Credibility for the Chain Ladder Reserving Method

Alois Gisler and Mario Wüthrich
AXA-Winterthur and ETH Zürich
Switzerland

Introduction

- claims reserving still one of the basic actuarial tasks
- based on claims development triangles (or trapezoids) predict the ultimate claim
- problems often encountered in practise
  - scarce data
    data from individual triangle do not allow to make a reliable forecast;
    actuaries then often rely on industry-wide development patterns;
    question: when to rely on industry-wide data and to what extent
  - line of business with little data
    then one often considers the development of other similar lines
    question: how much to rely on the other lines and how much on the data of the line in question ?

- **credibility: answer to such kind of questions**
  how to combine portfolio information and individual data

Introduction

- in the paper we concentrate on the Chain-Ladder reserving method, but the basic idea could also be transformed to other reserving methods

- **basic idea of the paper:**
  to use a Bayesian set-up and credibility to combine portfolio information with the individual triangle information

- **link to classical chain ladder**
  taking an non informative prior we arrive at the classical chain ladder estimate.
  However: the estimate of the mean square error differs from the ones found in the literature. The paper also throws a new light upon the estimation of the mean square error.

classical chain ladder

At time I, observations (upper left trapezoid)

$$D_I = \{C_{i,j} : 0 \le i \le I,\ 0 \le j \le J,\ i+j \le I\}$$

Usually,

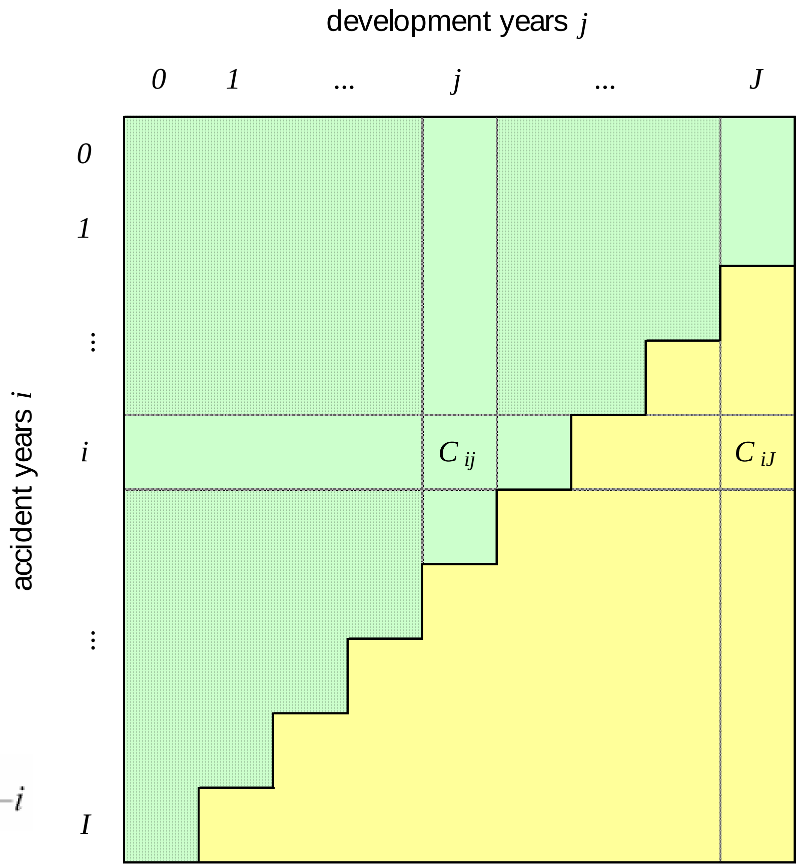
$C_{i,j}$ = cumulative claims payments or incurred claims

$C_{i,J}$ = ultimate claim of accident year i

want to estimate for each accident year i :

