Optimal Strategies for Ruin Probabilities and Expected Gains

Dr. Cary Chi-Liang Tsai and Dr. Gary Parker
Department of Statistics and Actuarial Science
Simon Fraser University

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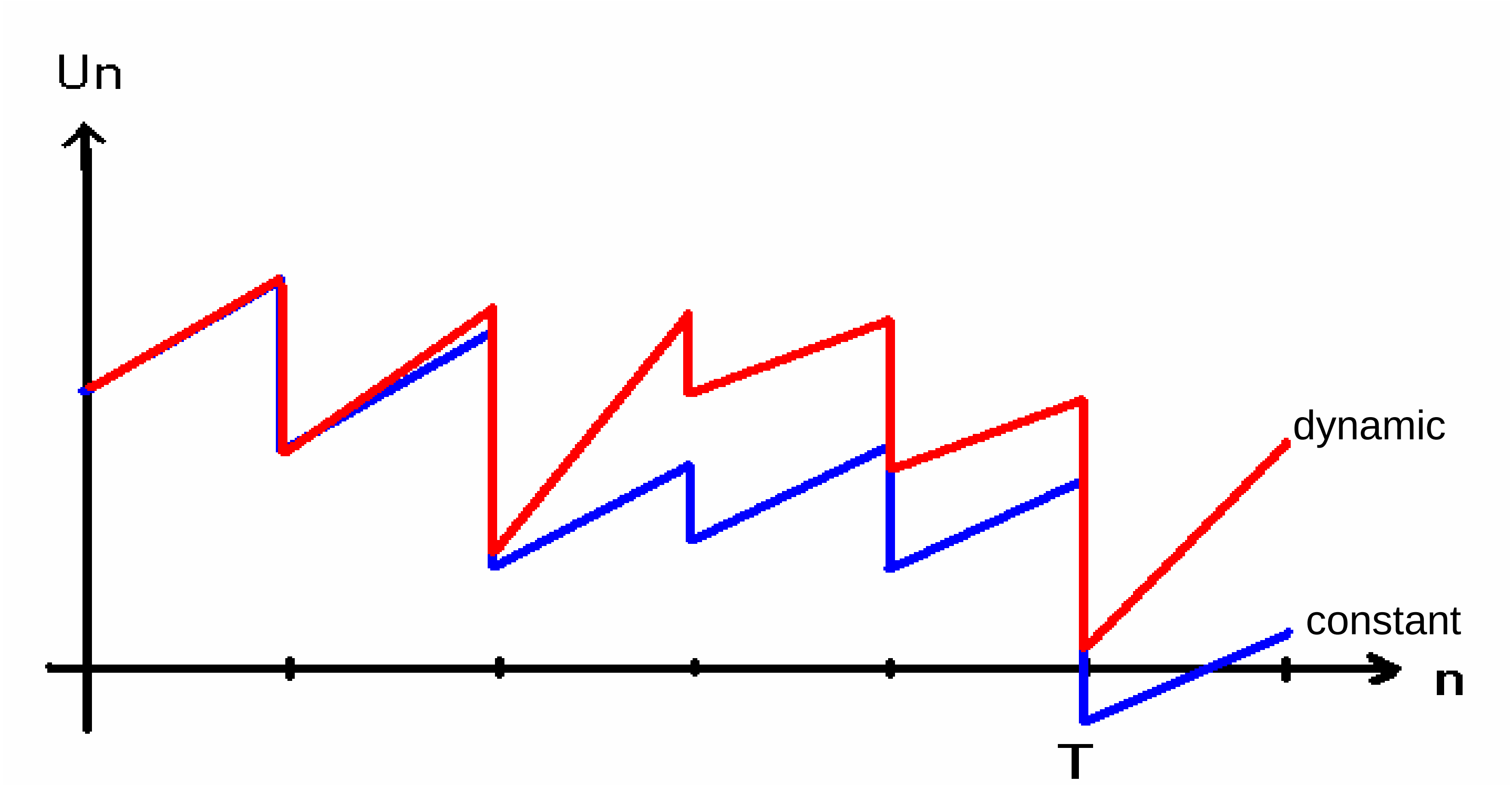
Classical Surplus Process (discrete time)

- $U_n = u + n \times c - S_n$: the surplus at time n
- $u = U_0$: the initial surplus
- c: the amount of premium received each period
- $S_n = W_1 + W_2 + \ldots + W_n$: the aggregate claims up to time n
- W_k: the sum of the claims in the kth period
- $W_1, W_2, \ldots W_n$ are i.i.d. r.v. distributed as W
- $c = (1+\theta) \times E[W]$
- θ : the relative security loading
- $E(W) = E(N) \times E[X]$, and N~Poisson(λ)

Probability of Ruin

- $T = \min\{n: U_n < 0\}$ ($T = \infty$ if $U_n \geqq 0$ for all n):
  the time of ruin (the first time that the surplus becomes negative)
- $\Psi(u) = \Pr\{T < \infty \mid U_0 = u\}$:
  the probability of ruin
- $\Psi(u,n) = \Pr\{T \leq n \mid U_0 = u\}$:
  the probability of ruin before or at time n (the dist. function of T)

Constant and Dynamic Premiums for Surplus Process



Buhlmann's Credibility Premium

- **Case 1**: $U_{n+1} = U_n + c - W_{n+1}$, $c = (1+\theta) \times E[W]$
- **Case 2**: $U_{n+1} = U_n + c_{n+1} - W_{n+1}$,

  $c_{n+1} = (1+\theta) \times [Z_n \times \overline{W}_n + (1-Z_n) \times E(W)]$,

  $Z_n = n / (n + v/a)$, $\overline{W}_n = \frac{1}{n}\sum_{i=1}^{n} W_i$
- **Case 3**: $U_{n+1} = U_n + c_{n+1,m} - W_{n+1}$,

  $c_{n+1,m} = (1+\theta) \times [Z_{n,m} \times \overline{W}_{n,m} + (1-Z_{n,m}) \times E(W)]$,

