Credibility for additive and multiplicative models

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Introduction

- **multivariate tariffs**
  - tariffs depending on several rating factors common since a long time
  - in recent years, tendency to more and more refined tariffs using more and more rating factors

- **"standard" actuarial techniques for multivariate tariffs**
  - data cross classified
  - additive or multiplicative tariff structure
  - estimation of tariff parameters:
    - weighted least squares, marginal totals, etc.
    - multivariate statistical techniques, in particular generalized linear models (GLM)

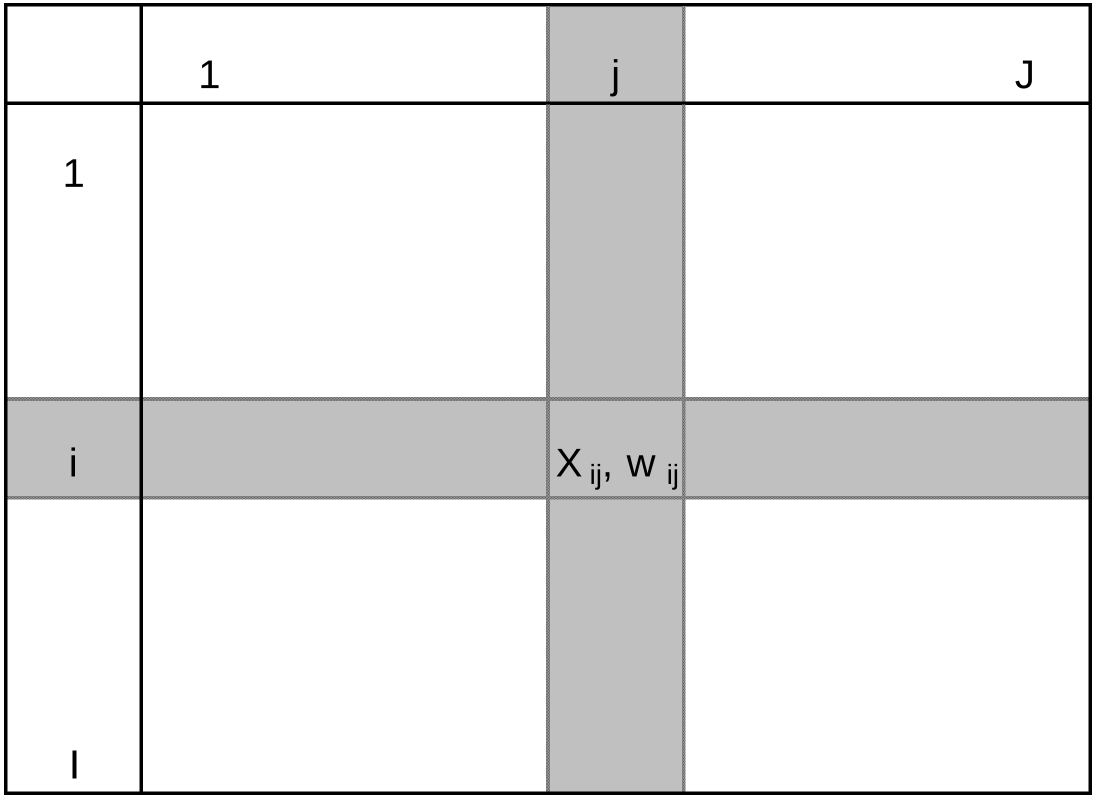
Introduction

- **Credibility: another established technique for ratemaking**
  - Credibility = linear Bayes estimator
  - individual risk embedded into a collective of similar risk, credibility estimate = weighted mean between individual and collective claims experience
  - in particular suited for scarce data
  - However
credibility hardly used so far for multivariate rate-making, although problem of scarce data also in this case;
main reason: suitable credibility models for this case not available (or not known by actuaries)

- **Aim of the paper**
  - to fill this gap
  - to show how credibility can be used for cross-classified tariffs (development of the models, derivation of the cred. estimators)

Introduction

- For didactical reasons: consider the case with 2 rating factors. However, generalisation to any number of rating factors obvious.



	1	j	J
1			
i		X_{ij}, w_{ij}	
I			

X_{ij} observable r. v. (claim freq., aver. claim size, aver. burning cost)

w_{ij} exposure measures

$$P_{ij} = E[X_{ij}]$$

classical additive model

Model Assumptions

$$X_{ij} = \mu_0 + \psi_i + \varphi_j + \varepsilon_{ij},$$

where μ_0, ψ_i, φ_j are real numbers and

where ε_{ij} are independent r. v. with $E[\varepsilon_{ij}] = 0,\ \mathrm{Var}(\varepsilon_{ij}) = \frac{\sigma^2}{w_{ij}}$.

problem: to determine $P_{ij} = E[X_{ij}] = \mu_0 + \psi_i + \varphi_j$ for each $cell_{ij}$

resp. to find estimators $\widehat{P}_{ij} = \hat{\mu}_0 + \widehat{\psi}_i + \widehat{\varphi}_j$.

weighted least square estimator:

minimise $Q = \sum_{i,j} w_{ij}\left(X_{ij} - \widehat{P}_{ij}\right)^2$.

method of marginal totals

$$\sum_j w_{ij}(\hat{\mu}_0 + \widehat{\psi}_i + \widehat{\varphi}_j) = \sum_j w_{ij}X_{ij},$$

$$\sum_i w_{ij}(\hat{\mu}_0 + \widehat{\psi}_i + \widehat{\varphi}_j) = \sum_i w_{ij}X_{ij}.$$

classical additive model

Theorem (estimators: w.l.s. and marginal totals)

$$\hat{\mu}_0 = \overline{X}_{\bullet\bullet}, \qquad \hat{\psi}_i = (\overline{X}_{i\bullet} - \overline{X}_{\bullet\bullet}) - \sum_{j=1}^{J} \frac{w_{ij}}{w_{i\bullet}} \hat{\varphi}_j \qquad i = 1, \ldots, I,$$

$$\hat{\varphi}_j = (\overline{X}_{\bullet j} - \overline{X}_{\bullet\bullet}) - \sum_{i=1}^{I} \frac{w_{ij}}{w_{\bullet j}} \hat{\psi}_i \qquad j = 1, \ldots, J,$$

where $\overline{X}_{i\bullet} = \sum_i \frac{w_{ij}}{w_{i\bullet}} X_{ij}, \ \overline{X}_{\bullet j} = \sum_i \frac{w_{ij}}{w_{\bullet j}} X_{ij}, \ \overline{X}_{\bullet\bullet} = \sum_{i,j} \frac{w_{ij}}{w_{\bullet\bullet}} X_{ij}.$

properties of above estimators:

- they minimize the mse among all linear unbiased estimators of μ_0, ψ_i, φ_j

$$mse = E\left[\sum_{i,j} w_{ij}\left(X_{ij} - \hat{P}_{ij}\right)^2\right]$$

- if in addition the r. v. X_{ij} are normally distributed then
  - they minimize the mse error among all unbiased estimators
  - they are also the maximum likelihood estimators of μ_0, ψ_i, φ_j

Bayesian additive model

Model Assumptions

i) The r.v. X_{ij} are conditionally, given $\Theta_{ij} = (\Psi_i, \Phi_j)$, independent with

$$E[X_{ij}|\Theta_{ij}] = \mu_0 + \Psi_i + \Phi_j, \quad \mathrm{Var}(X_{ij}|\Theta_{ij}) = \frac{\sigma^2}{w_{ij}}.$$

ii) The r.v. Ψ_i, $i = 1, \ldots, I$, are i.i.d. with

$$E[\Psi_i] = \mu_\Psi = 0, \quad \mathrm{Var}(\Psi_i) = \tau_\Psi^2.$$

iii) The r.v. Φ_j, $j = 1, 2, \ldots, J$, are i.i.d. with

$$E[\Phi_j] = \mu_\Phi = 0, \quad \mathrm{Var}(\Phi_j) = \tau_\Phi^2.$$

iv) Ψ_i, Φ_j are independent.

Problem: find for each $cell_{ij}$ the credibility estimator of

$$P_{ij} = E[X_{ij}|\Theta_{ij}] = \mu(\Theta_{ij}) = \mu_0 + \Psi_i + \Phi_j.$$

Credibility Estimators additive model

Theorem

i) $$P_{ij}^{Cred} = \widehat{\widehat{\mu(\Theta_{ij})}} = \mu_0 + \Psi_i^{Cred} + \Phi_j^{Cred},$$

ii) Ψ_i^{Cred} and Φ_j^{Cred} are given by the following system of equations:

$$\Psi_i^{Cred} = \alpha_i(\overline{X}_{i\bullet} - \mu_0) - \alpha_i \sum_{j=1}^{J} \frac{w_{ij}}{w_{i\bullet}} \Phi_j^{Cred},$$

$$\Phi_j^{Cred} = \beta_j(\overline{X}_{\bullet j} - \mu_0) - \beta_j \sum_{i=1}^{I} \frac{w_{ij}}{w_{\bullet j}} \Psi_i^{Cred},$$

where $$\alpha_i = \frac{w_{i\bullet}}{w_{i\bullet} + \frac{\sigma^2}{\tau_\Psi^2}}, \quad \beta_j = \frac{w_{\bullet j}}{w_{\bullet j} + \frac{\sigma^2}{\tau_\Phi^2}}.$$

Credibility Estimators additive model

Remarks:

- As the in the classical additive model the equation system can be solved iteratively.
- If all credibility weights α_i and β_j were equal to one (non-informative prior), one obtains the same solution as in the classical model.
- The above estimators are the credibility pendant to the classical estimators and hence also the credibility pendant to the GLM estimators in the case of normally distributed X_{ij}.

explicit solution if all weights are equal ($w_{ij} = 1$ for all i and j)

Corollary

$$P_{ij}^{Cred} = \widehat{\widehat{\mu(\Theta_{ij})}} = \mu_0 + \alpha(\overline{X}_{i\bullet} - \mu_0) + \beta(\overline{X}_{\bullet j} - \mu_0) - \gamma(\overline{X}_{\bullet\bullet} - \mu_0),$$

where $$\alpha = \frac{J}{J + \frac{\sigma^2}{\tau_\Psi^2}}, \quad \beta = \frac{I}{I + \frac{\sigma^2}{\tau_\Phi^2}}, \quad \gamma = \frac{\alpha\beta}{1 - \alpha\beta}(2 - \alpha - \beta).$$

Credibility Estimators additive model

Derivation

Conditionally, given $\boldsymbol{\Phi} = (\Phi_1, \Phi_2, \ldots, \Phi_J)'$, consider $X_{ij}^* = X_{ij} - \mu_0 - \Phi_j$.

It holds that $E[X_{ij}^* | \Psi_i, \Phi_j] = \Psi_i$; Var$(X_{ij}^* | \Psi_i, \Phi_j) = \dfrac{\sigma^2(\Theta_{ij})}{w_{ij}}$;

=> X_{ij}^* fulfill conditions of Bü-Straub modell

=> Credibility estimator based on X_{ij} and $\boldsymbol{\Phi}$

$$\Psi_i^* = \alpha_i(\overline{X}_{i\bullet} - \mu_0) - \alpha_i \sum_{j=1}^{J} \frac{w_{ij}}{w_{i\bullet}} \Phi_j$$

where

$$\alpha_i = \frac{w_{i\bullet}}{w_{i\bullet} + \frac{\sigma^2}{\tau_\Psi^2}}$$

from iterativity property follows

$$\Psi_i^{Cred} = \alpha_i(\overline{X}_{i\bullet} - \mu_0) - \alpha_i \sum_{j=1}^{J} \frac{w_{ij}}{w_{i\bullet}} \Phi_j^{Cred}.$$