ultimate claim $C_{i,J}$ and

the outstanding liabilities $R_i = C_{i,J} - C_{i,I-i}$



classical chain ladder

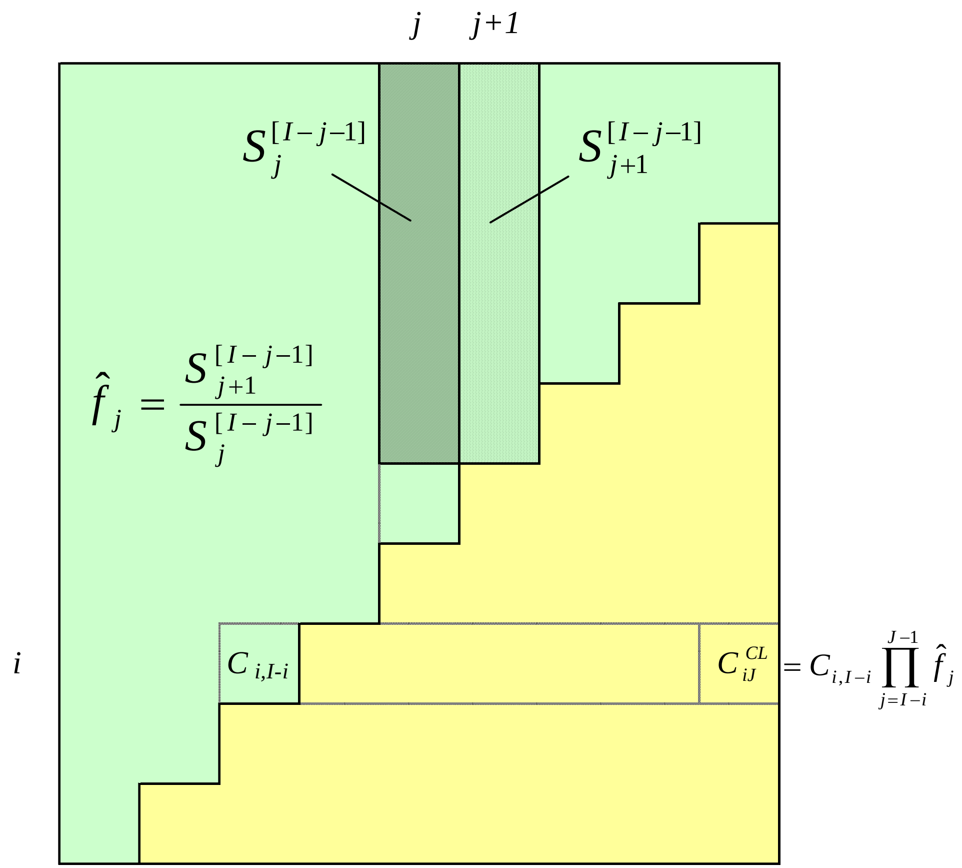
chain ladder algorithm:

$$C_{i,J}^{CL} = C_{i,I-i} \prod_{j=I-i}^{J-1} \widehat{f}_j$$

where
$$\widehat{f}_j = \frac{S_{j+1}^{[I-j-1]}}{S_j^{[I-j-1]}}$$

$$S_j^{[k]} = \sum_{i=0}^{k} C_{i,j}$$

$$R_i^{CL} = C_{i,J}^{CL} - C_{i,I-i}$$



classical chain ladder

Model Assumptions of Mack

M1 $C_{i,j}$ belonging to different accident years are independent

M2 There exist constants $f_j > 0$ and $\sigma_j^2 > 0$, such that

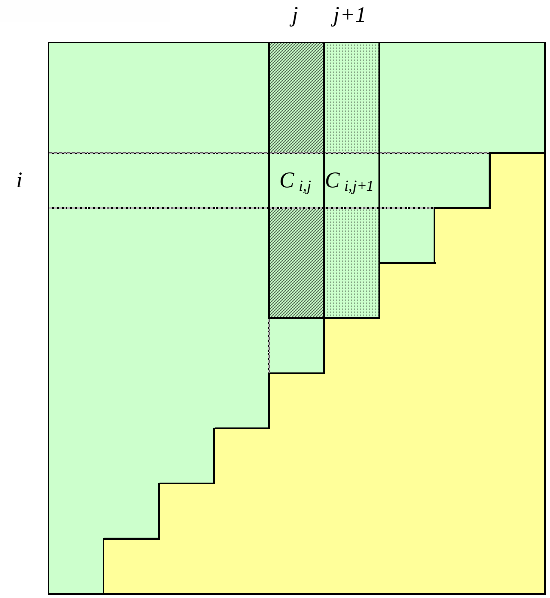
$$E[C_{i,j+1} | C_{i,1}, \ldots, C_{i,j}] = f_j \, C_{i,j},$$