  $Z_{n,m} = m / (m + v/a)$, $\overline{W}_{n,m} = \frac{1}{m}\sum_{i=h}^{n} W_i$

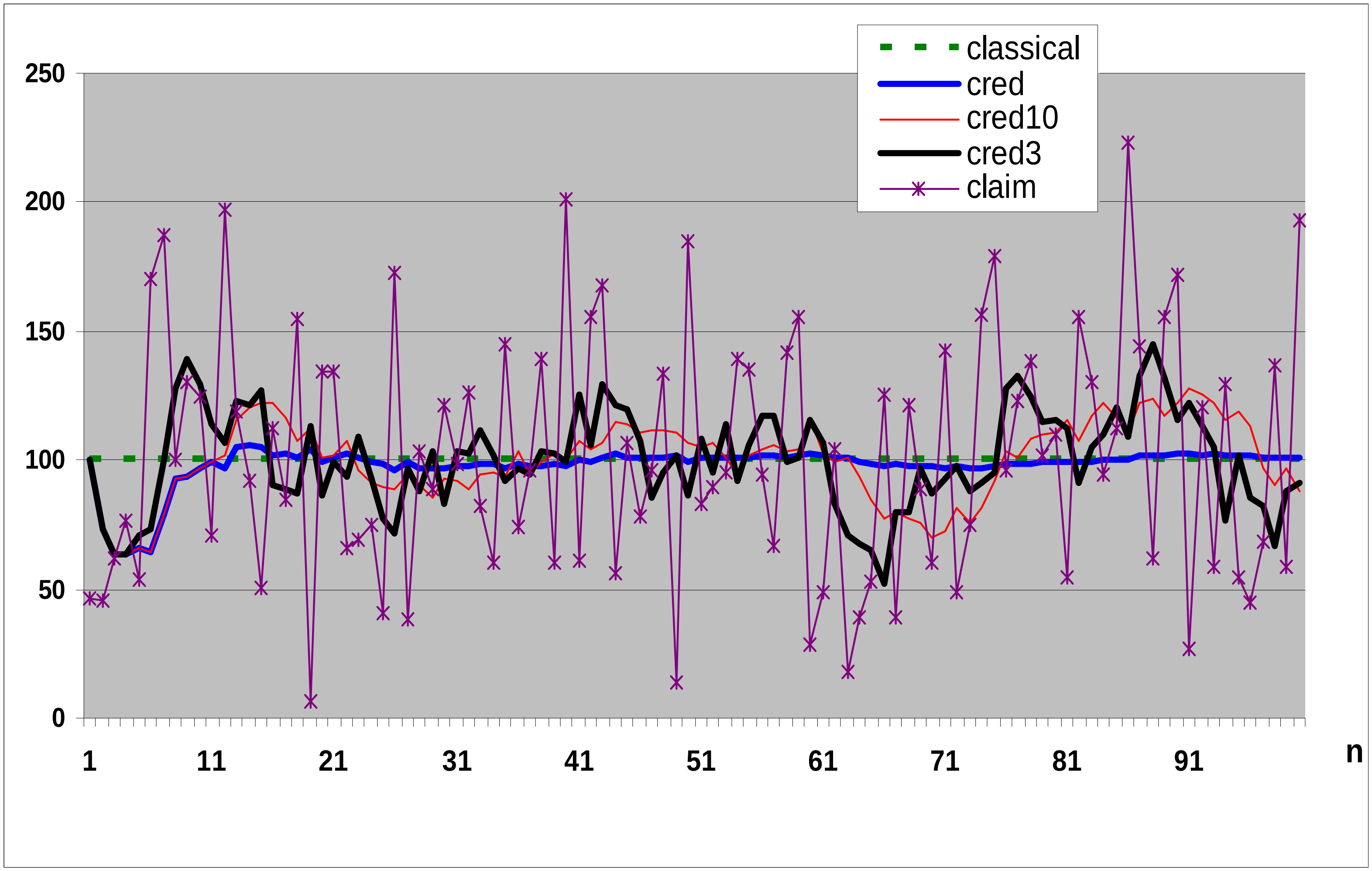
  where $m = \min(n,k)$, $h = \max(n-k,0)+1 = n-m+1$

- v: the expected process variance, a: the variance of the hypothetical means
- Note: $k = \infty$, $m = n$, $Z_{n,m} = Z_n$ and $h = 1$, Case 3 => Case 2
  $k = 0$, $m = 0$, $Z_{n,m} = 0$ and $h = n+1$, Case 3 => Case 1

Strategies

- Premium scheme:
  1:constant, 2:cred, 3:cred10, 4:cred3
- DL modifier: deductible and policy limit
  1:none, 2:deductible, 3: policy limit, 4: both
- Size modifier: sizes of deductible and policy limit
  1:k=∞, 3:M=3, 4:M=4, 5:M=5
  where deductible, D= E[X] / M and
  policy limit L= E[X] $\times$ M
  $c_1 = (1+\theta) \times E[N] \times E[Y]$, Y is the r.v. with D/L on X
- 40 strategies in total
- Goal: which strategies reduce ruin probability most?

A sample path



Claims Distributions and Parameters

- 3 mixtures for frequency(F) and severity(S):
  Low F / High S: λ = E[N]=1 and E[X]=100
  Mid F / Mid S: λ = E[N]=10 and E[X]=10
  High F / Low S: λ = E[N]=100 and E[X]=1
  All three mixtures have equal mean E[W]=100
- 3 distributions for individual claim size X
  LT (Light-Tailed) : Weibull(α,θ) with $\alpha > 1$
  RT (Neutral-Tailed) : Exponential(β)
  HT (Heavy-Tailed) : Pareto(τ,θ) with $\tau > 1$
  All three distributions have equal mean E[X]

Underlying Severity Distributions

Severity dist'n X	Light -tailed	Neutral -tailed	Heavy -tailed	E[X]
Low severity	Weibull $(2, 1/\Gamma(1.5))$	Exponential (1)	Pareto (3, 2)	1
Mid severity	Weibull $(2,10/\Gamma(1.5))$	Exponential (10)	Pareto (3, 20)	10
High severity	Weibull $(2,100/\Gamma(1.5))$	Exponential (100)	Pareto (3, 200)	100

