Optimal Strategies for Ruin Probabilities and Expected Gains

Dr. Cary Chi-Liang Tsai and Dr. Gary Parker
Department of Statistics and Actuarial Science
Simon Fraser University

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Classical Surplus Process
(discrete time )

• $U_n = u + n \cdot c - S_n$: the surplus at time $n$
• $u = U_0$: the initial surplus
• $c$: the amount of premium received each period
• $S_n = W_1 + W_2 + \ldots + W_n$: the aggregate claims up to time $n$
• $W_k$: the sum of the claims in the $k$th period
• $W_1, W_2, \ldots W_n$ are i.i.d. r.v. distributed as $W$
• $c = (1 + \theta ) \cdot \mathbb{E}[W]$
• $\theta$: the relative security loading
• $\mathbb{E}(W) = \mathbb{E}(N) \cdot \mathbb{E}[X]$, and $N \sim \text{Poisson}(\lambda)$
Probability of Ruin

• $T = \min\{n: U_n < 0\}$ ($T = \infty$ if $U_n \geq 0$ for all $n$): the time of ruin (the first time that the surplus becomes negative)

• $\Psi(u) = \Pr\{T < \infty | U_0 = u\}$: the probability of ruin

• $\Psi(u,n) = \Pr\{T \leq n | U_0 = u\}$: the probability of ruin before or at time $n$ (the dist. function of $T$)
Constant and Dynamic Premiums for Surplus Process
Buhlmann’s Credibility Premium

- **Case 1**: \( U_{n+1} = U_n + c - W_{n+1}, \) \( c = (1 + \theta) \; \mathbb{E}[W] \)

- **Case 2**: \( U_{n+1} = U_n + c_{n+1} - W_{n+1}, \)
  \[
c_{n+1} = (1 + \theta) \; \mathbb{E}[W_n] + \mathbb{E}[W] = \frac{1}{n} \sum_{i=1}^{n} W_i
  \]

- **Case 3**: \( U_{n+1} = U_n + c_{n+1,m} - W_{n+1}, \)
  \[
c_{n+1,m} = (1 + \theta) \; \mathbb{E}[W_{n,m}] + \mathbb{E}[W] = \frac{1}{m} \sum_{i=h}^{m} W_i
  \]

where \( m = \min(n,k), \) \( h = \max(n-k,0)+1 = n-m+1 \)

- \( v: \) the expected process variance, \( a: \) the variance of the hypothetical means

- **Note**: \( k = \infty, \) \( m = n, \) \( Z_{n,m} = Z_n \) and \( h = 1, \) **Case 3** => **Case 2**
  \( k = 0, \) \( m = 0, \) \( Z_{n,m} = 0 \) and \( h = n+1, \) **Case 3** => **Case 1**
Strategies

• Premium scheme:
  1: constant, 2: cred, 3: cred10, 4: cred3

• DL modifier: deductible and policy limit
  1: none, 2: deductible, 3: policy limit, 4: both

• Size modifier: sizes of deductible and policy limit
  1: k=∞, 3: M=3, 4: M=4, 5: M=5

where deductible, \( D = \frac{E[X]}{M} \) and policy limit \( L = E[X] \times M \)

\[ c_1 = (1 + \theta) \frac{E[N] \cdot E[Y]}{D/L} \]

\( \theta \) is the r.v. with \( D/L \) on \( X \)

• 40 strategies in total

• Goal: which strategies reduce ruin probability most?
A sample path
Claims Distributions and Parameters

• 3 mixtures for frequency(F) and severity(S):
  Low F / High S: \[\lambda = E[N]=1 \quad \text{and} \quad E[X]=100\]
  Mid F / Mid S: \[\lambda = E[N]=10 \quad \text{and} \quad E[X]=10\]
  High F / Low S: \[\lambda = E[N]=100 \quad \text{and} \quad E[X]=1\]
  All three mixtures have equal mean \(E[W]=100\)