$$\mathrm{Var}[C_{i,j+1} | C_{i,1}, \ldots, C_{i,j}] = \sigma_j^2 \, C_{i,j}.$$

"individual" chain ladder ratios: $Y_{i,j} = \dfrac{C_{i,j+1}}{C_{i,j}}$

model assumption M2 equivalent to

$$E[Y_{i,j+1} | C_{i,j}] = f_j, \quad \mathrm{Var}[Y_{i,j+1} | C_{i,j}] = \frac{\sigma_j^2}{C_{i,j}}$$



$$\widehat{f}_j = \frac{\sum_{i=0}^{I-j-1} C_{i,j} Y_{i,j}}{\sum_{i=0}^{I-j-1} C_{i,j}} = \frac{\sum_{i=0}^{I-j-1} C_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}} = \frac{S_{j+1}^{[I-j-1]}}{S_j^{[I-j-1]}}$$

$$E\left[\widehat{f}_j \middle| B_j\right] = f_j,$$

$$\mathrm{Var}\left(\widehat{f}_j \middle| B_j\right) = \frac{\sigma_j^2}{S_j^{[I-j-1]}}.$$

classical chain ladder

point estimate:

$$C_{i,J}^{CL} = C_{i,I-i} \prod_{j=I-i}^{J-1} \widehat{f}_j$$

mean square error

$$mse(R_i^{CL}) = E\left[(C_{i,J}^{CL} - C_{i,J})^2 \middle| \mathcal{D}_I \right]$$

$$= \underbrace{(C_{i,J} - E[C_{i,J}|\mathcal{D}_I])^2}_{\textit{process variance}} + \underbrace{(C_{i,J}^{CL} - E[C_{i,J}|\mathcal{D}_I])^2}_{\textit{estimation error}}$$

$$= C_{i,I-i}\Gamma_{I-i}^{CL} + C_{i,I-i}^2 \Delta_{I-i}^{CL}$$

where

$$\Gamma_{I-i}^{CL} = \sum_{j=I-i}^{J-1} \left\{ \prod_{m=I-i}^{j-1} f_m \cdot \sigma_k^2 \prod_{n=j+1}^{J-1} f_n^2 \right\}$$

$$\Delta_{I-i}^{CL} = \left(\prod_{j=I-i}^{J-1} f_j - \prod_{j=I-i}^{J-1} \widehat{f}_j \right)^2$$

classical chain ladder

estimation of mean square error:

Mack (2003); Buchwalder, Bühlmann, Merz, Wüthrich (2006)

process variance:

Mack and BBMW $$\widehat{\Gamma_{I-i}^{CL}} = \sum_{j=I-i}^{J-1} \left\{ \prod_{m=I-i}^{j-1} \hat{f}_m \cdot \hat{\sigma}_k^2 \prod_{n=j+1}^{J-1} \hat{f}_n^2 \right\}$$

estimation error:

Mack $$\widehat{\Delta_{I-i}^{CL}}^{Mack} = \left(\prod_{l=I-i}^{j-1} \hat{f}_l^2 \right) \left(\sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{S_j^{[I-j-1]}} \frac{1}{\hat{f}_j^2} \right)$$

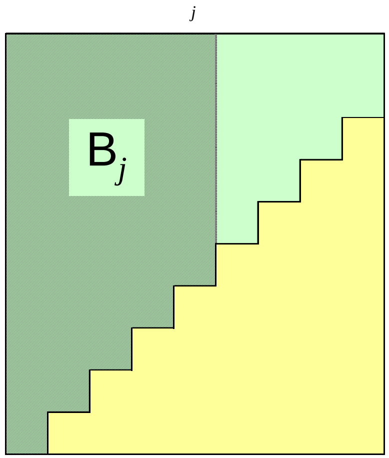
BBMW $$\widehat{\Delta_{I-i}^{CL}}^{BBMW} = \prod_{j=I-i}^{J-1} \left(\hat{f}_j^2 + \frac{\hat{\sigma}_j^2}{S_j^{[I-j-1]}} \right) - \left(\prod_{j=I-i}^{J-1} \hat{f}_j^2 \right)$$

Bayes chain ladder

$f_j,\ j = 0, 1, \ldots, J-1,$ realizations of independent r. v. F_j

$\mathbf{F} = (F_0, F_1, \ldots, F_{J-1})'$

$\mathrm{B}_j = \{C_{i,k};\ i + k \leq I,\ k \leq j\}$



Model Assumptions

B1 Conditionally, given $\mathbf{F}$, r.v. $C_{i,j}$ belonging to different accident years i are independent.