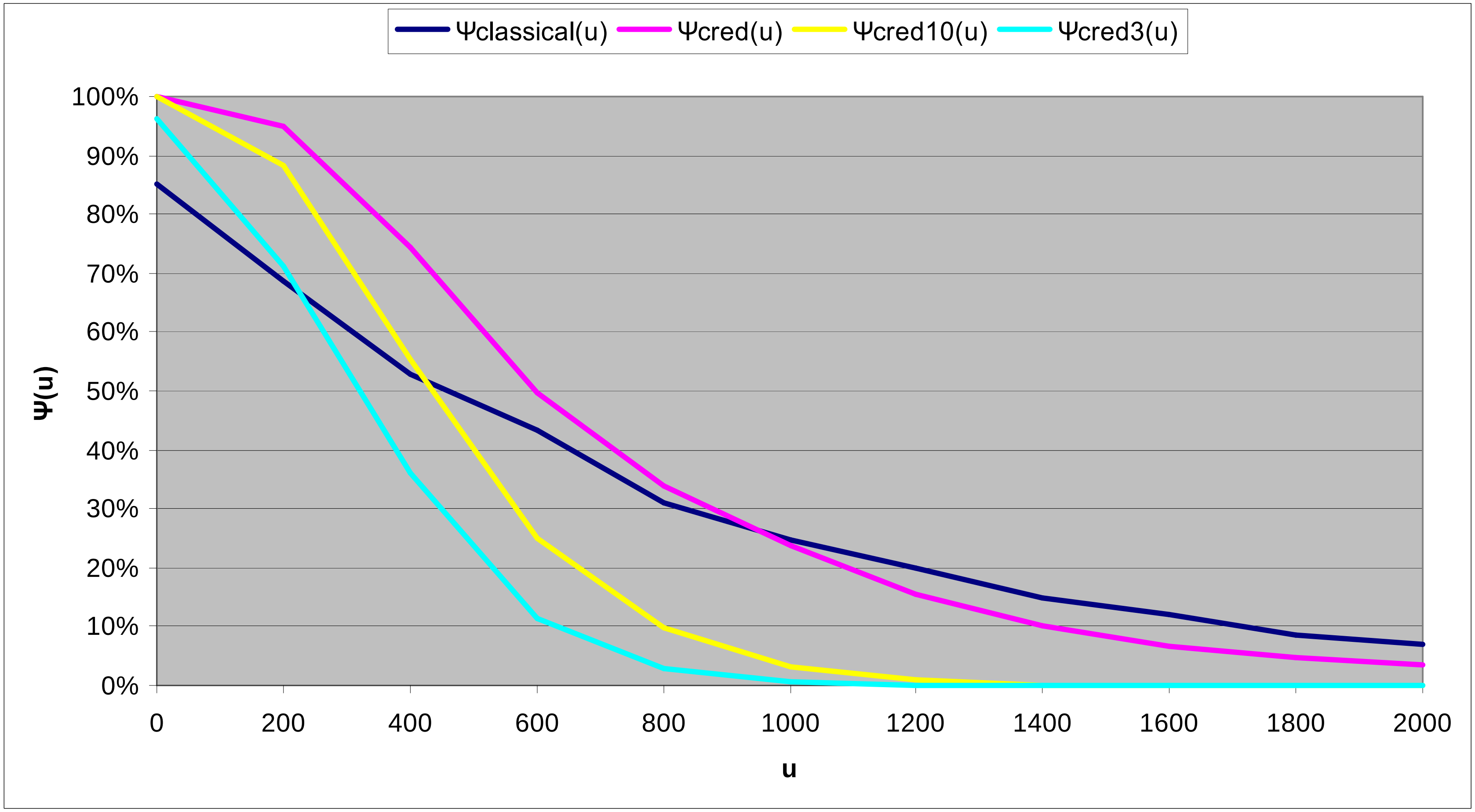
Probabilities for claims with D/L

	Probabilities	Weibull	Exponential	Pareto
M=2	P(X<D)	0.178	0.393	0.488
	P(D<X<L)	0.779	0.471	0.387
	P(X>L)	0.043	0.135	0.125
M=3	P(X<D)	0.083	0.283	0.370
	P(D<X<L)	0.916	0.667	0.566
	P(X>L)	0.001	0.050	0.064
M=4	P(X<D)	0.048	0.222	0.298
	P(D<X<L)	0.952	0.760	0.665
	P(X>L)	0.000	0.018	0.037

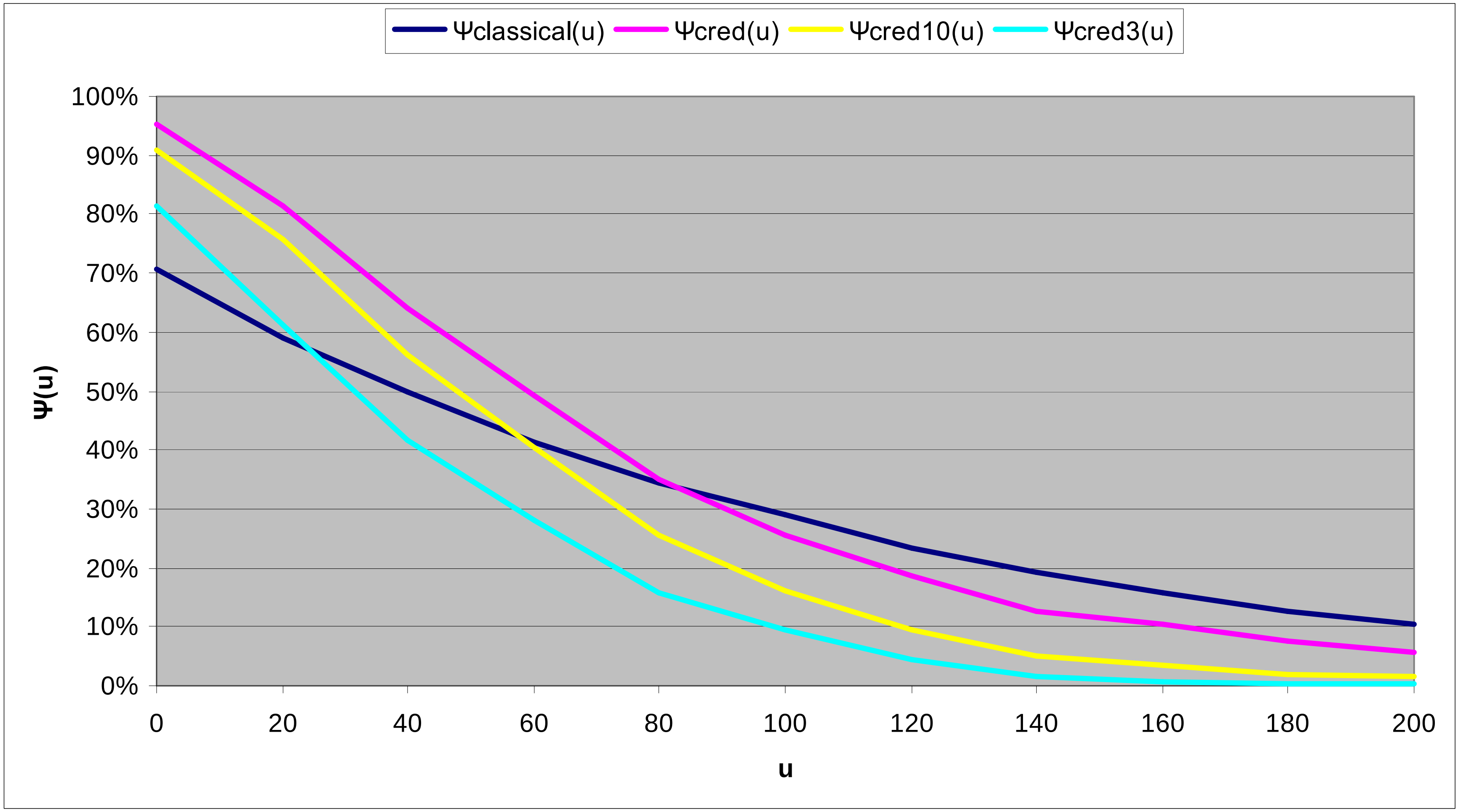
Monte Carlo Simulations

- 11 initial surpluses u
  0, 200, …, 2000 for low frequency and high severity
  0, 20, ….., 200 for mid frequency and mid severity
  0, 2, ..., 20 for high frequency and low severity
- Generate number of claims n from Poisson(λ) first, then claims X_1, X_2,…, X_n for each year
- 1000 paths for U_n up to n=100 for each case
- Ruin probability,
  $\Psi(u)$ = # of $\{U_k < 0$ for some $k \leq 100 | U_0 = u\}$ / 1000

