

Optimal reinsurance for variance related premium calculation principles

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- Y - Aggregate claims (non-negative) for a given period of time
- The aggregate claims over consecutive periods are assumed to be i.i.d.
- Z - ceded claims through a reinsurance contract.

Z is a function $Z : [0, +\infty[\mapsto [0, +\infty[$, mapping each possible claims aggregate value, in a given period, into the corresponding value ceded under the reinsurance contract.

- \mathcal{Z} is the set of all possible reinsurance policies is, i.e.

$$\mathcal{Z} = \{Z : [0, +\infty[\mapsto \mathbb{R} \mid Z \text{ is measurable and } 0 \leq Z(y) \leq y, \forall y \geq 0\}.$$

- We do not distinguish between functions which differ only on a set of zero probability.

- c - gross premiums per unit of time, $c > E[Y]$.
- $P = P(Z)$ - Reinsurance premium per unit of time
- L_Z - insurer's net profit per unit of time

$$L_Z = c - P(Z) - (Y - Z(Y)).$$

Assumptions

- A1** Y is a continuous random variable with density function f , and $E[Y^2] < +\infty$.
- A2** No reinsurance policy exists that guarantees a nonnegative profit, i.e., $\Pr\{L_Z < 0\} > 0$ holds for every $Z \in \mathcal{Z}$.
- A3** The reinsurance premium is a non negative and convex functional such that $P(0) = 0$. It is continuous in the mean-squared sense, i.e., $\lim P(Z_k) = P(Z')$ holds for every sequence $\{Z_k \in \mathcal{Z}\}$ such that $\lim \int_0^{+\infty} (Z_k(y) - Z'(y))^2 f(y) dy = 0$.
- A4** $P(Z) = E[Z] + g(\text{Var}(Z))$, where $g : [0, +\infty[\mapsto [0, +\infty[$ is a function smooth in $]0, +\infty[$ such that $g(0) = 0$ and $g'(x) > 0, \forall x \in]0, +\infty[$.

Examples are:

- the standard deviation principle, $g(x) = \beta\sqrt{x}, \beta > \frac{c-E[Y]}{(\text{Var}[Y])^{1/2}}$
- the variance principle, $g(x) = \beta x, \beta > \frac{c-E[Y]}{\text{Var}[Y]}$.

The adjustment coefficient

- Consider the map $G : \mathbb{R} \times \mathcal{Z} \mapsto [0, +\infty]$, defined by

$$G(R, Z) = \int_0^{+\infty} e^{-RL_Z(y)} f(y) dy, \quad R \in \mathbb{R}, Z \in \mathcal{Z}.$$

- R_Z - adjustment coefficient of the retained risk for a particular reinsurance policy, $Z \in \mathcal{Z}$, i.e. R_Z is defined as the strictly positive value of R which solves the equation

$$G(R, Z) = 1. \tag{1}$$

- Equation (1) can not have more than one positive solution. This means the map $Z \mapsto R_Z$ is a well defined functional in the set

$$\mathcal{Z}^+ = \{Z \in \mathcal{Z} : (1) \text{ admits a positive solution}\}.$$

The Adjustment Coefficient Problem

In Guerra and Centeno (2007) we have solved the following problem.

Problem

Find $(\hat{R}, \hat{Z}) \in]0, +\infty[\times \mathcal{Z}^+$ such that

$$\hat{R} = R_{\hat{Z}} = \max \{ R_Z : Z \in \mathcal{Z}^+ \}.$$

Definition

Two strategies $Z_1, Z_2 \in \mathcal{Z}$ are said to be economically equivalent if and only if

$$\Pr \{Z_1 - P(Z_1) = Z_2 - P(Z_2)\} = 1. \quad \square \quad (2)$$

Notice that (2) implies that two economically equivalent policies differ (up to null sets) only by an additive constant and this constant must be the difference between the two premiums. That is, for variance related premium calculation principles, Z_1 and Z_2 are economically equivalent if and only if there exists a constant x such that

$$Z_2 = Z_1 + x. \quad (3)$$

When the infimum of the support of the distribution of Y is zero the concept is not relevant. Let ν be that number, i.e.

$$\nu = \sup \{y \geq 0 : \Pr\{Y < y\} = 0\} .$$

$\nu = 0$ implies that if $Z \in \mathcal{Z}$ and $x \neq 0$, then $(Z + x) \notin \mathcal{Z}$ must hold. Hence the existence of optimal equivalent policies, against a unique optimal policy, only can happen when $\nu > 0$.

Theorem 1

There is an optimal policy for the adjustment coefficient criterion. The optimal policy is economically equivalent to one of the following policies:

- a) *when g' is a bounded function in the neighbourhood of zero, a contract satisfying*

$$y = Z(y) + \frac{1}{R} \ln \frac{Z(y) + \alpha}{\alpha}, \quad \text{a.e. } y \geq 0, \quad (4)$$

where $\alpha > 0$ is a constant such that

$$h(\alpha) = 0, \quad (5)$$

$$h(\alpha) = \alpha + E[Z] - \frac{1}{2g'(\text{Var}(Z))}, \quad (6)$$

and R is the the maximal adjustment coefficient.

- b) *when g' is an unbounded function in the neighbourhood of zero and (5) has no positive solution, $Z \equiv 0$ (no risk is reinsured). \square*

Theorem 1 leaves some ambiguity about the number of roots of equation (5). In the paper presented here we show that this equation has at most one solution. For the proof we use the following Property, which follows easily from Theorems 5 and 6 in Deprez and Gerber (1985).

Proposition 1

Assume that g is twice differentiable. $P[Z] = E[Z] + g(\text{Var}[Z])$ is a convex functional if and only if

$$\frac{g''(x)}{g'(x)} \geq -\frac{1}{2x}, \quad \forall x > 0. \quad \square \quad (7)$$

Note that (7) holds as an equality for the standard deviation principle and that the left hand side is zero for the variance principle.

Proposition 2

For any $R > 0$, $\lim_{\alpha \rightarrow +\infty} h(\alpha) = +\infty$, with $h(\alpha)$ given by (6), and equation (5) has at most one positive solution. Let $\hat{\alpha}$ be the root of (5) (assuming it exists). Then $h(\alpha) < 0, \forall \alpha \in]0, \hat{\alpha}[$ and $h(\alpha) > 0, \forall \alpha \in]\hat{\alpha}, +\infty[$. \square

Proof.

The proof is made using the above Proposition and the Cauchy–Schwarz inequality to show that

$$h'(\alpha) \Big|_{h(\alpha)=0} \geq 0.$$



Definition

$$\Phi_k(R, \alpha) = \int_0^{+\infty} (1 + R(Z(y) + \alpha))^k f(y) dy, \quad k \in \mathbb{Z}, \quad (8)$$

where $Z(y)$ is such that (4) holds for the particular (R, α) indicated. \square

Proposition 3

For any $R > 0$, the expected value and the variance of Z , when Z is such that (4) holds, can be calculated by

$$E[Z] = \frac{1}{R} (\Phi_1 - (1 + R\alpha)), \quad (9)$$

$$\text{Var}[Z] = \frac{1}{R^2} (\Phi_2 - \Phi_1^2). \quad \square \quad (10)$$

Proof.

The proof is made using the change of variable $\phi = 1 + R(Z + \alpha)$, i.e.

$$Z = \frac{1}{R} (1 + R(Z + \alpha) - R\alpha - 1) = \frac{1}{R} (\phi - (1 + R\alpha)).$$

□

Numerical Issues

Let $G(R, \alpha)$ be defined as $G(R, Z)$ with Z satisfying (4) for that particular (R, α) . We can also calculate $G(R, \alpha)$ easily.

Proposition 4

$G(R, \alpha)$ can be computed by

$$G(R, \alpha) = \frac{1}{\alpha} (E[Z] + \alpha) e^{R(P[Z]-c)}. \square$$

Proof.

$$\begin{aligned} G(R, \alpha) &= e^{R(P[Z]-c)} \int_0^{+\infty} e^{R(y-Z(y))} f(y) dy = \\ &= e^{R(P[Z]-c)} \int_0^{+\infty} \frac{Z(y) + \alpha}{\alpha} dy = \\ &= \frac{1}{\alpha} (E[Z] + \alpha) e^{R(P[Z]-c)}. \end{aligned}$$

Proposition 5

Φ_k can be represented as the integral :

$$\Phi_k(R, \alpha) = \frac{1}{R} \int_0^{+\infty} \frac{(1 + R(\zeta + \alpha))^{k+1}}{\zeta + \alpha} f\left(\zeta + \frac{1}{R} \ln \frac{\zeta + \alpha}{\alpha}\right) d\zeta. \quad \square \quad (11)$$

Proof.

