{\alpha}{\beta}$$

Then the resulting distribution of the number of claims in the portfolio is called a mixed Poisson distribution

$$p_k = \int_0^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} u(\lambda) d\lambda,$$

and p_k is a negative binomial distribution.

An optimal BMS for the Negative Binomial Model

- Consider a policyholder observed for t years, and denotes by k_j the number of accidents in which he was at fault incurred during year j. So the information concerning the policyholder is a vector $(k_1, \cdots, k_t)$.
- By Bayes theorem, given claims history $(k_1, \cdots, k_t)$, the posterior expectation of Λ is

$$E(\Lambda \mid k_1, \cdots, k_t) = \frac{\alpha + k}{\beta + t},$$

where $k = \sum_{i=1}^{t} k_i$, which is the best estimate of Λ for a quadratic loss function at time t+1.

- The above means that the policyholder who underwent claims history $(k_1, \cdots, k_t)$ will have to pay a premium

$$P_{k+1}(k_1, \cdots, k_t) = BP \times BMF(k,t) = \frac{\alpha + k}{\beta + t}, \quad \text{let} \quad C_{(k,t)} = \frac{\alpha + k}{\beta + t} \cdot \frac{\beta}{\alpha}$$

Some Properties of the Optimal BMS

- The system is fair in a Bayesian sense.
- The system is financially balanced: Every single year, the average of all premiums, collected from all policyholders, remains constant at the initial level $E\Lambda = \dfrac{\alpha}{\beta}$. Let

$$\alpha = 1.0923183, \quad \beta = 7.70077, \quad C_{(k,t)} = \frac{\alpha + k}{\beta + t} . \frac{\beta}{\alpha}$$

Table 1

OPTIMAL WEIGHT $100 \times C_{(N, t)}$

$t \setminus N$	0	1	2	3	4
0	100 00				
1	88 51	169 53	250 56	331 59	412 61
2	79 38	152 06	224 73	297 41	370 08
3	71 96	137 85	203 73	269 61	335 50
4	65 81	126 07	186 32	246 57	306 82
5	60 63	116 14	171 65	227 16	282 66
6	56 21	107 66	159 12	210 58	262 03
7	52 38	100 34	148 30	196 25	244 21
8	49 05	93 95	138 35	183 75	228 65
9	46 11	88 32	130 54	172 75	214 96

1.3 An example of the BMS from Coene and Doray (1996, ASTIN Bulletin)

- The BMS is a system with 18 classes. $C_i, i = 1, \cdots, 18,$ denotes the premium scale who is in class i. The premium corresponding to class 10 is equal to the base premium, so that $C_{10} = 1$. A new driver will start in this class. The class of a driver will be modified each year according to the following transition rules.

- Y_t denotes the class of a driver for the period [t,t+1), N_t denotes the number of claims in the preceding period. This process Y_t is thus defined by the following equation

$$Y_0 = 10, Y_t = Y_{t-1} + 3N_t - 1, t > 0, and \quad 1 \le Y_t \le 18.$$

- The system is Markovian. And the Makov chain with finite states has a long run stationary distribution.

Table 2: An example of the BMS from Coene and Doray (1996 ASTIN Bulletin) defined by some transition rules.

$t \setminus N$	0	1	2	3	4
0	C_{10}				
1	C_9	C_{12}	C_{15}	C_{18}	C_{18}
2	C_8	C_{11}	C_{14}	C_{17}	C_{18}
3	C_7	C_{10}	C_{13}	C_{16}	C_{18}
4	C_6	C_9	C_{12}	C_{15}	C_{18}
5	C_5	C_8	C_{11}	C_{14}	C_{17}
6	C_4	C_7	C_{10}	C_{13}	C_{16}
7	C_3	C_6	C_9	C_{12}	C_{15}
8	C_2	C_5	C_8	C_{11}	C_{14}
9	C_1	C_4	C_7	C_{10}	C_{13}

Allow a reduction of one class for each claim-free year , and penalize policyholder by two classes for the first accident and by three classes for each subsequent accident .

The lowest level is c_1, the highest level is c_{18}.