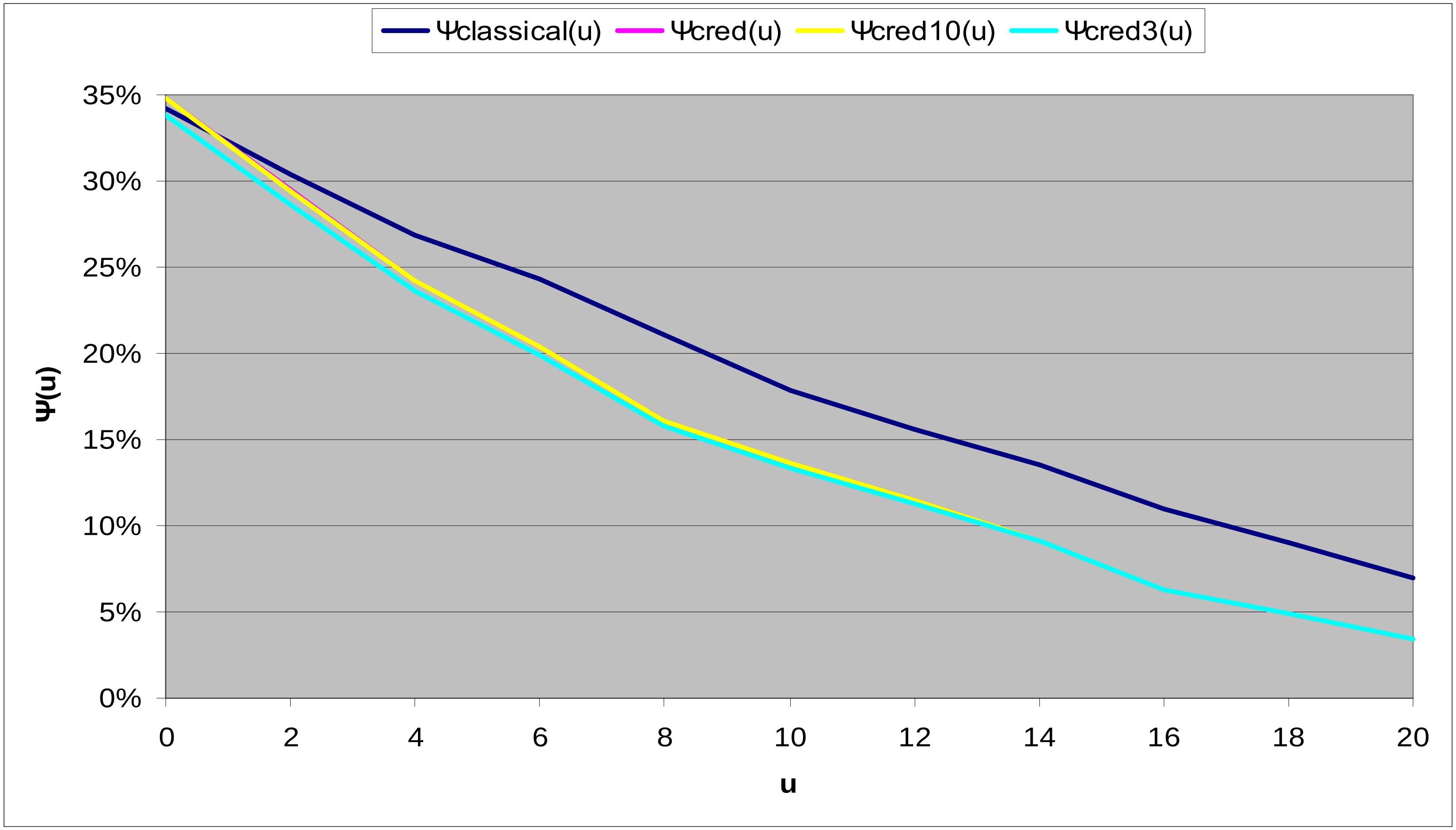
Poisson(1)/EXP(100) Neutral-Tailed
Low Frequency/High Severity without D/L



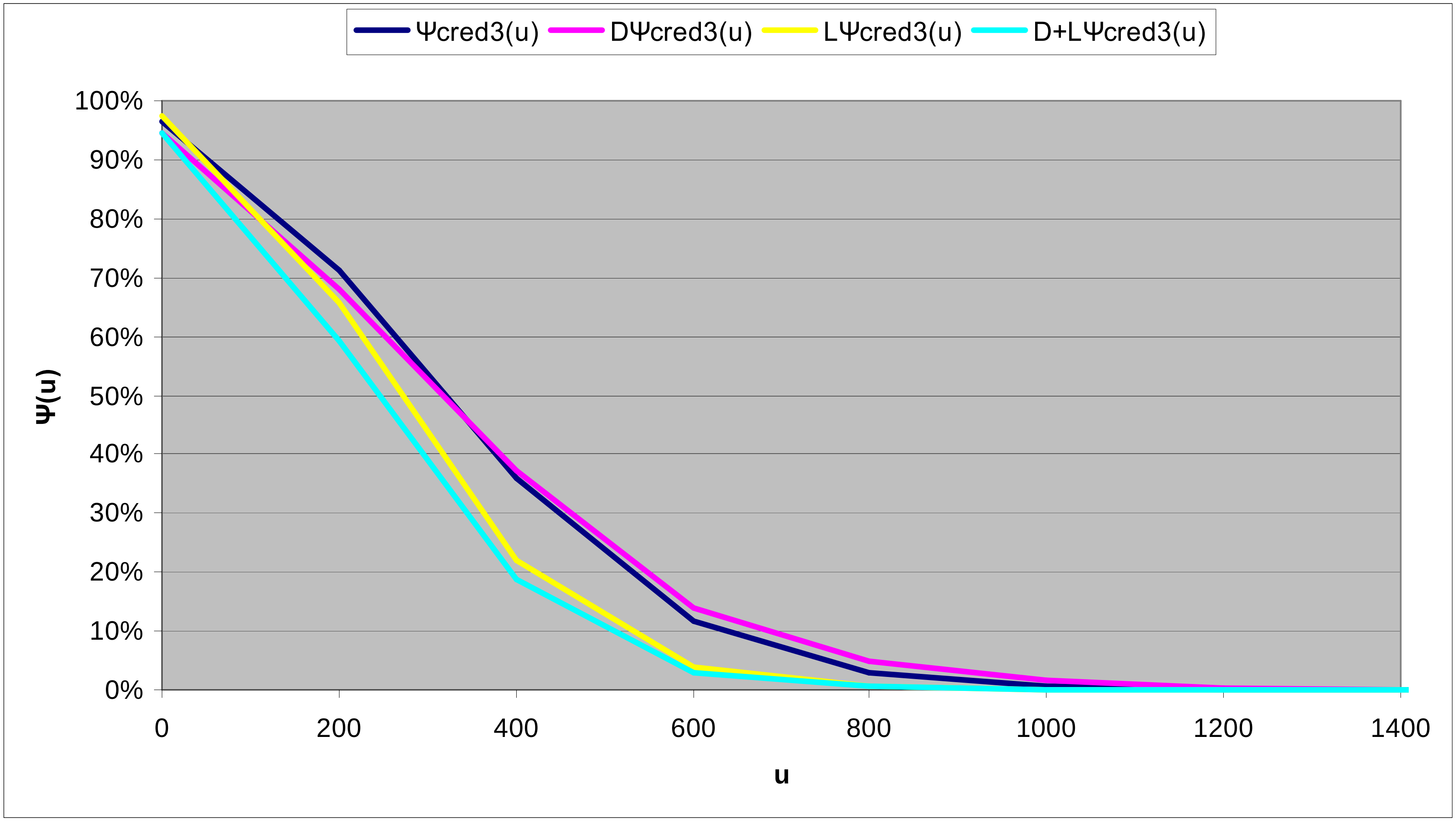
Poisson(10)/EXP(10) Neutral-Tailed Mid Frequency/Mid Severity without D/L