• 3 distributions for individual claim size \(X\)
  LT (Light-Tailed): \(\text{Weibull}(\alpha,\theta) \) with \(\alpha > 1\)
  RT (Neutral-Tailed): \(\text{Exponential}(\beta)\)
  HT (Heavy-Tailed): \(\text{Pareto}(\tau,\theta) \) with \(\tau > 1\)
  All three distributions have equal mean \(E[X]\)
# Underlying Severity Distributions

<table>
<thead>
<tr>
<th>Severity dist’n X</th>
<th>Light-tailed</th>
<th>Neutral-tailed</th>
<th>Heavy-tailed</th>
<th>E[X]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low severity</td>
<td>Weibull (2, 1/Γ (1.5))</td>
<td>Exponential (1)</td>
<td>Pareto (3, 2)</td>
<td>1</td>
</tr>
<tr>
<td>Mid severity</td>
<td>Weibull (2,10/Γ (1.5))</td>
<td>Exponential (10)</td>
<td>Pareto (3, 20)</td>
<td>10</td>
</tr>
<tr>
<td>High severity</td>
<td>Weibull (2,100/Γ (1.5))</td>
<td>Exponential (100)</td>
<td>Pareto (3, 200)</td>
<td>100</td>
</tr>
</tbody>
</table>
# Probabilities for claims with D/L

<table>
<thead>
<tr>
<th></th>
<th>Probabilities</th>
<th>Weibull</th>
<th>Exponential</th>
<th>Pareto</th>
</tr>
</thead>
<tbody>
<tr>
<td>M=2</td>
<td>P(X&lt;D)</td>
<td>0.178</td>
<td>0.393</td>
<td>0.488</td>
</tr>
<tr>
<td></td>
<td>P(D&lt;X&lt;L)</td>
<td>0.779</td>
<td>0.471</td>
<td>0.387</td>
</tr>
<tr>
<td></td>
<td>P(X&gt;L)</td>
<td>0.043</td>
<td>0.135</td>
<td>0.125</td>
</tr>
<tr>
<td>M=3</td>
<td>P(X&lt;D)</td>
<td>0.083</td>
<td>0.283</td>
<td>0.370</td>
</tr>
<tr>
<td></td>
<td>P(D&lt;X&lt;L)</td>
<td>0.916</td>
<td>0.667</td>
<td>0.566</td>
</tr>
<tr>
<td></td>
<td>P(X&gt;L)</td>
<td>0.001</td>
<td>0.050</td>
<td>0.064</td>
</tr>
<tr>
<td>M=4</td>
<td>P(X&lt;D)</td>
<td>0.048</td>
<td>0.222</td>
<td>0.298</td>
</tr>
<tr>
<td></td>
<td>P(D&lt;X&lt;L)</td>
<td>0.952</td>
<td>0.760</td>
<td>0.665</td>
</tr>
<tr>
<td></td>
<td>P(X&gt;L)</td>
<td>0.000</td>
<td>0.018</td>
<td>0.037</td>
</tr>
</tbody>
</table>
Monte Carlo Simulations

• 11 initial surpluses \( u \)
  0, 200, \ldots, 2000 for low frequency and high severity
  0, 20, \ldots., 200 for mid frequency and mid severity
  0, 2, \ldots., 20 for high frequency and low severity

• Generate number of claims \( n \) from \textit{Poisson}(\( \lambda \)) first, then claims \( X_1, X_2, \ldots, X_n \) for each year

• 1000 paths for \( U_n \) up to \( n=100 \) for each case

• Ruin probability,
  \[ \Psi(u) = \# \text{ of } \{ U_k < 0 \text{ for some } k \leq 100 \mid U_0 = u \} / 1000 \]
Poisson(1)/EXP(100) Neutral-Tailed Low Frequency/High Severity without D/L
Poisson(10)/EXP(10) Neutral-Tailed
Mid Frequency/Mid Severity without D/L
Poisson(100)/EXP(1) Neutral-Tailed
High Frequency/Low Severity without D/L
Poisson(1)/EXP(100) Neutral-Tailed Low Frequency/High Severity, Cred_3 and K=3
Poisson(10)/EXP(10) Neutral-Tailed Mid Frequency/Mid Severity, Cred_3 and K=3
Poisson(100)/EXP(1) Neutral-Tailed High Frequency/Low Severity, Cred_3 and K=3
Poisson(1)/EXP(100) Neutral-Tailed
Low Frequency/High Severity, Cred_3 and D+L
Poisson(10)/EXP(10) Neutral-Tailed
Mid Frequency/Mid Severity, Cred_3 and D+L
Poisson(100)/EXP(1) Neutral-Tailed
High Frequency/Low Severity, Cred_3 and D+L
Ruin Probability Reduction

<table>
<thead>
<tr>
<th>Tail Type</th>
<th>Heavy-Tailed</th>
<th>Regular-Tailed</th>
<th>Light-Tailed</th>
</tr>
</thead>
<tbody>
<tr>
<td>LF / HS</td>
<td>1+1+1</td>
<td>0.3439</td>
<td>1+1+1</td>
</tr>
<tr>
<td></td>
<td>4+1+1</td>
<td>0.2582</td>
<td>4+1+1</td>
</tr>
<tr>
<td></td>
<td>4+4+3</td>
<td>0.0894</td>
<td>4+4+3</td>
</tr>
<tr>
<td>MF / MS</td>
<td>1+1+1</td>
<td>0.4525</td>
<td>1+1+1</td>
</tr>
<tr>
<td></td>
<td>4+1+1</td>
<td>0.3552</td>
<td>4+1+1</td>
</tr>
<tr>
<td></td>
<td>4+4+3</td>
<td>0.1348</td>
<td>4+4+3</td>
</tr>
<tr>
<td>HF / LS</td>
<td>1+1+1</td>
<td>0.2960</td>
<td>1+1+1</td>
</tr>
<tr>
<td></td>
<td>4+1+1</td>
<td>0.2594</td>
<td>4+1+1</td>
</tr>
<tr>
<td></td>
<td>4+4+3</td>
<td>0.1213</td>
<td>4+4+3</td>
</tr>
</tbody>
</table>

• Factor_1: Premium Scheme: 1-constant, 2-cred, 3-cred10, 4-cred3
• Factor_2: DL Modifier: 1-None, 2-D, 3-L, 4-D+L
• Factor_3: Size Modifier: 1-M=∞, 3-M=3, 4-M=4, 5-M=5

• Conclusion: in general, the strategy with cred3, high deductible and lower policy limit (4+4+3) can significantly reduce classical ruin probability for all cases, especially for Heavy-Tailed case.
Conclusion for Ruin Probabilities

- \( \Psi_{(HF,LS)}(u) \leq \Psi_{(MF,MS)}(u) \leq \Psi_{(LF,HS)}(u) \)
- credibility premium schemes can reduce the ruin probability except some small \( u \)
- \( \Psi_{(4,1,1)}(u) \leq \Psi_{(3,1,1)}(u) \leq \Psi_{(2,1,1)}(u) \)
- strategy (4,4,3) produces lower ruin probability than (4,1,1), (4,2,3) and (4,3,3) for most \( u \) for MS/MF and LF/HS risks
- strategy (4,4,3) yields smaller ruin probability than (4,4,4) and (4,4,5) except for very few low \( u \).
Ruin Ratio, Gain Ratio and Index (I)

- **S**: the set of forty strategies \{ (i, j, k) | i, j = 1, 2, 3, 4 and k = 1, 3, 4, 5 \}
- \( \bar{U}_s(u,n) \): the average surplus at time \( n \) over 1000 simulations for \( u > 0 \) and strategy \( s \) in \( S \);
- \( \bar{G}_s(n) = \bar{U}_s(u,n) - u \): the average gain at time \( n \) over 1000 simulations for strategy \( s \) in \( S \)
- \( \bar{G}(n) = \max \{ \bar{G}_s(n) : s \in S \} \): the largest average gain at time \( n \) over 1000 simulations among \( s \) in \( S \)
- \( \bar{\Psi}_s(u,n) \): the average ruin probability by or at time \( n \) over 1000 simulations for \( u > 0 \) and strategy \( s \) in \( S \)
Ruin Ratio, Gain Ratio and Index (II)

- \( \overline{\Psi}_s(n) = \frac{1}{10} \sum_{u} \overline{\Psi}_s(u,n) \): the average ruin probability by or at time \( n \) over 10 \( u > 0 \) for strategy \( s \) in \( S \)
- \( \overline{\Psi}(n) = \min \{ \overline{\Psi}_s(n) : s \in S \} \): the smallest average ruin prob. by or at time \( n \) over 10 \( u > 0 \) among \( s \) in \( S \)
- \( GR_s(n) = \frac{\overline{G}_s(n)}{G(n)} \leq 1 \): a gain ratio for the study period \( n \) and strategy \( s \) in \( S \)
- \( RR_s(n) = \frac{\overline{\Psi}(n)}{\overline{\Psi}_s(n)} \leq 1 \): a ruin ratio for the study period \( n \) and strategy \( s \) in \( S \)
- \( \text{Index}_s(n) = GR_s(n) \times RR_s(n) \leq 1, s \in S \)
Conclusions for RR, GR and Index

• to maximize the average gain $\bar{G}_s(n)$, strategies w/o D or L imposed should be adopted; strategies w/o D produce higher gains than strategies with D; strategy $(2,1,1)$ is the overall best;

• to minimizing the average ruin probability $\bar{\Psi}_s(n)$ adopt a modified credibility premium (cred3); the best DL indicator and DL size modifier depends on the type of risk and tail distribution; strategy $(4,4,3)$ is the overall best;

• to maximize $Index_s(n)$, strategy $(4,3,3)$ is the overall best choice
Value at Risk: Definition

- \( \text{VaR}(\alpha) = \inf\{t: S_X(t) \leq \alpha\} = 100(1-\alpha) \text{th percentile of } X \) where \( S_X \) is the survival function (sf) of \( X \).
- \( \text{VaR}(\alpha) \) is non-increasing in \( \alpha \)
- for the continuous surplus process, \( \Psi(u) = S_Z(u) \), the sf of the maximal aggregate loss \( Z \).
- in our discrete time case, we similarly define \( \text{VaRs}(\alpha, n) = \inf\{u: \Psi_s(u,n) \leq \alpha\} \) for the confidence level \( 1-\alpha \), study period \( n \) and strategy \( s \)
Value at Risk: Figure
Rates of Return

- total rates of return for \( u \) and \( n \) years

\[
TRR_s(u, n) = \frac{G_s(n)}{u} = \frac{U_s(u, n) - u}{u}
\]

- annualized rates of return for \( u \) and \( n \) years

\[
ARR_s(u, n) = \sqrt[n]{\frac{U_s(u, n)}{u}} - 1 = n\sqrt{TRR_s(u, n)} + 1 - 1
\]

- given \( \alpha \) and \( n \), we want

\[
\max_{s \in S} \frac{U(VaR_s(\alpha, n), n) - VaR_s(\alpha, n)}{VaR_s(\alpha, n)}
\]
Conclusions for ARR

- \( ARR_s(VaR_s(\alpha, n), n) \) is decreasing in \( n \) for most types of risks
- \( ARR_s(LF / HS) < ARR_s(MF / MS) < ARR_s(HF / LS) \) for all three tail types
- \( ARR_s(HT) < ARR_s(NT) < ARR_s(LT) \) for most cases of for all three mixtures of frequency and severity
Ordering Diagram Based on ARR

- HT / LF / HS < HT / MF / MS < HT / HF / LS
  - ^  ^  ^
  - NT / LF / HS < NT / MF / MS < NT / HF / LS
  - ^  ^  ^
  - LT / LF / HS < LT / MF / MS < LT / HF / LS

strategy (4, 3, 3) (cred3, L only, and M=3) is the overall best based on ARR for all cases
Overall Conclusions

- the schemes we have proposed can be applied by property and casualty insurers in a variety of business lines with individual claims following specific loss distributions.
- first identifies the risk attributes of the nine combinations of tail type, frequency and severity that best corresponds to its line of business;
- then decides which strategy should be adopted based on the maximization of gain, the minimization of ruin probability or both.