1.4 A financially balanced BMS defined by Coene and Doray (1996, ASTIN Bulletin)

$$\min \sum_{l=1}^{18} \sum_{(0\le N\le 4, 1\le t\le 9)} f_l \left[C_l - C_{(N,t)} \right]^2$$

$$\begin{cases} C_{l+1} - C_l \ge 0, \quad l = 1,\dots,17 \\ C_{10} = 1 \\ \sum_{l=1}^{18} f_l C_l \ge 1 \end{cases}$$

$$C_{(N,t)} = \frac{\alpha + N}{\beta + t} . \frac{\beta}{\alpha}$$

Table 3 Premium Scales

Class	f_1	$100C_l$
1	0 66229	79 2
2	0 05658	82 2
3	0 07675	85 5
4	0 02246	88 8
5	0 01985	93 8
6	0 02487	99 6
7	0 00992	100 0
8	0 00823	100 0
9	0 01703	100 0
10	0 00648	100 0
11	0 00661	180 2
12	0 01433	195 1
13	0 00676	220 8
14	0 00840	237 9
15	0 01349	258 1
16	0 01078	282 4
17	0 01492	306 6
18	0 02025	357 9

1.5 The open model

- Centeno and Andrade (2001, IME) studied bonus systems in an open portfolio, i.e. considering that a policyholder could transfer his policy to a different insurance company at any time in Portugal. They named this situation by “open model” as opposed to the model without exits, which was named by “closed model”.

- Under the assumptions that the market shares of the company were exogenously given and the probabilities that the policyholders exited the company did not depend on the premiums, they proved the long run distribution of the policyholders among the classes of the bonus system was independent of the market shares, and was easily calculated.

- But assuming that the exits probabilities were independent of the premium scales should be questionable. Particularly in China, by the lack of data disclosure among insurers, the cost of evasion is very low. When the malus is twice than the initial premium, the insured will exit the company with the probability nearly one.

- Here we assume that the insured will exit the company with the probability one when the malus is beyond the level of the tolerance, it means that all the policyholders only have a single pattern to exit the system.

2. Extension models of a financially balanced BMS in an open portfolio

Model I of general model in an open portfolio

$$\min\left\{\sum_{l=1}^{K}\sum_{(N,t)} f_l^{(n)}(C,G)\left[C_l - C_{(N,t)}\right]^2\right\}$$

subject to

$$(1)\begin{cases} C_{l+1}-C_l \geq 0, \quad l=1,\dots,K-1 \\ C_k = 1 \\ \sum_{l=1}^{K} f_l^{(n)}(C,G)C_l \geq 1 \end{cases} \qquad (2)\begin{cases} C_{l+1}-C_l \geq 0, \quad l=1,\dots,K-1 \\ C_k = 1 \\ \sum_{l=1}^{K} f^{(n)}{}_l(C,G)C_l \geq \dfrac{E\Lambda_n}{E\Lambda} \end{cases}$$

where G is the the level of the tolerance , $E\Lambda_n$ is the average premium, in period n.

Remark: $\sum_{l=1}^{K} f^{(n)}{}_l(C,G)C_l \geq \dfrac{E\Lambda_n}{E\Lambda}$ is a new financially-balanced condition in open portfolio. It will change dynamically because the distribution of Λ_n depends on the new entrants rate of bad drivers and the exits rate of older bad drivers that is related with the premium C and the level of the tolerance G.

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Model II of general model in an open portfolio

1) $$\min\left\{\sum_{l=1}^{K}\sum_{(N,t)} f_l^{(n)}(C,G)\left[C_l - C_{(N,t)}\right]^2 + M\times\left[\sum_{l=1}^{K} f_l^{(n)}(C,G)C_l - 1\right]^2\right\}$$

$$\begin{cases} C_{l+1} - C_l \geq 0, & l = 1,\dots,K-1 \\ C_k = 1 \end{cases}$$

2) $$\min\left\{\sum_{l=1}^{K}\sum_{(N,t)} f_l^{(n)}(C,G)\left[C_l - C_{(N,t)}\right]^2 + M\times\left[\sum_{l=1}^{K} f_l^{(n)}(C,G)C_l - \frac{E\Lambda_n}{E\Lambda}\right]^2\right\}$$

$$\begin{cases} C_{l+1} - C_l \geq 0, & l = 1,\dots,K-1 \\ C_k = 1 \end{cases}$$

where *M* is a penalty factor.

Remark: The idea is from the method of penalty function in nonlinear optimization.

Model III of a feasible simplification model

- First, assuming $C_K \le G$, that is keeping all drivers in the portfolio.
- Second, assuming the entering rate is 0, then the portfolio is close. Therefore $\frac{E\Lambda_n}{E\Lambda} = 1$, the stationary probability is existent. We get

$$\min\left\{\sum_{l=1}^{K}\sum_{(N,t)} f_l\left[C_l - C_{(N,t)}\right]^2 + M \times \left[\sum_{l=1}^{K} f_l C_l - 1\right]^2\right\}$$

$$\begin{cases} C_{l+1} - C_l \ge 0, \quad l = 1,\ldots,K-1 \\ C_k = 1 \\ C_K \le G \end{cases}$$

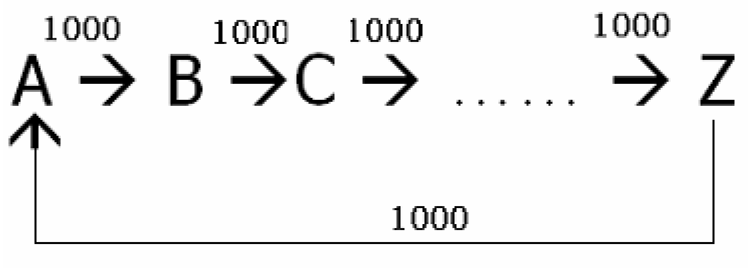
With lacking data disclosure among insurers, one possible reason of keeping the bad drivers in the open portfolio rather than adopting tougher systems, comes from game theory.

- Subramanian (1998, NAAJ) developed a model illustrating that the most aggressive insurer eventually manages to drive its competitor out of the market. The most severe BMS won in all cases.

- However, the folk theorem for infinitely repeated games assert that if the players are sufficiently patient then any feasible, individually rational payoffs can be enforced by an equilibrium.

- Analogous to the Prisoner's Dilemma game, if the game is repeated finitely many times then the unique equilibrium is for all players to always play selfishly; if there is an infinite time horizon then the actual outcome is always the cooperative, unless someone makes a mistake.

An imaginary case that all insurers penalize most severely



After one year, 1000 bad drivers in company A will exit and enter B, at the same time, bad drivers in company B will exit and enter C, and so on. Then all insurers should only obtain the base premium.

However, if they adopt modest penalty and keep the bad drivers in the portfolio, then the profit is higher.

Therefore, if one insurer penalize more severely, the others have to follow it, then the result is that all will lost more than the cooperative.

Note: This is imaginary and over-simplistic case without any test.

3 simulation

Transition rule:

$$Y_0 = 10, Y_t = Y_{t-1} + 4N_t - 1,$$
$$t > 0, \quad \text{and} \quad 1 \le Y_t \le 24.$$

f_l is the stationary distribution, the third column is financially balanced BMS defined by Coene and Doray (1996).

The last row is the average premium (AP).

G is the the level of the tolerance, M is a penalty factor. $a = 5\%$ means that

$$C_{l+1} - C_l \ge a, \quad l = 1, \ldots, 9.$$

Table 4 Premium Scales of BMS with 24 classes for an Open Portfolio

	f_l	Balanced BMS	$G=2$ $M=1$ $a=0\%$	$G=2$ $M=1$ $a=5\%$	Belgian BMS (1992)
1	0.5602	54.10	**58.38**	55.00	54
2	0.0514	57.04	61.32	60.00	54
3	0.0449	60.38	64.65	65.00	54
4	0.05	64.20	68.48	70.00	57
5	0.0175	78.47	80.61	75.00	60
6	0.0354	83.88	86.02	80.00	63
7	0.015	90.15	92.28	85.00	66
8	0.0108	97.52	99.66	90.00	69
9	0.0073	100.00	100.00	95.00	73
10	0.0277	100.00	100.00	100.00	77
11	0.006	147.08	149.21	150.40	81
12	0.0054	159.60	161.72	162.91	85
13	0.0049	173.97	175.40	176.20	90
14	0.0226	189.02	191.17	192.36	95
15	0.005	204.00	**200.00**	200.00	100
16	0.0049	221.64	200.00	200.00	105
17	0.0059	233.55	200.00	200.00	111
18	0.0197	241.61	200.00	200.00	117
19	0.0078	260.91	200.00	200.00	123
20	0.0103	283.72	200.00	200.00	130
21	0.013	311.12	200.00	200.00	140
22	0.0224	314.81	200.00	200.00	160
23	0.0214	343.49	200.00	200.00	200
24	0.0305	395.34	200.00	200.00	
100AP		100	87.73	85.35	

4.Conclusion

- The result of the model is closer to the BMS scales that we see in practice, i.e., both bonus and malus scales are reduced to some milder levels than would be indicated in an optimal BMS system.

- The stationary probability of being in classes in open portfolio is heavily related with the tolerance degree of a driver who wants to evade the BMS to avoid maluses. In fact, the best way to describe the behavior of evasion is the probability logic, but most difficult.

- The rationality of keeping the bad drivers in the portfolio from a viewpoint of infinitely repeated games should be proved thoroughly.

Thanks!