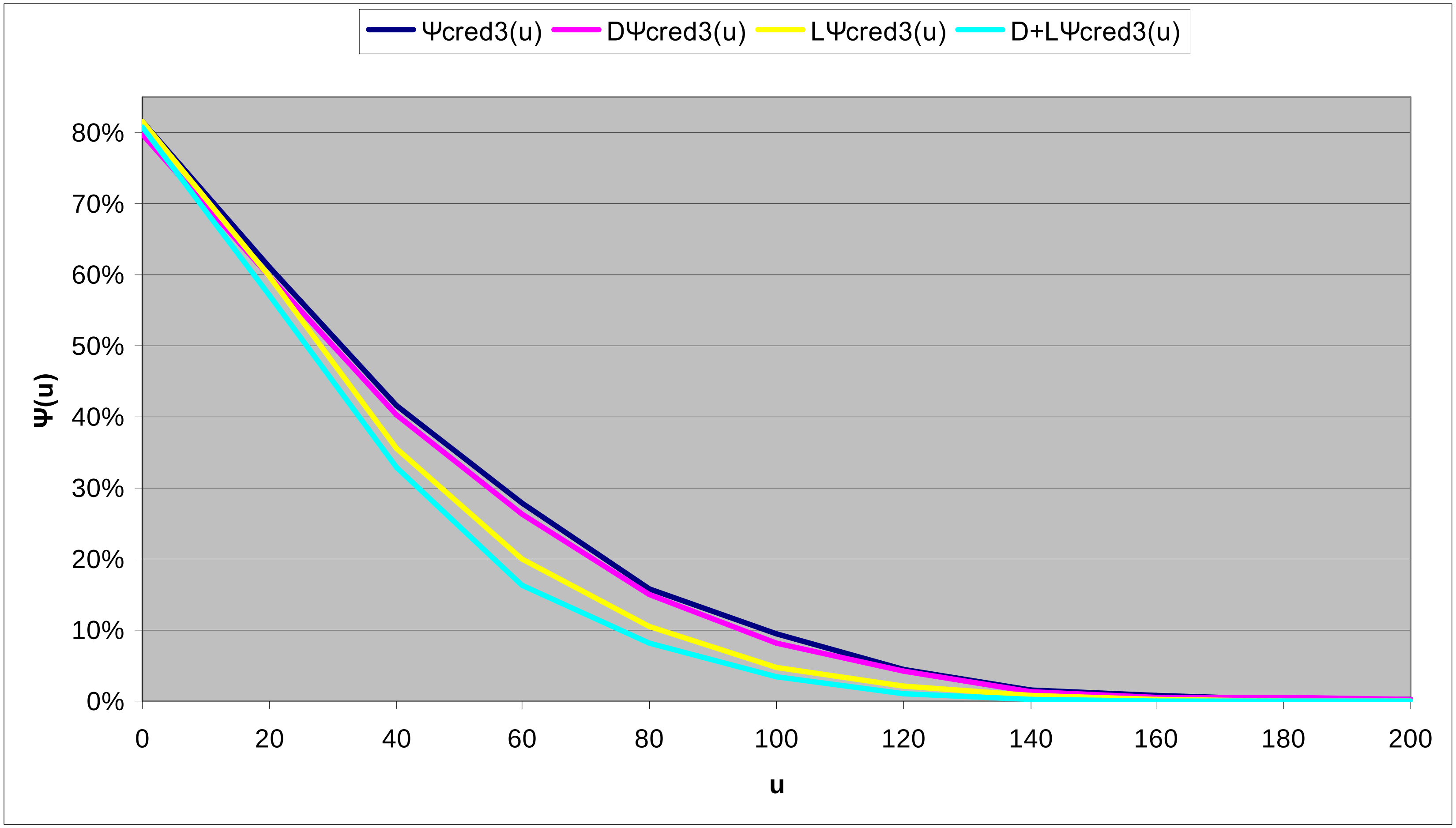
Poisson(100)/EXP(1) Neutral-Tailed High Frequency/Low Severity without D/L



Poisson(1)/EXP(100) Neutral-Tailed
Low Frequency/High Severity, Cred_3 and K=3



Poisson(10)/EXP(10) Neutral-Tailed
Mid Frequency/Mid Severity, Cred_3 and K=3



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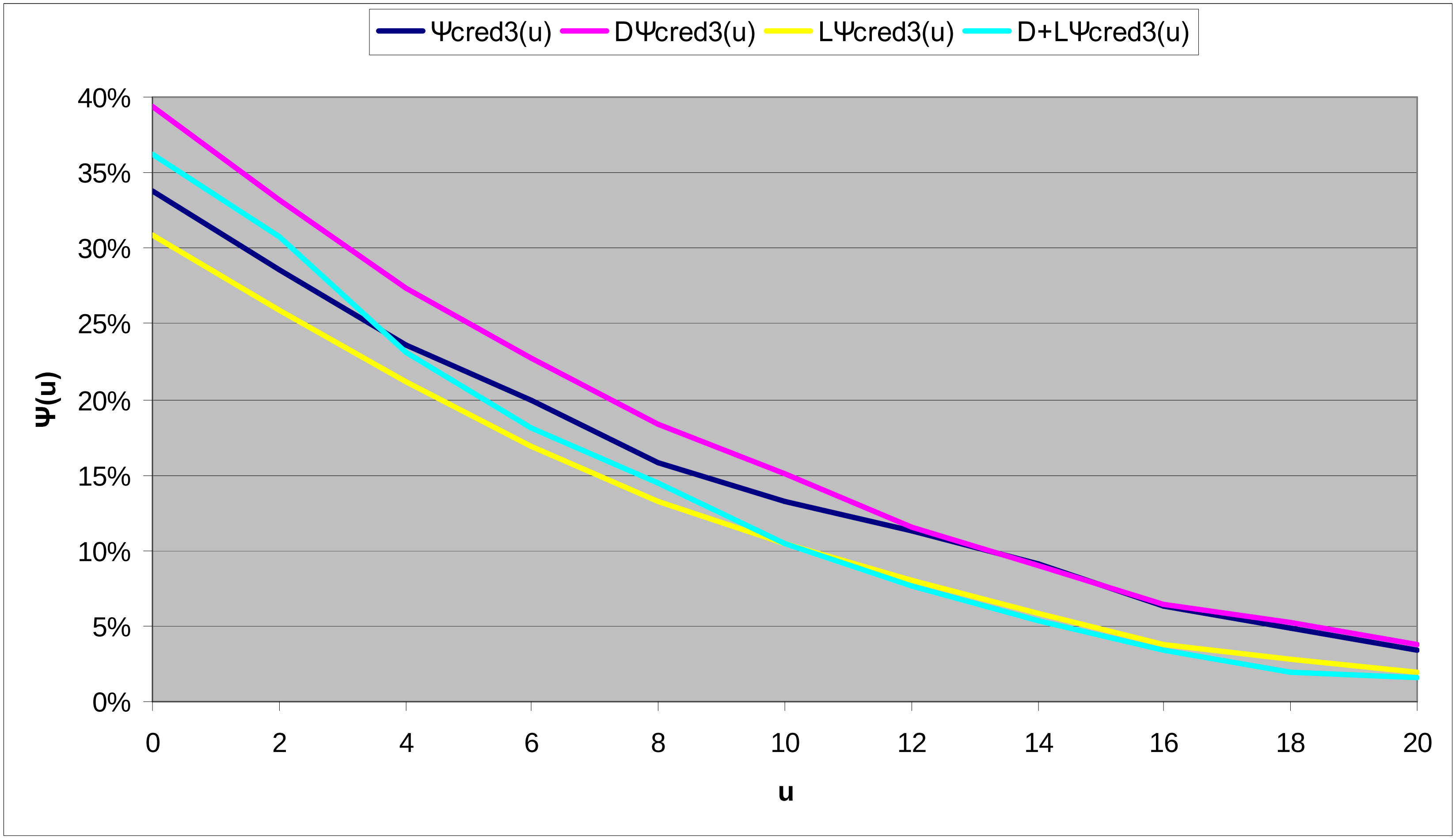
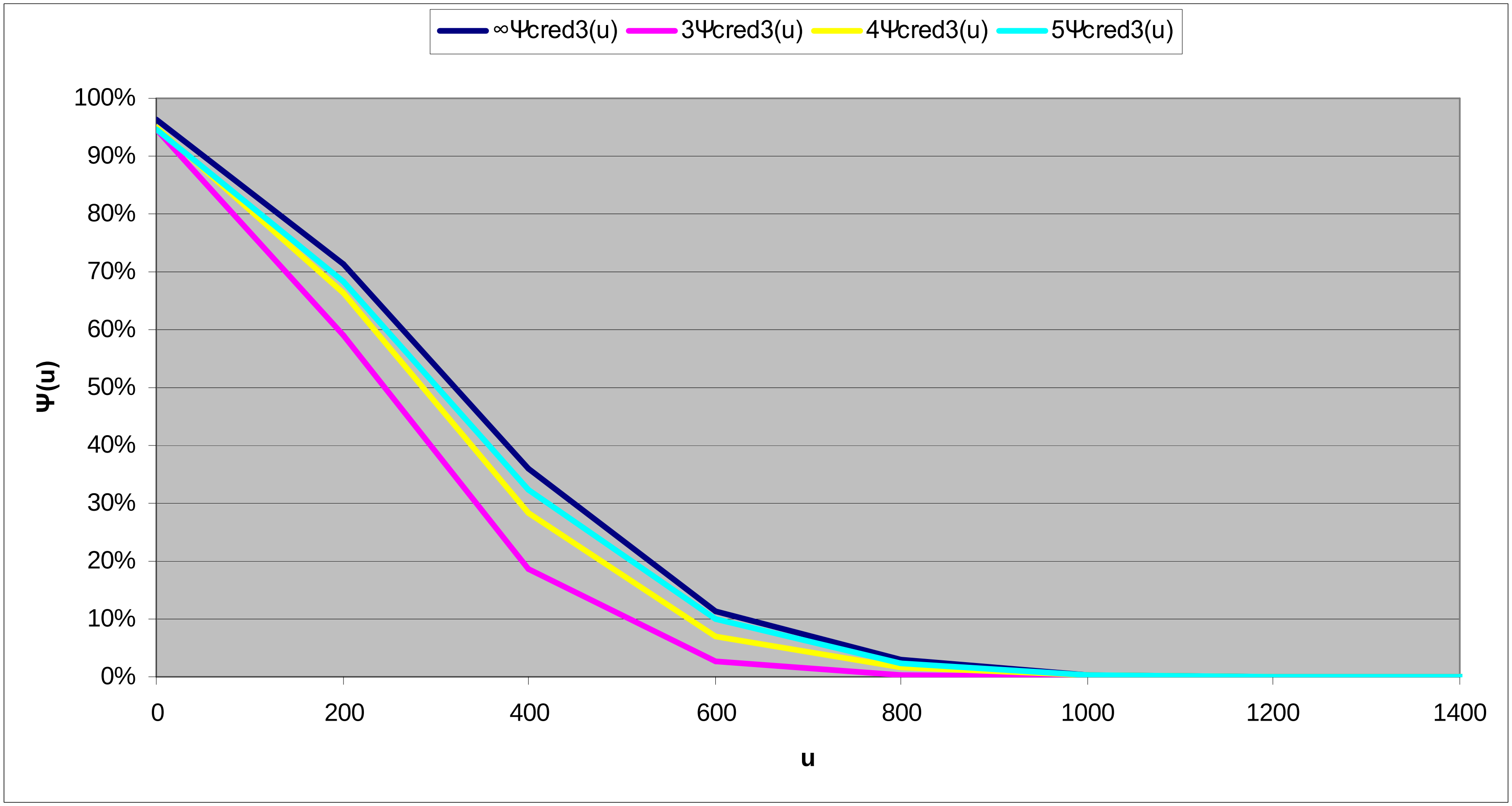


Poisson(1)/EXP(100) Neutral-Tailed
Low Frequency/High Severity, Cred_3 and D+L



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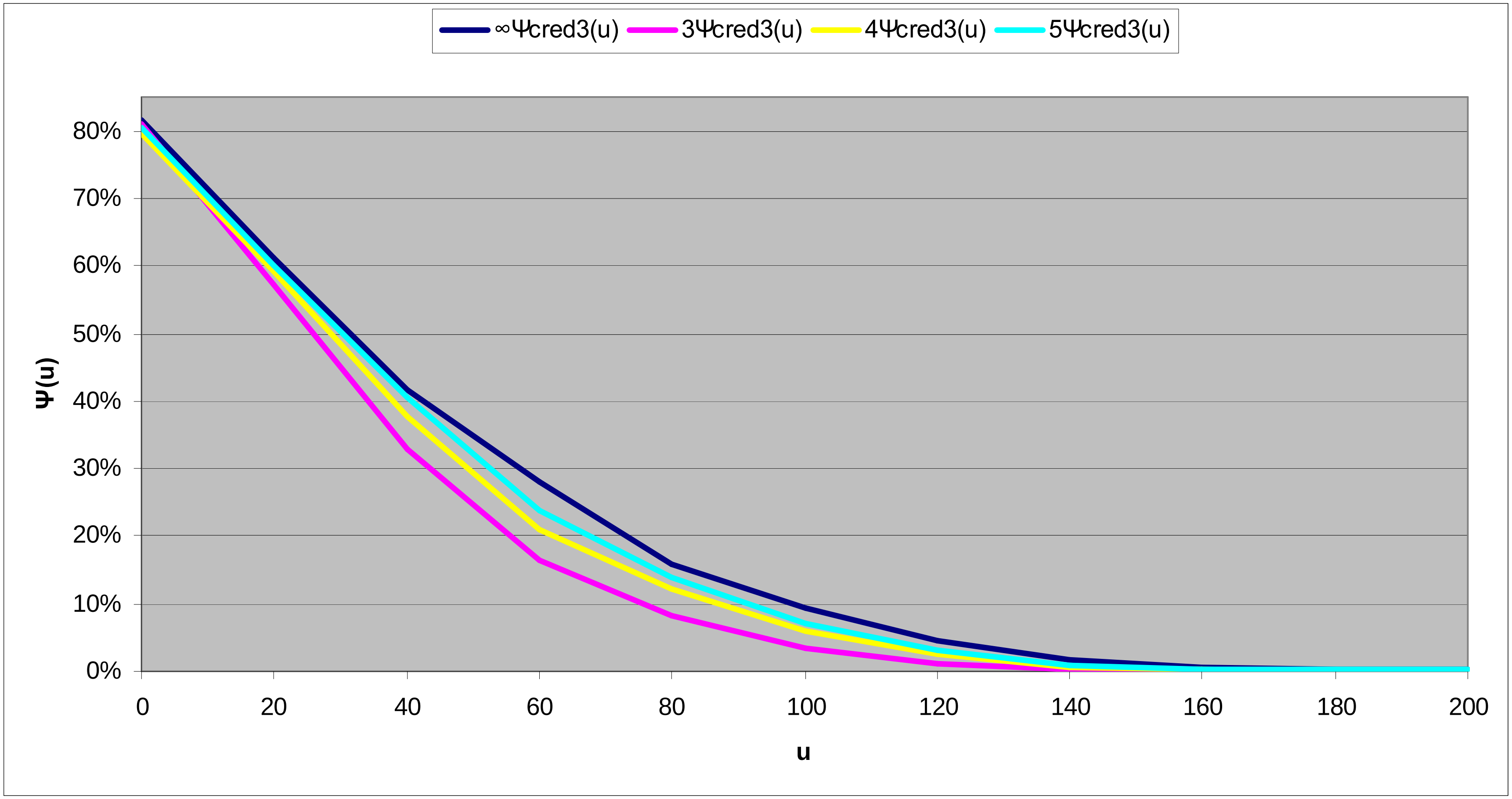