B2 Conditionally, given $\mathbf{F}$ and B_j, the conditional distribution of $Y_{i,j}$ depend only on $C_{i,j}$ and

$$E[Y_{i,j}|\mathbf{F}, \mathrm{B}_j] = F_j, \quad ,\mathrm{Var}[Y_{i,j}|\mathbf{F}, \mathrm{B}_j] = \frac{\sigma_j^2(F_j)}{C_{i,j}}.$$

B3 $\{F_0, F_1, \ldots, F_{J-1}\}$ are independent with distributions $U_j(f_j)$

Bayes chain ladder

Remark:

$$\widehat{F}_j = \frac{S_{j+1}^{[I-j-1]}}{S_j^{[I-j-1]}} \qquad E\left[\widehat{F}_j \middle| \mathbf{F}, \mathrm{B}_j\right] = F_j, \quad \mathrm{Var}\left[\widehat{F}_j \middle| \mathbf{F}, \mathrm{B}_j\right] = \frac{\sigma_j^2(F_j)}{S_j^{[I-j-1]}}.$$

Theorem:

A posteriori, given D_I, the r.v. $F_0, F_1, \ldots, F_{J-1}$ are independent with posterior distributions $U_j(f_j|\mathrm{D}_I)$

$$C_{i,J}^{Bayes} = C_{i,I-i} \prod_{j=I-i}^{J-1} F_j^{Bayes}, \quad \text{where } F_j^{Bayes} = \text{Bayes estimator of } F_j.$$

Bayes chain ladder

Theorem:

$$mse\left(C_{i,J}^{Bayes}\right) = E\left[\left(C_{i,J}^{Bayes} - C_{i,J}\right)^2 \middle| \mathrm{D}_I\right]$$

$$= \underbrace{E[\mathrm{Var}[C_{i,J}|\mathbf{F}, \mathrm{D}_I]|\mathrm{D}_I]}_{\text{"average" process variance}} + \underbrace{E\left[\left(C_{i,J}^{Bayes} - E[C_{i,J}|\mathbf{F}, \mathrm{D}_I]\right)^2 \middle| \mathrm{D}_I\right]}_{\text{"average" estimation error}}$$

$$= C_{i,I-i}\Gamma_{I-i} + C_{i,I-i}^2 \Delta_{I-i}^B,$$

where

$$\Gamma_{I-i} = \sum_{k=I-i}^{J-1} \left\{ \prod_{m=I-i}^{k-1} F_m^{Bayes} \cdot E[\sigma^2(F_k)|\mathrm{D}_I] \cdot \prod_{n=k+1}^{J-1} E[F_n^2|\mathrm{D}_I] \right\},$$

$$\Delta_{I-i}^B = \mathrm{Var}\left(\prod_{j=I-i}^{J-1} F_j \middle| \mathrm{D}_I \right).$$

Credibility for chain ladder

basic idea: replace F_j^{Bayes} by a credibility estimator F_j^{Cred}

Definition

credibility based predictor of $C_{i,J}$ given D_I : $C_{i,J}^{(Cred)} = C_{i,I-i} \prod_{j=I-i}^{J-1} F_j^{Cred}$

Theorem

i) $$F_j^{Cred} = \alpha_j \widehat{F}_j + (1-\alpha_j) f_j,$$

where $$\widehat{F}_j = \frac{S_{j+1}^{[I-j-1]}}{S_j^{[I-j-1]}}, \qquad \alpha_j = \frac{S_j^{[I-j-1]}}{S_j^{[I-j-1]} + \frac{\sigma_j^2}{\tau_j^2}},$$

$$f_j = E[F_j], \quad \sigma_j^2 = E[\sigma_j^2(F_j)], \quad \tau_j^2 = \mathrm{Var}[F_j].$$

ii) mean square error of F_j^{Cred}

$$E\left[(F_j^{Cred} - F_j)^2 \,\middle|\, \mathrm{B}_j \right] = \alpha_j \frac{\sigma_j^2}{S_j^{[I-j-1]}} = (1-\alpha_j)\tau_j^2.$$