numerical example: additive model

Observed average claim size

Factor B

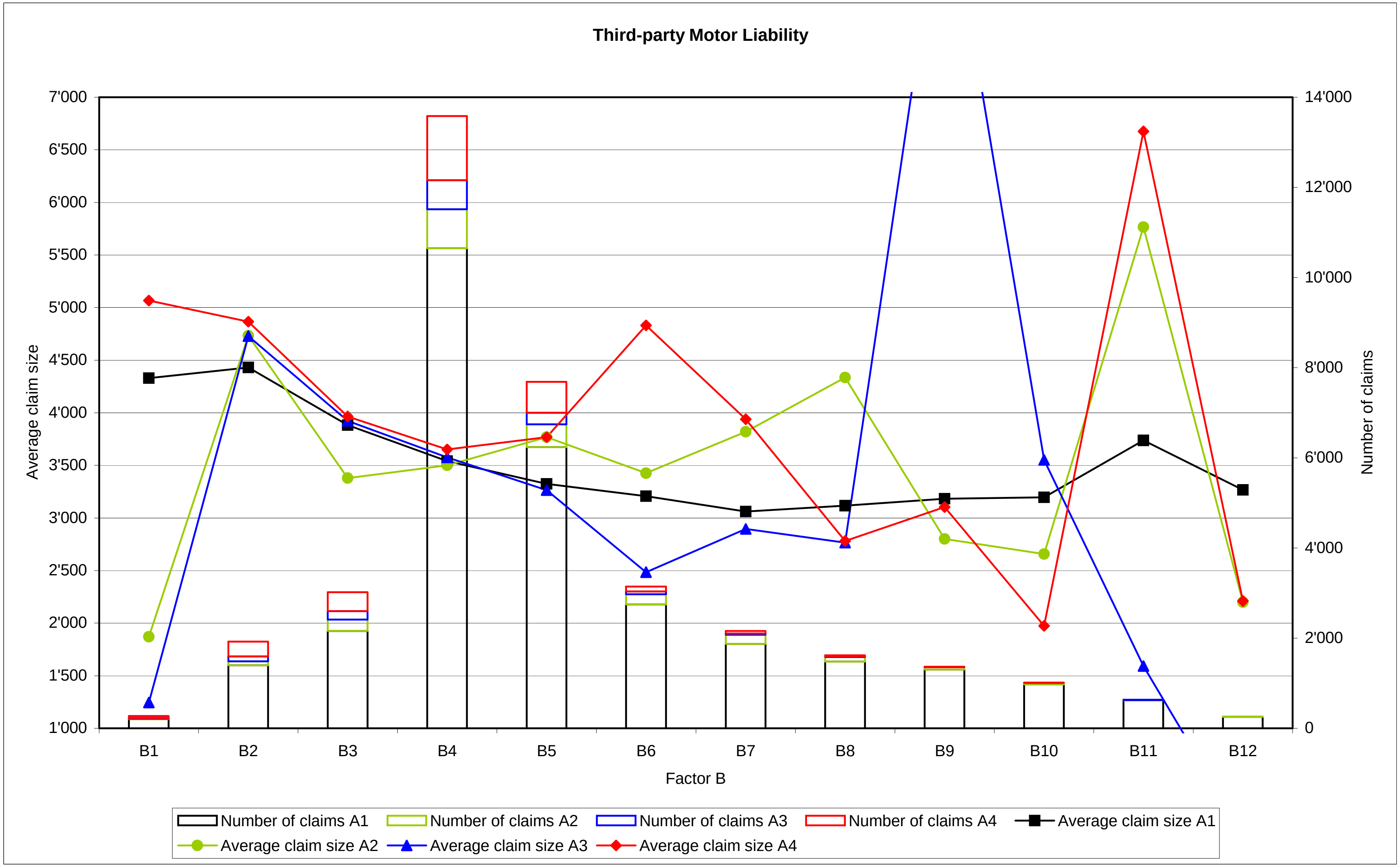
		B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11	B12	Total
Factor A	A1	4'330	4'432	3'883	3'540	3'324	3'206	3'062	3'117	3'182	3'197	3'738	3'267	3'460
	A2	1'871	4'733	3'379	3'501	3'769	3'426	3'818	4'335	2'800	2'657	5'765	2'200	3'619
	A3	1'246	4'729	3'923	3'575	3'265	2'484	2'896	2'764	9'169	3'553	1'592		3'575
	A4	5'066	4'866	3'966	3'652	3'769	4'830	3'939	2'780	3'103	1'974	6'674	2'211	3'903
	Total	4'252	4'536	3'855	3'551	3'390	3'262	3'156	3'186	3'181	3'169	3'771	3'226	3'514

Number of claims

Factor B

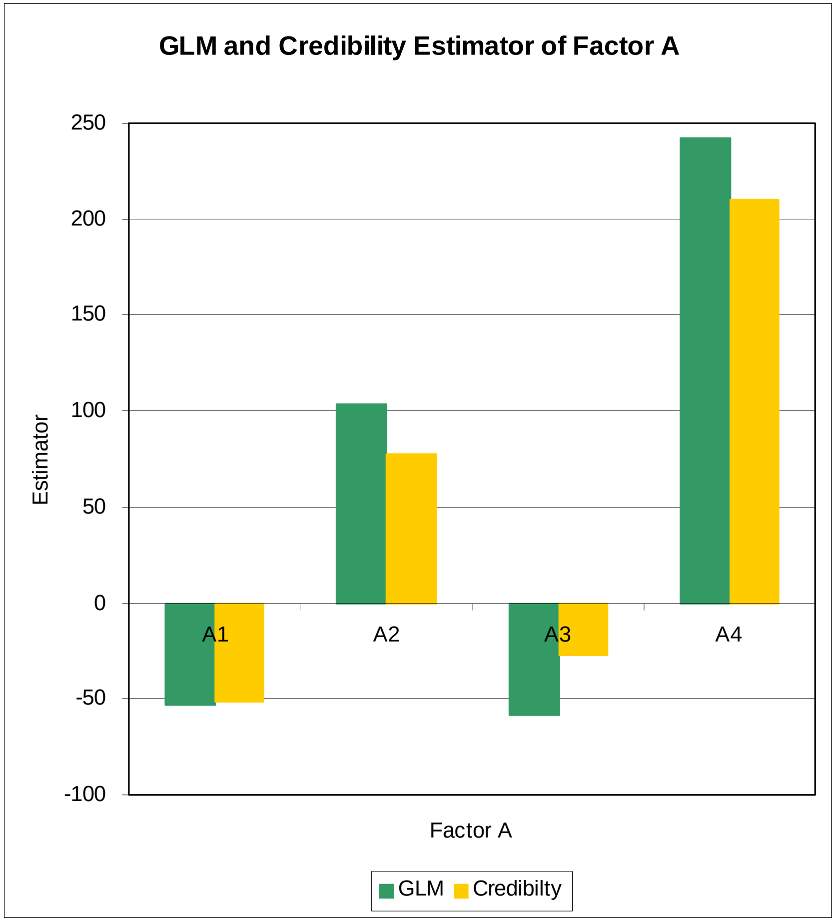
		B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11	B12	Total
Factor A	A1	204	1'397	2'161	10'650	6'239	2'746	1'870	1'478	1'306	974	620	252	2'451
	A2	12	89	251	864	501	228	209	103	48	31	9	9	193
	A3	9	108	184	644	261	64	23	15	3	2	3	0	112
	A4	49	327	427	1'427	683	105	56	24	9	10	3	1	266
	Total	68	475	745	3'330	1'879	764	525	393	331	246	154	63	3'022

numerical example: additive model

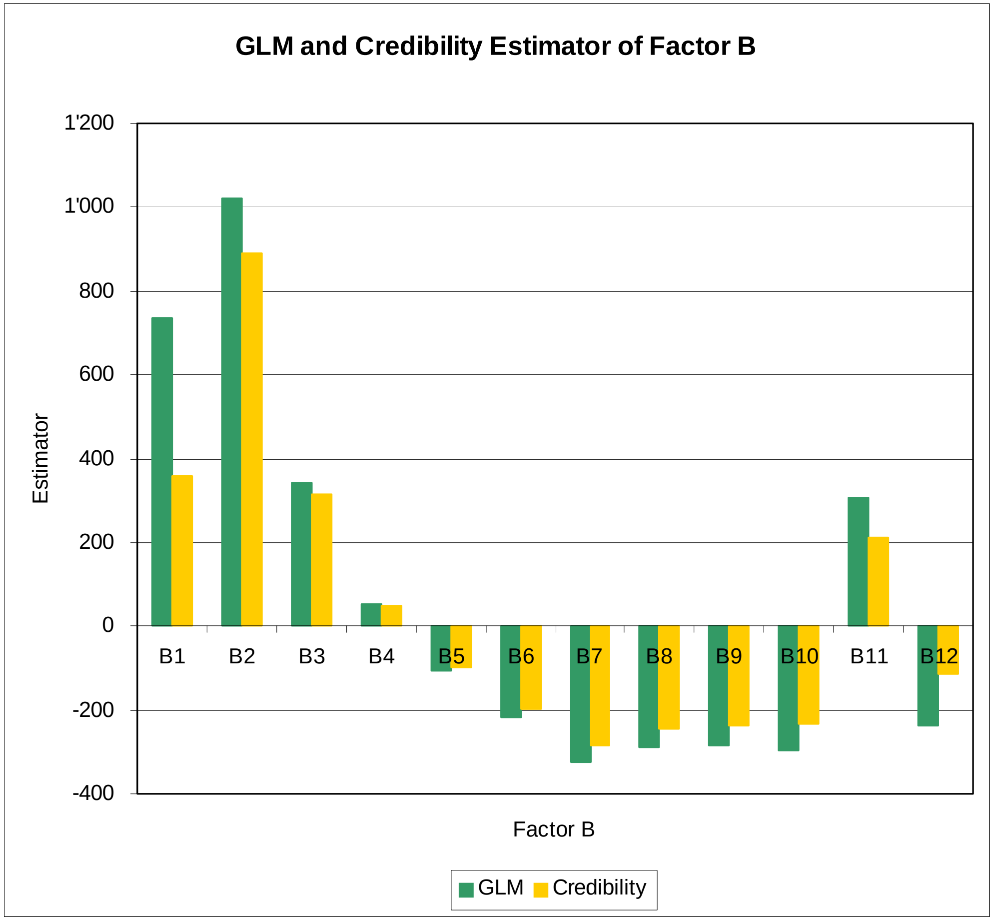


numerical example: additive model

$\widehat{\mu_0}$	$\widehat{\sigma^2}$	$\widehat{\tau^2_\Psi}$	$\widehat{\tau^2_\Phi}$
3’512	32’647’932	41’434	112’348



GLM and Credibility Estimator of Factor A



GLM and Credibility Estimator of Factor B

classical multiplicative model

Model Assumptions

The observable r. v. X_{ij} satisfy

$$E[X_{ij}] = \mu_0 \cdot \psi_i \cdot \varphi_j$$

where μ_0, ψ_i and φ_j are real numbers

$$\mathrm{Var}(X_{ij}) \propto w_{ij}^{-1}$$

Problem:

to determine for each $cell_{ij}$ $P_{ij} = E[X_{ij}] = \mu_0 \cdot \psi_i \cdot \varphi_j$

resp. to find estimators $\widehat{P}_{ij} = E[X_{ij}] = \hat{\mu}_0 \cdot \widehat{\psi}_i \cdot \widehat{\varphi}_j$

classical multiplicative model

method of marginal totals

$$\sum_j w_{ij}(\widehat{\mu_0} \cdot \widehat{\psi_i} \cdot \widehat{\varphi_j}) = \sum_j w_{ij} X_{ij} \qquad \sum_i w_{ij}(\widehat{\mu_0} \cdot \widehat{\psi_i} \cdot \widehat{\varphi_j}) = \sum_i w_{ij} X_{ij}$$

which is equivalent to

$$\widehat{\mu_0} = \overline{X}_{\bullet\bullet} \qquad \widehat{\psi_i} = \frac{\sum_j w_{ij} \frac{X_{ij}}{\overline{X}_{\bullet\bullet}}}{\sum_j w_{ij} \widehat{\varphi_j}} \qquad \widehat{\varphi_j} = \frac{\sum_i w_{ij} \frac{X_{ij}}{\overline{X}_{\bullet\bullet}}}{\sum_i w_{ij} \widehat{\psi_i}}$$

Properties

Maximum Likelihood estimators in the case where X_{ij} are independent and Poisson or overdispersed Poisson distributed

Bayesian multiplicative model

Model Assumptions

i) The r.v. X_{ij}, are conditionally, given $\Theta_{ij} = (\Psi_i, \Phi_j)$, independent with

$$E[X_{ij} | \Theta_{ij}] = \mu_0 \cdot \Psi_i \cdot \Phi_j,$$

$$\mathrm{Var}(X_{ij} | \Theta_{ij}) = \frac{\sigma^2(\Theta_{ij})}{w_{ij}} = \frac{\eta \cdot (\mu_0 \cdot \Psi_i \cdot \Phi_j)^p}{w_{ij}}, \quad \text{where } \eta \text{ and } p \in \mathrm{R}^+$$

ii) The r.v. Ψ_i, $i = 1, \ldots, I$, are i.i.d. with

$$E[\Psi_i] = \mu_\Psi = 1, \quad \mathrm{Var}(\Psi_i) = \tau_\Psi^2$$

iii) The r.v. Φ_j, $j = 1, 2, \ldots, J$, are i.i.d. with

$$E[\Phi_j] = \mu_\Phi = 1, \quad \mathrm{Var}(\Phi_j) = \tau_\Phi^2$$

Credibility estimator multiplicative model

Problem: find for each $cell_{ij}$ an estimator of

$$P_{ij} = E[X_{ij} | \Theta_{ij}] = \mu(\Theta_{ij}) = \mu_0 \bullet \Psi_i \bullet \Phi_j$$

Given $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_J)'$, consider "normalised" r.v. $X_{ij}^{(1)} = \frac{X_{ij}}{\Phi_j \mu_0}$

$$E\left[X_{ij}^{(1)} \middle| \Theta_{ij}\right] = \Psi_i,$$

$$\text{Var}\left(X_{ij}^{(1)} \middle| \Theta_{ij}\right) = \frac{\eta \bullet \Psi_i^p}{w_{ij}^{(1)}}, \quad \text{where } w_{ij}^{(1)} = w_{ij}(\Phi_j \mu_0)^{2-p}$$

Given $\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_I)'$, consider "normalised" r.v. $X_{ij}^{(2)} = \frac{X_{ij}}{\Psi_j \mu_0}$

$$E\left[X_{ij}^{(1)} \middle| \Theta_{ij}\right] = \Psi_i,$$

$$\text{Var}(X_{ij}^{(1)} | \Theta_{ij}) = \frac{\eta \bullet \Phi_j^p}{w_{ij}^{(1)}}, \quad \text{where } w_{ij}^{(2)} = w_{ij}(\Psi_i \mu_0)^{2-p}$$

Credibility estimator multiplicative model

=> **Pseudo-Estimators**

$$\Psi_i^*(\Phi) = 1 + \alpha_i\left(\overline{X}_{i\bullet}^{(1)} - 1\right),$$

$$\Phi_j^*(\Psi) = 1 + \beta_j\left(\overline{X}_{\bullet j}^{(2)} - 1\right),$$

where

$$X_{ij}^{(1)} = \frac{X_{ij}}{\Phi_j\mu_0}, \quad w_{ij}^{(1)} = w_{ij}(\Phi_j\mu_0)^{2-p}, \quad \overline{X}_{i\bullet}^{(1)} = \sum_{j=1}^{J}\frac{w_{ij}^{(1)}}{w_{i\bullet}^{(1)}}X_{ij}^{(1)},$$

$$X_{ij}^{(2)} = \frac{X_{ij}}{\Psi_i\mu_0}, \quad w_{ij}^{(2)} = w_{ij}(\Psi_i\mu_0)^{2-p}, \quad \overline{X}_{\bullet j}^{(2)} = \sum_{i=1}^{I}\frac{w_{ij}^{(2)}}{w_{\bullet j}^{(2)}}X_{ij}^{(2)},$$

$$\alpha_i = \frac{w_{i\bullet}^{(1)}}{w_{i\bullet}^{(1)} + \frac{\sigma_\Psi^2}{\tau_\Psi^2}}, \quad \sigma_\Psi^2 = \eta \cdot E[\Psi_i^p]$$

Credibility estimator multiplicative model

Credibility based estimator

i)

$$\Psi_i^{(Cred)} = \Psi_i^*(\boldsymbol{\Phi}^{(Cred)})$$

$$\Phi_j^{(Cred)} = \Phi_j^*(\boldsymbol{\Psi}^{(Cred)})$$

where $\boldsymbol{\Phi}^{(Cred)} = (\Phi_1^{(Cred)}, \ldots, \Phi_J^{(Cred)})'$

$$\boldsymbol{\Psi}^{(Cred)} = (\Psi_1^{(Cred)}, \ldots, \Psi_I^{(Cred)})'$$

ii)

$$P_{ij}^{(Cred)} = \mu_0 \bullet \Psi_i^{(Cred)} \bullet \Phi_j^{(Cred)}$$

Remark:

for $p = 1$ ((overdispersed) Poisson) and if we put $\alpha_i = \beta_j = 1$ (non informative prior), then we obtain the same results as with GLM resp. with the method of marginal totals

Thank you for your attention

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