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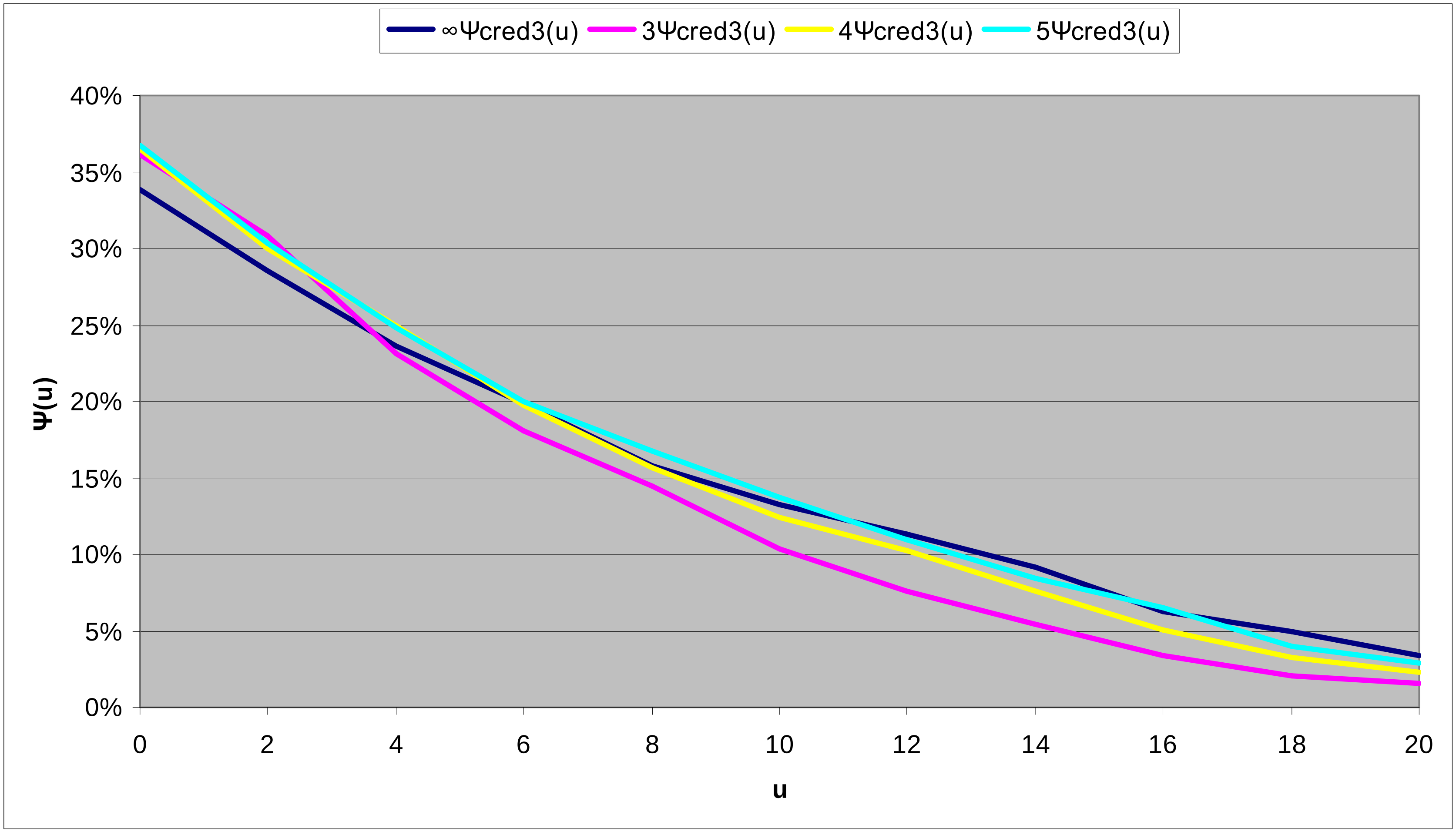


Ruin Probability Reduction

Tail Type	Heavy-Tailed		Regular-Tailed		Light-Tailed	
Frequency/Severity	Strategy	Avg Ruin Prob	Strategy	Avg Ruin Prob	Strategy	Avg Ruin Prob
LF / HS	1+1+1	0.3439	1+1+1	0.2825	1+1+1	0.2038
	4+1+1	0.2582	4+1+1	0.1223	4+1+1	0.0634
	4+4+3	0.0894	4+4+3	0.0812	4+4+3	0.0385
MF / MS	1+1+1	0.4525	1+1+1	0.2947	1+1+1	0.1943
	4+1+1	0.3552	4+1+1	0.1636	4+1+1	0.1038
	4+4+3	0.1348	4+4+3	0.1195	4+4+3	0.0728
HF / LS	1+1+1	0.2960	1+1+1	0.1766	1+1+1	0.0809
	4+1+1	0.2594	4+1+1	0.1362	4+1+1	0.0567
	4+4+3	0.1213	4+4+3	0.1169	4+4+3	0.0526

- Factor_1: Premium Scheme: 1-constant, 2-cred, 3-cred10, 4-cred3
  Factor_2: DL Modifier: 1-None, 2-D, 3-L, 4-D+L
  Factor_3: Size Modifier: 1-M=∞ 3-M=3 4-M=4 5-M=5
- Conclusion: in general, the strategy with cred3, high deductible and lower policy limit (4+4+3) can significantly reduces classical ruin probability for all cases, especially for Heavy-Tailed case.

Conclusion for Ruin Probabilities

- $\Psi_{(HF,LS)}(u) \leq \Psi_{(MF,MS)}(u) \leq \Psi_{(LF,HS)}(u)$
- credibility premium schemes can reduce the ruin probability except some small *u*
- $\Psi_{(4,1,1)}(u) \leq \Psi_{(3,1,1)}(u) \leq \Psi_{(2,1,1)}(u)$
- strategy (4,4,3) produces lower ruin probability than (4,1,1), (4,2,3) and (4,3,3) for most *u* for MS/MF and LF/HS risks
- strategy (4,4,3) yields smaller ruin probability than (4,4,4) and (4,4,5) except for very few low *u*.