Credibility for chain ladder

Theorem

$$mse\left(R_i^{(Cred)}\right) \cong \underbrace{C_{i,I-i}\Gamma_{I-i}^{Cr}}_{\textit{process variance}} + \underbrace{C_{i,I-i}^2\Delta_{I-i}^{Cr}}_{\textit{estimation error}}$$

where

$$\Gamma_{I-i}^{Cr} = \sum_{k=I-i}^{J-1}\left\{\prod_{m=I-i}^{k-1} F_m^{Cred} \cdot \sigma_k^2 \prod_{n=k+1}^{J-1}\left((F_n^{Cred})^2 + \alpha_n \frac{\sigma_n^2}{S_n^{[I-n-1]}} \right)\right\}$$

$$\Delta_{I-i}^{Cr} = \prod_{j=I-i}^{J-1}\left((F_j^{Cred})^2 + \alpha_j \frac{\sigma_j^2}{S_j^{[I-j-1]}} \right) - \prod_{j=I-i}^{J-1}(F_j^{Cred})^2 .$$

Credibility for chain ladder

Link to classical chain ladder

For $\tau_j^2 \to \infty$ (non informative prior)

- $F_j^{Cred} = \widehat{f}_j$ and $C_{i,J}^{((Cred))} = C_{i,J}^{CL}$
- mean square error:

process variance $C_{i,I-i}\Gamma_{I-i}^{Cr}$:

$$\Gamma_{I-i}^{Cr} = \sum_{k=I-i}^{J-1}\left\{\prod_{m=I-i}^{k-1}\widehat{f}_m \cdot \widehat{\sigma}_k^2 \prod_{n=k+1}^{J-1}\left(\widehat{f}_n^2 + \frac{\widehat{\sigma}_n^2}{S_n^{[I-n-1]}}\right)\right\}$$

$$\Gamma_{I-i}^{Cr} \neq \widehat{\Gamma_{I-i}^{CL}} \quad \text{(of Mack and BBMW)}$$

estimation error $C_{i,I-i}^2\Delta_{I-i}^{Cr}$:

$$\Delta_{I-i}^{Cr} = \prod_{j=I-i}^{J-1}\left(\widehat{f}_j^2 + \frac{\widehat{\sigma}_j^2}{S_j^{[I-j-1]}}\right) - \prod_{j=I-i}^{J-1}\left(\widehat{f}_j\right)^2$$

$$\Delta_{I-i}^{Cr} = \widehat{\Delta_{I-i}^{CL}}^{BBMW} \neq \widehat{\Delta_{I-i}^{CL}}^{Mack}$$

Exact Credibility

Best reserve estimate would be the Bayes reserves R_i^{Bayes}

Are the credibility reserves a good approximation ?

Are there reasonable Bayes models, where $C_{i,j}^{Cred} = C_{i,j}^{Bayes}$?

The answer is yes !

basic assumptions:

- conditionally on $\mathbf{F}$ and B_j, the r.v. $Y_{0,j}, \ldots, Y_{I,j}$ are independent with a distribution belonging to the one-parameter exponential dispersion family
- the a priori distribution of F_j belongs to the family of the natural conjugate priors

Then $C_{i,j}^{Cred} = C_{i,j}^{Bayes}$

Exact Credibility

of particular interest:

Tweedie models = subclass of the exponential family with $V(\mu) = \mu^p$

Defined only for p outside the interval $0 < p < 1$

include

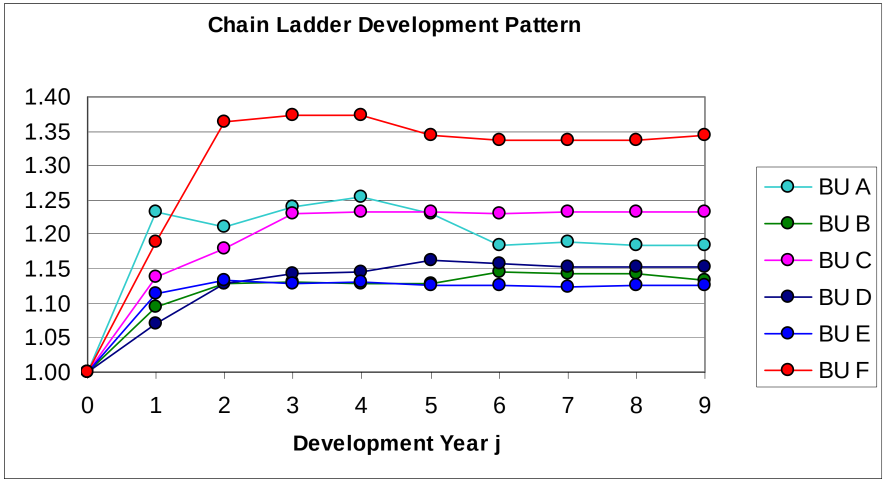
- $p = 0$: Normal-distribution
- $p = 1$: (Overdispersed) Poisson distribution
- $1 < p < 2$: Compound Poisson with Gamma distributed claim amounts
- $p = 2$: Gamma distribution