Using the change of variable $y = \zeta + \frac{1}{R} \ln \frac{\zeta + \alpha}{\alpha}$, $\zeta \in [0, +\infty[$, we obtain

$$\begin{aligned} \Phi_k &= \int_0^{+\infty} (1 + R(Z(y) + \alpha))^k f(y) dy = \\ &= \int_0^{+\infty} (1 + R(\zeta + \alpha))^k f\left(\zeta + \frac{1}{R} \ln \frac{\zeta + \alpha}{\alpha}\right) \left(1 + \frac{1}{R(\zeta + \alpha)}\right) d\zeta = \\ &= \frac{1}{R} \int_0^{+\infty} \frac{(1 + R(\zeta + \alpha))^{k+1}}{\zeta + \alpha} f\left(\zeta + \frac{1}{R} \ln \frac{\zeta + \alpha}{\alpha}\right) d\zeta. \end{aligned}$$

Summarizing:

1. When the premium is the variance premium principle, i.e. when $P[Z] = E[Z] + \beta \text{Var}[Z]$, the adjustment coefficient of the retained aggregate claims is maximized when $(\hat{Z}(y), \hat{R}, \hat{\alpha}) = (Z(y), R, \alpha)$ is the only solution to

$$\begin{cases} y = Z(y) + \frac{1}{R} \ln \frac{Z(y)+\alpha}{\alpha}, \forall y > 0, \\ \alpha = \frac{1}{2\beta} e^{R(P[Z]-c)}, \\ \alpha = \frac{1}{2\beta} - E[Z], \end{cases}$$

with $E[Z]$, $\text{Var}[Z]$ computed by (9), (10) respectively.

2. When the premium follows the standard deviation principle, i.e. when $P[Z] = E[Z] + \beta\sqrt{\text{Var}[Z]}$,
- a. if $\exists \alpha > 0 : h(\alpha) < 0$, with

$$h(\alpha) = \alpha + E[Z] - \frac{\sqrt{\text{Var}[Z]}}{\beta}, \quad (12)$$

the adjustment coefficient of the retained aggregate claims is maximized when $(\widehat{Z}(y), \widehat{R}, \widehat{\alpha}) = (Z(y), R, \alpha)$ is the only solution to

$$\left\{ \begin{array}{l} y = Z(y) + \frac{1}{R} \ln \frac{Z(y)+\alpha}{\alpha}, \forall y > 0, \\ \alpha = (E[Z] + \alpha) e^{R(P[Z]-c)}, \\ \alpha = \frac{\sqrt{\text{Var}[Z]}}{\beta} - E[Z], \end{array} \right.$$

with $E[Z]$, $\text{Var}[Z]$ computed by (9), (10) respectively;

- b. if $h(\alpha) \geq 0, \forall \alpha > 0$, with $h(\alpha)$ given by (12), then the adjustment coefficient of the retained aggregate claims is maximized when $\widehat{Z}(y) = 0, \forall y$ (in this case \widehat{R} is the adjustment coefficient associated to the gross claim amount).
- 3. If $\nu = 0$, the solution to the problem is unique. If $\nu > 0$ the optimal solution to the problem is not unique, but they are all of the form $\widehat{Z}(y) + x$, with x constant such that $-\widehat{Z}(\nu) \leq x \leq \nu - \widehat{Z}(\nu)$.

Examples

In this section we give two examples for the standard deviation principle. In the first example we consider that Y follows a Pareto distribution. In the second example we consider a generalized gamma distribution. The parameters of these distributions were chosen such that $E[Y] = 1$ and both distributions have the same variance (which was set to $\text{Var}[Y] = \frac{16}{5}$, for convenience of the choice of parameters). Notice that though they have the same mean and variance, the tails of the two distributions are rather different. However, none of them has a moment generating function. Hence the optimal solution must always be different than no reinsurance.

In both examples we consider the same premium income $c = 1.2$ and the same loading coefficient $\beta = 0.25$.

Example 1

Example 1

We consider that Y follows the Pareto distribution

$$f(y) = \frac{32 \times 21^{32/11}}{(21 + 11y)^{43/11}}, \quad y > 0.$$

Example 1

The first column of Table 1 shows the optimal value of α and the corresponding values of R , $E[Z]$, $Var[Z]$, $P[Z]$, and $E[L_Z]$, while the second column shows the corresponding values for the best (in terms of the adjustment coefficient) stop loss treaty. The optimal policy improves the adjustment coefficient by 16.1% with respect to the best stop loss treaty, at the cost of an increase of 111% in the reinsurance premium. However, notice that the relative contribution of the loading to the total reinsurance premium is much smaller in the optimal policy, compared with the best stop loss. Hence, though a larger premium is ceded under the optimal treaty than under the best stop loss, this is made mainly through the pure premium, rather than the premium loading, so the expected profits not very different.

Example 1

Table 1: Y - Pareto random variable

	Optimal Treaty	Best Stop Loss
	$\alpha = 1.74411$	$M = 67.4436$
R	0.055406	0.047703
$E[Z]$	0.098018	0.001050
$Var[Z]$	0.212089	0.160269
$P[Z]$	0.213151	0.101134
$E[L_Z]$	0.084867	0.099916

Example 1

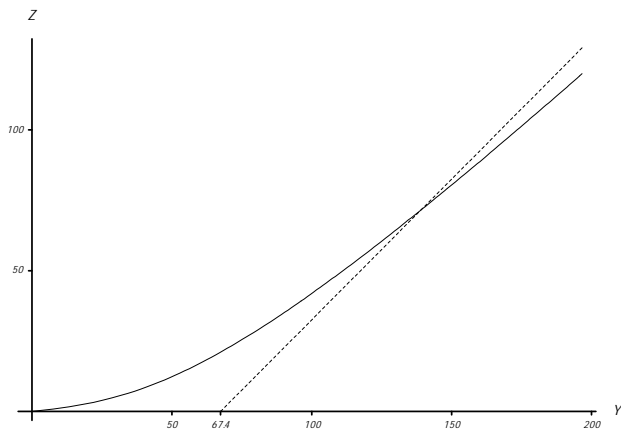


Figure 1: Optimal policy (full line) versus best stop loss (dashed line): the Pareto case

Example 2

In this example, Y follows the generalized gamma distribution with density

$$f(y) = \frac{b}{\Gamma(k)\theta} \left(\frac{y}{\theta}\right)^{kb-1} e^{-\left(\frac{y}{\theta}\right)^b}, \quad y > 0,$$

with $b = 1/3$, $k = 4$ and $\theta = 3!/6!$.

Example 2

Table 2: Y - Generalized gamma random variable

	Optimal Treaty	Best Stop Loss
	$\alpha = 0.813383$	$M = 47.8468$
R	0.084709	0.078571
$E[Z]$	0.076969	0.000204
$Var[Z]$	0.049546	0.004950
$P[Z]$	0.132616	0.017794
$E[L_Z]$	0.144353	0.182410

Example 2

Table 2 shows the results for this example. The general features are similar to Example 1 but the improvement with respect to the best stop loss is smaller (the optimal policy increases the adjustment coefficient by about 7.8% with respect to the best stop loss). The optimal policy presents a larger increase in the sharing of risk and profits and a sharp increase in the reinsurance premium (more than seven-fold) with respect to the best stop loss. However, in both cases the amount of the risk and of the profits which is ceded under the reinsurance treaty is substantially smaller than in the Pareto case.

Example 2

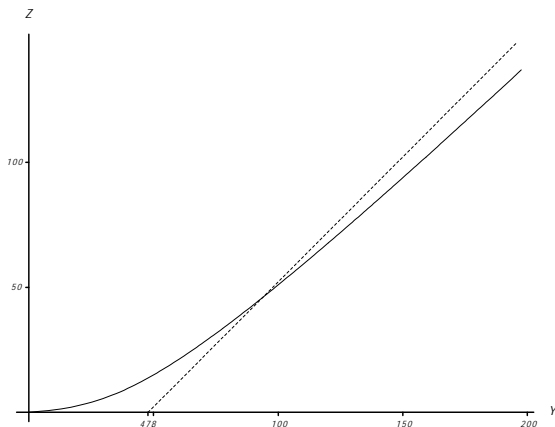




Figure 2: Optimal policy (full line) versus best stop loss (dashed line): the generalized gamma case

-  Deprez, O. and Gerber, H.U. (1985), On convex principles of premium calculation. *Insurance: Mathematics and Economics*, 4, 179-189.
-  Guerra, M. and Centeno, M.L. (2007), Optimal reinsurance policy: the adjustment coefficient and the expected utility criteria. *Insurance Mathematics and Economics*.
<http://dx.doi.org/10.1016/j.insmatheco.2007.02.008>