Ruin Ratio, Gain Ratio and Index (I)

- *S*: the set of forty strategies {(*i, j, k*) | *i, j* = 1, 2, 3, 4 and *k* = 1, 3, 4, 5}
- $\overline{U}_s(u,n)$: the average surplus at time *n* over 1000 simulations for *u*>0 and strategy *s* in *S*;
- $\overline{G}_s(n)=\overline{U}_s(u,n)-u$: the average gain at time *n* over 1000 simulations for strategy *s* in *S*
- $\overline{G}(n)$ = max $\{\overline{G}_s(n) : s \in S\}$ the largest average gain at time *n* over 1000 simulations among *s* in *S*
- $\overline{\Psi}_s(u,n)$: the average ruin probability by or at time *n* over 1000 simulations for *u*>0 and strategy *s* in *S*

Ruin Ratio, Gain Ratio and Index (II)

- $\overline{\Psi}_s(n) = \frac{1}{10}\sum_u \overline{\Psi}_s(u,n)$: the average ruin probability by or at time *n* over 10 u>0 for strategy *s* in *S*
- $\overline{\Psi}(n)$ = min$\{\overline{\Psi}_s(n) : s \in S\}$ the smallest average ruin prob. by or at time n over 10 u>0 among *s* in *S*
- $GR_s(n) = \frac{\overline{G}_s(n)}{\overline{\overline{G}}(n)} \leqq 1$: a gain ratio for the study period *n* and strategy *s* in *S*
- $RR_s(n) = \frac{\overline{\Psi}(n)}{\overline{\overline{\Psi}}_s(n)} \leqq 1$: a ruin ratio for the study period *n* and strategy *s* in S
- $Index_s(n) = GR_s(n) \times RR_s(n) \leqq 1$, *s* in *S*

Conclusions for RR, GR and Index

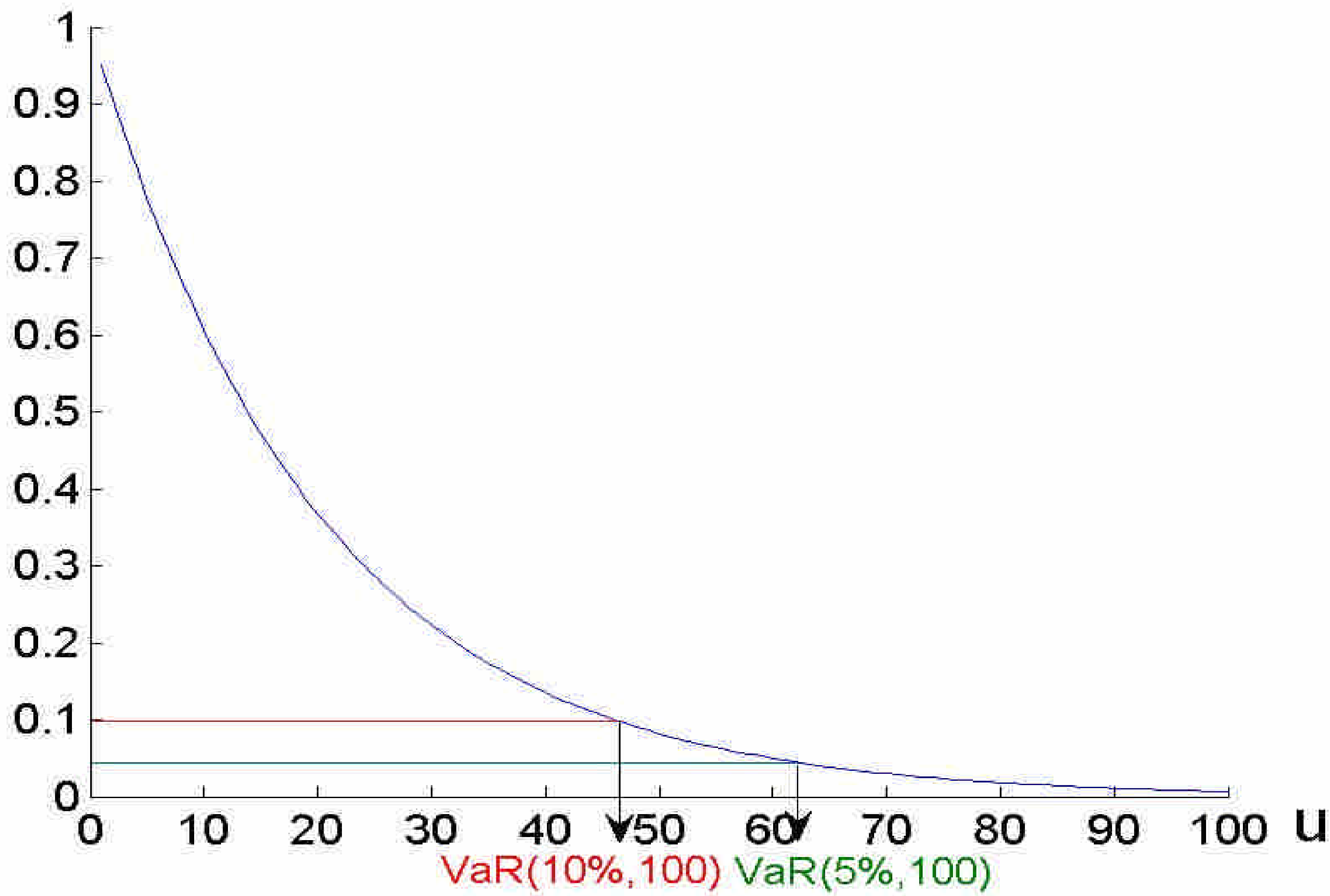
- to maximize the average gain $\overline{G}_s(n)$, strategies w/o D or L imposed should be adopted; strategies w/o D produce higher gains than strategies with D; strategy (2, 1, 1) is the overall best;
- to minimizing the average ruin probability $\overline{\Psi}_s(n)$ adopt a modified credibility premium (cred3); the best DL indicator and DL size modifier depends on the type of risk and tail distribution; strategy (4,4,3) is the overall best
- to maximize $Index_s(n)$, strategy (4,3,3) is the overall best choice

Value at Risk: Definition

- $VaR(\alpha) = \inf\{t: S_X(t) \leqq \alpha\}$ = 100(1-α)th percentile of X where S_X is the survival function (sf) of X.
- $VaR(\alpha)$ is non-increasing in α
- for the continuous surplus process, $\Psi(u) = S_Z(u)$, the sf of the maximal aggregate loss Z.
- in our discrete time case, we similarly define

$VaR\mathrm{s}(\alpha, n) = \inf\{u: \Psi\mathrm{s}(u,n) \leqq \alpha\}$ for the confidence level 1-α, study period n and strategy s

Value at Risk: Figure



Rates of Return

- total rates of return for *u* and *n* years

$$TRR_s(u,n) = \frac{\overline{G}_s(n)}{u} = \frac{\overline{U}_s(u,n) - u}{u}$$

- annualized rates of return for *u* and *n* years

$$ARR_s(u,n) = \sqrt[n]{\frac{\overline{U}_s(u,n)}{u}} - 1 = \sqrt[n]{TRR_s(u,n) + 1} - 1$$

- given α and *n*, we want

$$\underset{s \in S}{Max}\{\frac{U(\mathrm{VaR}_s(\alpha,n),n) - \mathrm{VaR}_s(\alpha,n)}{\mathrm{VaR}_s(\alpha,n)}\}$$

Conclusions for ARR

- $ARR_s(VaR_s(\alpha, n), n)$ is decreasing in n for most types of risks
- ARR_s(LF / HS) < ARR_s(MF / MS) < ARR_s(HF / LS) for all three tail types
- ARR_s(HT) < ARR_s(NT) < ARR_s(LT) for most cases of for all three mixtures of frequency and severity

Ordering Diagram Based on ARR

- HT / LF / HS < HT / MF / MS < HT / HF / LS

  ^ ^ ^

  NT / LF / HS < NT / MF / MS < NT / HF / LS

  ^ ^ ^

  LT / LF / HS < LT / MF / MS < LT / HF / LS

  strategy (4, 3, 3) (cred3, L only, and M=3) is the overall best based on ARR for all cases

Overall Conclusions

- the schemes we have proposed can be applied by property and casualty insurers in a variety of business lines with individual claims following specific loss distributions.
- first identifies the risk attributes of the nine combinations of tail type, frequency and severity that best corresponds to its line of business;
- then decides which strategy should be adopted based on the maximization of gain, the minimization of ruin probability or both.