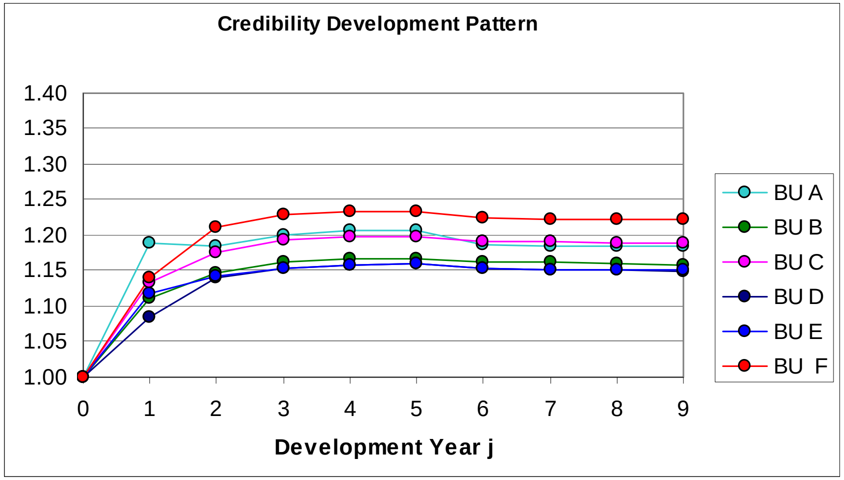
in particular, the case $1 < p < 2$ often a reasonable assumption

Numerical Example

table of results												
	j=	0	1	2	3	4	5	6	7	8	9	product
	σ_j^2	336.53	34.739	7.828	5.934	0.426	4.342	4.248	0.239	0.097	0.154	
all BU	$\mathbf{F_j^{CL}}$	**2.111**	**1.119**	**1.031**	**1.013**	**1.004**	**1.001**	**0.993**	**0.998**	**1.000**	**0.999**	**2.452**
A	F_j^{CL}	2.270	1.233	0.982	1.024	1.012	0.981	0.962	1.003	0.996	1.000	2.686
	F_j^{cred}	2.111	1.189	0.996	1.015	1.004	1.001	0.984	0.998	1.000	0.999	2.499
B	F_j^{CL}	2.133	1.094	1.032	1.002	0.998	1.000	1.014	0.999	1.000	0.990	2.416
	F_j^{cred}	2.111	1.111	1.033	1.012	1.003	1.001	0.997	0.998	1.000	0.997	2.444
C	F_j^{CL}	2.189	1.138	1.037	1.042	1.003	1.000	0.999	1.002	1.000	1.000	2.700
	F_j^{cred}	2.111	1.134	1.036	1.016	1.004	1.001	0.994	0.998	1.000	0.999	2.509
D	F_j^{CL}	2.108	1.070	1.054	1.013	1.004	1.015	0.996	0.995	1.000	1.000	2.429
	F_j^{cred}	2.111	1.084	1.050	1.013	1.004	1.001	0.994	0.998	1.000	0.999	2.426
E	F_j^{CL}	1.930	1.114	1.018	0.995	1.002	0.997	0.999	0.997	1.002	1.000	2.172
	F_j^{cred}	2.111	1.119	1.021	1.010	1.004	1.001	0.995	0.998	1.000	0.999	2.429
F	F_j^{CL}	3.008	1.190	1.146	1.006	1.000	0.979	0.996	1.000	1.000	1.004	4.041
	F_j^{cred}	2.111	1.139	1.064	1.013	1.004	1.001	0.993	0.998	1.000	0.999	2.578

Numerical Example





BU	estimated reserves		estimated mse$^{1/2}$			
	CL	Cred	CL		Cred	
A	486	504	657	135.1%	498	98.9%
B	235	244	288	122.7%	402	164.3%
C	701	517	411	58.6%	520	100.6%
D	1'029	899	844	82.1%	729	81.1%
E	495	621	397	80.2%	596	95.9%
F	40	25	140	347.0%	149	595.8%
sum	**2'987**	**2'810**	1'254	42.0%	1'261	44.9%

Thank you for your attention

Alois Gisler and Mario Wüthrich
AXA-Winterthur and ETH Zürich
Switzerland

AXA winterthur
FINANCIAL PROTECTION

ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich