

Catastrophe Insurance Products in Markov Jump Diffusion Model

(Topic of paper: **Risk management of an insurance enterprise**)

Lin, Shih-Kuei, Assistant Professor
Department of Finance
National University of Kaohsiung
No. 700, Kaohsiung University Rd.,
Nan Tzu Dist., 811, Kaohsiung, Taiwan, R.O.C.
E-mail: square@nuk.edu.tw
Tel: 886-7-5919496
Fax: 886-7-5919329

Chang, Chia-Chien, Ph.D Student
Department of Finance
National Sun Yat-sen University
No. 70, Lien-hai Rd, 804, Kaohsiung, Taiwan, R.O.C.
E-mail: u7501247@yahoo.com.tw
Tel: 886-7-5252000
Fax: 886-7-5254899

Abstract

For catastrophic events, the assumption that catastrophe claims occur in terms of the Poisson process is inadequate as it has constant intensity. This article proposes Markov Modulated Poisson process (MMPP) to model the arrival process for catastrophic events. Under this process, the underlying state is governed by a homogenous Markov chain, and it is the generalization of Cummins and Geman (1993), Chang, Chang, and Yu (1996) and Geman and Yor (1997). We apply Markov jump diffusion model to derive pricing formulas and hedging formulas for catastrophe insurance products, included futures call option, catastrophe PCS call spread and catastrophe bond. The numerical analysis shows how the catastrophe insurance products prices are related to jump rate of catastrophe events, standard deviation of jump size, and mean of jump size.

Keywords: Markov modulated Poisson process, Markov jump diffusion model, futures call option, catastrophe PCS call spread, catastrophe bond.

I. Introduction

Insurance companies traditionally often limit their liabilities given the large capital requirements required to cover high-loss-severity catastrophe risk by reinsurance. However, the reinsurance industry is also limit in size relative to the magnitude of these damages, creating large fluctuations in the price and availability of reinsurance during years as multiple catastrophes occur. During the last decade, the high level of worldwide catastrophe losses in terms of frequency and severity had a marked effect on the reinsurance market. The catastrophes such as Storm Daria (Europe 1990), Hurricane Andrew (USA 1992), Northridge earthquake (USA 1994) and the Kobe earthquake (Japan 1995) have impacted the profitability and capital bases of reinsurance companies. Therefore, because of worldwide capacity shortage and the increment number of natural catastrophes in recent years, the illustration of a viable substitute to reinsurance is viewed by the insurance industry to be both timely and desirable.

The Chicago Board of Trade (CBOT) launched catastrophe insurance futures contract in 1992, options on catastrophe insurance futures in 1993. These contract values link to the loss ratios compiled by the Insurance Service Office (ISO). Due to the low trading volume in these derivatives trading was given up in 1995. They are replaced by a new generation of options called Property Claim Services (PCS) options which is introduced at the CBOT in September 1995. PCS Options are based on catastrophe loss indices provided daily by PCS – a US industry authority which estimates catastrophic property damage since 1949. Unlike reinsurance that is costly, lengthy, and irreversible, these catastrophe insurance derivatives have the advantages of reversibility, transactional efficiency and anonymify because they are standardized and exchange-traded. These innovative catastrophe derivatives provide the insurance industry access to financial markets to transfer catastrophe exposure to contract counterparties willing to bear the risk. This access allows the insurance industry to stabilize losses and increases overall capacity in the insurance or reinsurance market. Besides underwriters, mainstream institutional investors may also enjoy the diversification effect of these derivatives since catastrophic losses display no correlation with the price movements of stocks or bonds.

Another innovation product for catastrophe risk management is the catastrophe bond (CAT bond) (or named as ‘Act of God bond’ or ‘insurance-liked bond’), which is a liability hedged instrument for insurance companies. CAT bond provisions have debt-forgiveness triggers whose generic design grants the interest payment and/or the return of principal

forgiveness. Meanwhile, the debt forgiveness can be triggered by the insurer's actual losses or a composite index of insurer's losses during the specific period. Under this structure, the insurance company could transfer the catastrophe risk to increase the ability to provide insurance protection. The first successful CAT bond was issued in 1997 by Swiss Re. In 1999 the first CAT bond by a non-financial firm was issued to cover earthquake losses for in the Tokyo region for Oriental Land Company Ltd. Guy Carpenter & Company show that the CAT bond market recorded total issuance of \$1.99 billion in 2005, a 74 percent increase over the \$1.14 billion issuance in 2004 and 15 percent higher than the previous record of \$1.73 billion issued in 2003. During the period 1997-2005, Guy Carpenter and MMC Securities Corporation reported that 69 catastrophe bonds have been issued with total risk limits of \$10.65 billion, whose predominant sponsors were insurers and reinsurers.

Certain articles focus on the pricing of catastrophe-linked securities. For example, Cox and Schwebach (1992), Cummins and Geman (1993), Cummins and Geman (1995), and Chang, Chang, and Yu (1996) price the CAT futures and CAT call spreads under deterministic interest rate and PCS loss process. Litzenberger, Beaglehole, and Reynolds (1996) price a zero-coupon CAT bond with hypothetical catastrophe loss distribution. Zajdenweber (1998) follows Litzenberger, Beaglehole, and Reynolds (1996) yet change the catastrophe loss distribution to Levy distribution. Louberge, Kellezi, and Gilli (1999) calculate the CAT bond under the assumption that the catastrophe loss stands for a pure Poisson process, the loss severity is an independently identical lognormal distribution, and the interest rate follows a binomial random process. Lee and Yu (2002) assume that the catastrophe aggregate loss follows the compound Poisson process and then compute risk-free and default-risky CAT under the consideration of default risk, basis risk, and moral hazard using Monte Carlo method. Vaugirard (2003a, 2003b) adapts the jump-diffusion model of Merton (1976) to develop a valuation framework that allows for catastrophic events, interest rate uncertainty, and non-traded underlying state variables and then reports the fairly simulations of insurance-linked securities. Dassios and Jang (2003) use the Cox process (or a doubly stochastic Poisson process) to model the claim arrival process for catastrophic events and then apply the model to the pricing of stop-loss catastrophe reinsurance contracts and catastrophe insurance derivatives under constant interest rate.

Therefore, expect for Dassios and Jang (2003), most pricing models for catastrophe insurance products assume that the loss claim arrival process follows Poisson process. This study intends to contribute to the literature by providing an alternative point process needs to be used to generate the arrival process. We will propose a new model, a Markov jump

diffusion model, which extends the Poisson process used in the jump diffusion model to be Markov modulated Poisson process. Markov modulated Poisson process stands for a doubly stochastic Poisson process where the underlying state is driven by a homogenous Markov chain (cf. Last and Brandt, 1995). More precisely, instead of constant average jump rate in the years in the jump diffusion model, under Markov modulated Poisson process, the arrival rates of new information are different from the abnormal vibrations of the loss dependent on current situation. The Markov jump diffusion model with two states, the so-called switched jump diffusion model, depends on the status of the economy. In switched jump diffusion model, the jump rates are different in different status, more precisely; the jump rates are large in one state and small in other status. Figure 1 show PCS loss quarterly in the United State from 1950 first quarterly to 2005 first quarterly, respectively ¹. In general, if the arrival process (jump rate) of natural catastrophes stands for Poisson process, then it could appear constant average jump rate in the years. However, figure 1 seems to reveal the smaller jump rate of natural catastrophes before 1990s and larger jump rate of natural catastrophes after 1990s. Therefore, we could infer that the arrival process (jump rate) of natural catastrophes could be different at different status and the Markov modulated Poisson process could be more fit than Poisson process to capture the arrival rate of natural catastrophes.

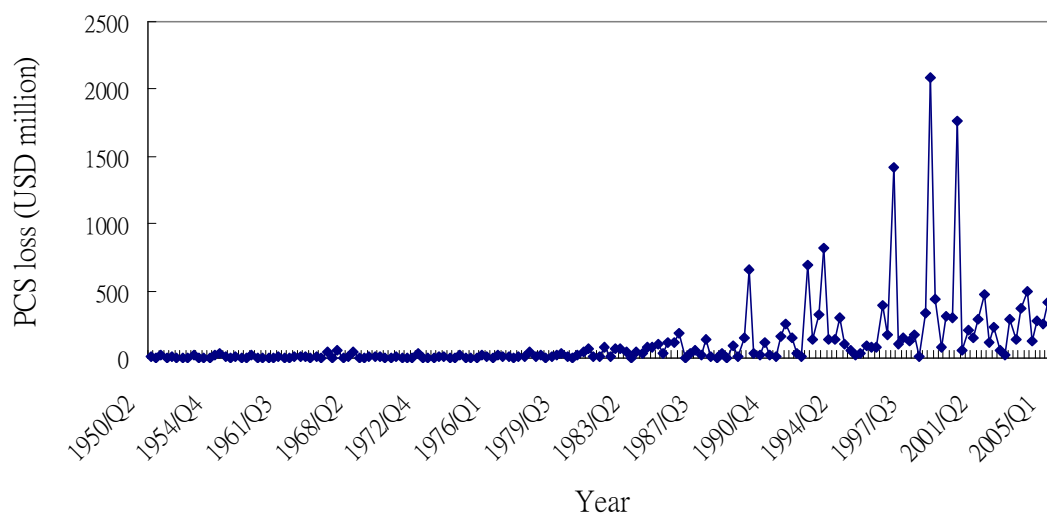


Figure 1: PCS loss in the United State during 1950 Q1 to 2005 Q1

The contributions of this paper are follows: (1) To capture the phenomenon of figure 1 that the arrival process of natural catastrophes could be different at different status, we propose a more general Markov jump diffusion model to model loss process and it could be the generalization of Cummins and Geman (1993), Chang, Chang, and Yu (1996) and Geman

and Yor (1997). (2) We apply the Markov jump diffusion model to evaluate accurately the valuation of catastrophe insurance products, including European futures call option, catastrophe PCS call spread, and CAT bond with the constant interest rate assumption. Meanwhile, we propose the hedging strategy for these catastrophe insurance products. We make use of Merton's (1976) assumption that the risk associated with jumps can be diversified away, and therefore the jump size distribution and transition rate are not altered by the measure changed. Our pricing formula of European futures call option can also reduce to Poisson sum of Black's prices of Chang, Chang, and Yu (1996) and Black's prices of Cox and Schwebach (1992).

The remainder of the paper is organized as follows. Section 2 presents the general framework of the model. Section 3 derives the pricing formulas and hedging strategy of three catastrophe insurance products: European futures call option, catastrophe PCS call spread, and CAT bond. Section 4 demonstrates numerical analysis. Section 5 summarizes the article and gives the conclusions. For ease of exposition, most proofs are in an appendix.

II. General Framework of The Model

1. Markov Modulated Poisson Process (MMPP)

The Markov modulated Poisson process $\Phi(t)$ stands for a doubly stochastic Poisson process where the underlying state is governed by a homogenous Markov chain. Since the Markov Chain has a finite number of states, the Poisson arrival rate takes discrete values corresponding to each state. More precisely, we consider a finite state space $\mathbf{X} = \{1, 2, \dots, I\}$ and let $\{X, P_i : i \in \mathbf{X}\}$ be a Markov jump process on a state space \mathbf{X} , with transition rate $\Psi(i, j)$ denoted as:

$$\Psi(i, j) = \begin{cases} \nu(i, j), & i \neq j, \\ -\sum_{j \neq i} \nu(i, j), & \text{otherwise.} \end{cases}$$

where $i, j \in \mathbf{X}$. Keeping company with X , we consider non-negative numbers $\lambda_1, \lambda_2, \dots, \lambda_I$ where λ_i is the intensity of a doubly stochastic Poisson process $\Phi(t)$ if X is at state i . That is, we consider a point process $\Phi(t) = (T_n)$, which is for all $i \in \mathbf{X}$ a $\{X, P_i\}$ -doubly stochastic Poisson process with $\sigma(X)$ -intensity kernel

$$v(dt) := E[\Phi(dt) | X] = 1_{\{t < T_\infty^X\}} \lambda_{X(t^-)} dt,$$

where T_∞^X is the point of explosion of X . More precisely, $P_i(\Phi \in \cdot | X)$ is P_i -a.s a distribution of a non-homogeneous Poisson process with intensity function $1_{\{t < T_\infty^X\}} \lambda_{X(t^-)}$.

For $0 < z < 1$, define

$$P^*(z, t) = \sum_{n=0}^{\infty} P(n, t) z^n \quad (2.1)$$

with $P(m, 0) = 1_{\{m=0\}} D_{ij}$, where $D_{ij} = 1$, if $i = j$; 0, otherwise.

By using Kolmogorov's forward equation, the derivative of $P(m, t)$ becomes

$$\frac{d}{dt} P(m, t) = (\Psi - \Lambda) P(m, t) + 1_{\{m \geq 1\}} \Lambda P(m-1, t),$$

and the derivative of $P^*(z, t)$ becomes

$$\frac{d}{dt} P^*(z, t) = (\Psi - \Lambda) P^*(z, t) + z P^*(z, t) \Lambda,$$

it's unique solution can be obtained as

$$P^*(z, t) = e^{[\Psi - (1-z)\Lambda]t} \quad (2.2)$$

where $P(m, t) := P_{ij}(m, t)$ denotes the transition probability at jump times m from state $X(0) = i$ to state $X(t) = j$. $\Psi := \Psi(i, j)$ represents transition rate, and Λ denotes $I \times I$ diagonal matrix with diagonal elements λ_i .

Finally, by using Laplace inverse transform (2.1) and the unique solution (2.2), we obtain the joint distribution of X and $\Phi(t)$ at time t when

$$P(m, t) = \frac{\partial^m}{\partial z^m m!} P^*(z, t) \Big|_{z=0}.$$

We assume that the jump rate under different status is unknown and then to consider the hidden switch Poisson process. Let the transition rate under hidden switch Poisson process is given by

$$\Psi = \begin{bmatrix} -\alpha_1 & \alpha_1 \\ \alpha_2 & -\alpha_2 \end{bmatrix}$$

and the jump rate under hidden switch Poisson process is

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

By using the unique solution (2.2) with $z = 1$, we have $P^*(1, t) = e^{\Psi(t)}$. Hence,

$$P^*(1, t) = \frac{1}{\alpha_1 + \alpha_2} \begin{bmatrix} \alpha_2 + \alpha_1 e^{-(\alpha_1 + \alpha_2)t} & \alpha_1 - \alpha_1 e^{-(\alpha_1 + \alpha_2)t} \\ \alpha_2 - \alpha_2 e^{-(\alpha_1 + \alpha_2)t} & \alpha_1 + \alpha_2 e^{-(\alpha_1 + \alpha_2)t} \end{bmatrix}.$$

It can be easy to get the limiting distribution as

$$\pi_1 = \lim_{t \rightarrow \infty} P_{i1}(1, t) = \frac{\alpha_2}{\alpha_1 + \alpha_2},$$

$$\pi_2 = \lim_{t \rightarrow \infty} P_{i2}(1, t) = \frac{\alpha_1}{\alpha_1 + \alpha_2},$$

where π_1 and π_2 denote the initial probability that the jump rate stays in state 1 and state 2, respectively.

Let $Q(m, t) = (\kappa + 1)^m \exp(-\Lambda \kappa t) P(m, t)$,

$$\begin{aligned} \text{thus } Q^*(z, t) &= \sum_{m=0}^{\infty} Q(m, t) z^m = \sum_{m=0}^{\infty} (\kappa + 1)^m \exp(-\Lambda \kappa t) P(m, t) z^m \\ &= \sum_{m=0}^{\infty} (z(\kappa + 1))^m \exp(-\Lambda \kappa t) P(m, t), \end{aligned}$$

and the unique solution of $Q^*(z, t)$ is

$$Q^*(z, t) = e^{\left[\Psi - (1-z)(\kappa+1)\Lambda \right] t} e^{-\Lambda \kappa t} = e^{\left[\Psi - ((1-z)(\kappa+1)\Lambda) \right] t}.$$

Therefore, under the original Markov modulated Poisson process $\Phi(t)$, the original transition probability is $P(m, t)$, with transition rate Ψ and $I \times I$ diagonal matrix Λ with diagonal elements λ_i . Through the change of measure, the risk neutral transition probability becomes $Q(m, t)$ with transition rate Ψ and $I \times I$ diagonal matrix $(\kappa + 1)\Lambda$ with diagonal elements λ_i .

2. Total Insured Loss process

Cox and Schwebach (1992) assume that the aggregate insured losses follow lognormal distribution and then derive the closed-form solution of catastrophe futures call option. The pricing formula is similar to Black's (1976) formula for pricing futures options, which assumes that the underlying futures price stands for a pure diffusion. However, option pricing based on a compound Poisson process has been well-developed in insurance literature. Cummins and Geman (1993) model the increments of the loss index as a geometric Brownian motion plus a jump process that is assumed to be a Poisson process with fixed loss sizes. Chang, Chang, and Yu (1996) use a randomized operational time approach to price European

futures options. The randomized operational time approach concept in probability theory indicates that a simple change of time scale will frequently reduce a general nonstationary process to a more tractable stationary one. More precisely, the time change transforms a general nonstationary process in the usual calendar-time scale to a stationary counterpart in an operational-time scale. They assume that parent process of the futures return is a lognormal diffusion and the directing process is a homogeneous Poisson and thus derive the option pricing formula as a risk-neutral Poisson sum of Black's prices. The extent of the usefulness of this pricing equation is determined only by a careful empirical evaluation using market data. Unfortunately, this approach is today inappropriate in the current situation since the futures contract stopped trading in 1995 and no "transaction time" can be derived from them in order to price options. Geman and Yor (1997) represent the dynamics of it directly modeled as a geometric Brownian motion plus a Poisson process with constant jump sizes and then derive the PCS call price. Aase (1995, 1999) takes a different modeling approach and uses a compound Poisson process with random jump sizes to describe the dynamics of the loss index. However, for catastrophic events, the assumption that resulting claims occur in terms of the Poisson process is inadequate as it has deterministic intensity. Dassios and Jang (2003) use the Cox process to model the arrival process for catastrophic events and price catastrophe insurance derivatives under constant interest rate.

On the other hand, only a few articles focus on the pricing of CAT bonds. Lee and Yu (2002) use an arbitrage-free framework to price catastrophe bonds, and assume that the catastrophe aggregate loss follows the compound Poisson process and then compute risk-free and default-risky CAT under the consideration of default risk, basis risk, and moral hazard using Monte Carlo method. Vaugirard (2003 a, 2003b) extends and adapts the jump diffusion model of Merton (1976) to develop a CAT bond valuation framework that allows for catastrophic events, interest rate uncertainty, and non-traded underlying state variables.

Therefore, when pricing catastrophe insurance products, most prior studies assume the arrival process of catastrophe event follows Poisson process with constant arrival rate. However, figure 1 displays that the arrival rate of catastrophe events could be significant difference at different situation. To capture this phenomenon, we extend the modeling frameworks of Cummins and Geman (1993), Geman and Yor (1997) and Vaugirard (2003a, 2003b) to assume that total insured loss index is driven by a Markov jump diffusion process, instead of Poisson jump diffusion process. Under Markov modulated Poisson process, the underlying state is governed by a homogeneous Markov chain. For simplicity, we assume the

interest rate is constant, thus under the risk neutral probability measure Q , the dynamic process of total insured loss can be written as:

$$L(t) = L(0) \exp \left\{ \left(r - \frac{1}{2} \sigma_L^2 \right) t + \sigma_L W^Q_t + \sum_{n=1}^{\Phi(t)} \ln Y_n - \Lambda \kappa t \right\}, \quad (2.3)$$

where $L(0)$ is the initial total insured loss, r is the drift parameter, and σ_L is the constant volatility of the Brownian component of the process. $\{W^Q(t):t>0\}$ is a standard Brownian motion and $\{Y_n:n=1,2,\dots\}$ are independent for the sequence and identically distributed nonnegative random variables representing the size of the n -th loss. $\{\Phi(t):t>0\}$ is a Markov modulated Poisson process under risk-neutral measure with arrival rate of catastrophe events $\Lambda(\kappa+1)$, where $\kappa \equiv E(Y_n - 1)$. The last term $\Lambda \kappa t$ in equation (2.3) is the compensator for the Markov modulated jump process under the risk-neutral measure. We assume that $\kappa < \infty$, which implies that the means of the jump sizes are finite for the total losses of the insured. In addition, all three sources of randomness, $W(t)$ standard Brownian motion, $\Phi(t)$ Markov modulated Poisson process and Y_n the jump size, are assumed to be independent.

Changes in the total insured loss comprise three components: the expected instantaneous total insured loss change conditional on no occurrences of catastrophes, the unanticipated instantaneous total insured loss change, which is the reflection of causes that have a marginal impact on the gauge, and the instantaneous change due to the arrival of the catastrophe. The total insured loss $L(t)$ follows the geometric Brownian motion during the time period $(0, t]$ given that no information of catastrophe event arrivals during the time period. When the information of catastrophe event arrivals at time t , the total insured loss changes instantaneously from $L(t)$ to $Y_n L(t_-)$. Various statistical distributions are used in actuarial models of insurance claims processes to describe the jump size of total insured loss, Y_n , such as lognormal, gamma, Pareto distribution. This paper follows prior studies, such as Chang, Chang, and Yu (1996), and Vaugirard (2003 a, 2003b), to adapt the lognormal distribution.

Note that for valuation purpose, we need to know the loss dynamic under the risk neutral probability measure. When the loss process has jumps, the market becomes incomplete, and then there is no unique pricing measure. Hence we follow Merton (1976) and suppose that investors acknowledge that natural catastrophe shock is idiosyncratic risk when it comes to

pricing contingent claims. The rationale underlying this stance is that natural catastrophes such as hurricanes and earthquakes are barely correlated to financial storms, which is supported for example by the empirical study of Hoyt and McCullough (1999). Therefore, CAT insurance products provide a valuable tool of diversification for investors because catastrophic losses are “zero-beta events” in the sense of the Capital Asset Pricing Model, as emphasized for example by Litzenberger, Beaglehole, and Reynolds (1996) or Canter, Cole, and Sandor (1997). By assuming that such the jump risk is nonsystematic and diversifiable, attaching a risk premium to the risk is unnecessary.

III. Pricing Catastrophe Insurance Product

1. European Catastrophe Call Futures Option Pricing

The structure of European Catastrophe Call Futures Option

In the early 1990s, CBOT launched catastrophe futures contracts and catastrophe futures call option, with contract values linked to the loss ratios compiled by PCS. These two catastrophe derivatives provide underwriters and risk managers an effective alternative to hedge and trade catastrophic losses. Unlike reinsurance negotiations that are costly, lengthy, and irreversible, these two catastrophe derivative contracts have the advantages of reversibility, transactional efficiency, and anonymity due to the standardized and exchange-traded properties. Besides, because catastrophic losses show no correlation with the price movements of stocks and bonds, underwriters and mainstream institutional investors may also enjoy the diversification effect of these derivatives

These two catastrophe derivatives trade on a quarterly basis: that is Jan-Mar, Apr-June, July-Sep and Oct-Dec. The CBOT devised a loss ratio index as the underlying instrument for catastrophe insurance futures and options contracts. The loss ratio index is the reported losses incurred in a given quarter and reported by the end of the following quarter divided by one fourth of the premiums received in the previous year. Further, it is likely that a cap is needed to limit the credit risk in the case of unusually large losses. However, to date there has not been an incident where the maximum loss ratio has been reached, thus we ignore the maximum loss ratio at 200%. Then the value of an insurance future at time T , $F(T)$, is the nominal contract value, US\$25,000, times the loss ratio index. The value of a catastrophe insurance call option on the future of the option, $C(T)$, at time T is given by

$$C(T) = \max(F(T, T) - K, 0) \quad (3.1)$$

where K denotes the pre-determined horizon exercise price.

From the equation (3.1), we know that the catastrophe call futures call option is much like a regular European futures call option. This right is only exercisable when the catastrophe futures price exceeds a constant critical value during the life time of the option; otherwise the call value is 0.

Pricing of European Catastrophe Call Futures Option

Under the risk-neutral measure Q , the value of the catastrophe call futures option can be obtained by discounted expectations. For simplicity, we assume the interest rate is constant. Thus the price of the European catastrophe call futures option at time t , $C(t)$, can be given as follows:

$$C(t) = e^{-r(T-t)} E^Q \left[(F(T) - K)^+ \right],$$

where r represents the risk-free interest rate.

To derive the pricing formula of European catastrophe call futures option, we assume that the size of the n -th loss, Y_n , follows a lognormal distribution with mean θ and variance δ^2 , and then apply the classical martingale method under risk neutral measure. A detailed proof is sketched in Appendix A, thus the formula of the European catastrophe call futures option can be obtained as Theorem 1:

Theorem 1 : The value of European catastrophe call futures option is

$$C(t) = \sum_{m=0}^{\infty} Q(m, T-t) \exp(-r_m(T-t)) \left[F'(t) \phi(d_{1m}^F) - K \phi(d_{2m}^F) \right] \quad (3.2)$$

where

$$(I \times I) - \text{matrix } d_{1,2m}^F = \frac{\ln[F(t)/K] \pm \frac{1}{2} \sigma_L^2 (T-t) + m\theta - \Lambda \kappa (T-t)}{\sqrt{\sigma_L^2 (T-t) + m \delta^2}},$$

$$F'(t) = F(t) \left(\exp \left[(-\Lambda \kappa)(T-t) + m\theta + \frac{1}{2} m \delta^2 \right] \right)^{-1}, \quad r_m = (r - \Lambda \kappa) + \frac{m\theta + m\delta^2/2}{(T-t)},$$

$\phi(\cdot)$ denotes the cumulative distribution function for a standard normal random variable.

This pricing formula can be viewed as the transition probability multiplied by Black's prices with jump component. The jump rate of catastrophe events depends on the different status. For example, if the jump rate of catastrophe events follows the switched jump

diffusion model, which is larger in one state and smaller in other status. Since a jump process gives rise to fatter tails for the underlying asset return distribution, it results in larger transition probability and volatility of jump size and then makes the value of the European catastrophe call futures option increase.

Note that when $\lambda_1 = \lambda_2 = \dots = \lambda_I = \lambda$, the Markov modulated Poisson reduces to the Poisson process with intensity λ . Hence, the pricing formula (3.2) can be rewritten as below:

$$C(t) = \sum_{m=0}^{\infty} \frac{e^{-\lambda(\kappa+1)(T-t)} (\lambda(\kappa+1)(T-t))^m}{m!} \exp(-r_m(T-t)) \left[F'(t) \phi(d_{1m}^F) - K \phi(d_{2m}^F) \right], \quad (3.3)$$

$$\text{where } d_{1,2m}^F = \frac{\ln[F(t)/K] \pm \frac{1}{2} \sigma_L^2 (T-t) + m\theta - \lambda\kappa(T-t)}{\sqrt{\sigma_L^2 (T-t) + m\delta^2}}, r_m = (r - \lambda\kappa) + \frac{m\theta + m\delta^2/2}{(T-t)}.$$

This implies that the expected number of jumps of catastrophe events per time unit is driven by the Poisson process. Chang, Chang, and Yu (1996) assume that the catastrophe futures price change follows a jump subordinated process in calendar-time. A subordinated process is obtained by random the time clock of the stationary process, called the parent process, using a new time clock, called the directing process or subordinator. Therefore, by changing the time scale, the subordinated process is transformed to a stationary process. Besides, the parent process of the futures return is a lognormal diffusion and the directing process is a homogeneous Poisson. If no information of catastrophe events arrives, the futures price stay put. On the other hand, if information of catastrophe events arrives, then futures price jumps instantaneously according to the lognormal distribution. Although our model setting is different with Chang, Chang, and Yu (1996), our pricing formula is similar to Chang, Chang, and Yu (1996) which is the Poisson sum of Black's prices.

If $\lambda_1 = \lambda_2 = \dots = \lambda_I = 0$, and total insured loss follows lognormal distribution, then European catastrophe call futures option price can be given by:

$$C(t) = \exp(-r(T-t)) \left[F(t) \phi(d_1^F) - K \phi(d_2^F) \right],$$

$$\text{where } d_{1,2}^F = \frac{\ln[F(t)/K] \pm \frac{1}{2} \sigma_L^2 (T-t)}{\sqrt{\sigma_L^2 (T-t)}}.$$

Hence, our pricing formula equation (3.2) will reduce to the pricing formula of Cox and Schwebach (1992), which is similar to Black's (1976) formula for pricing futures options. The diffusion assumption ignores the sporadic nature of catastrophes and jump in the number of claims.

Dynamic Hedging of European catastrophe call futures option

When the investors provide European catastrophe call futures option, they will simultaneously hedge their position to avoid taking on huge losses. Hence, this section will illustrate how to hedge against moderate changes in the total insured loss. In complete markets, Delta–Gamma hedging techniques will be used to measure the sensitivity of the option’s price to total insured loss movements at the first and second order. Hence, by partial differential of the equation (3.2), we have:

$$\Delta_F(t) = \frac{\partial}{\partial F(t)} C(t) = \sum_{m=0}^{\infty} Q(m, T-t) \exp(-r(T-t)) \phi(d_{1m}^F),$$

$$\Gamma_F(t) = \frac{\partial^2}{\partial F(t)^2} C(t) = \sum_{m=0}^{\infty} Q(m, T-t) \exp(-r(T-t)) \left[\frac{\phi'(d_{1m}^F)}{L(t) \sigma_L \sqrt{T-t}} \right].$$

where $\phi'(d_{1m}^F) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(d_{1m}^F)^2}{2}\right]$ is standard normal distribution function.

Since the underlying status follows Markov chain and different arrival rate of catastrophe events depends on different status under MMPP, our hedging formula are the sum of transition probability multiplied by hedging formula of Black’s price adding jump component. If no catastrophe occurs, our hedging formulas reduce to hedging formula of Black’s price.

2. European PCS Catastrophe Call Spread Pricing

The structure of European PCS Catastrophe Call Spread

PCS catastrophe insurance options were introduced at the CBOT in September 1995. They are standardized, exchange-traded contracts that are based on catastrophe loss indices provided daily by PCS. The PCS indices reflect estimated insured industry losses for catastrophes that occur over a specific period. Only cash options on these indices are available; no physical entity underlies the contracts. They can be traded as calls, puts, or spreads. Most of the trading activity occurs in call spreads, since they essentially work like aggregate excess-of-loss reinsurance agreements or layers of reinsurance that provide limited risk profiles to both the buyer and seller.

These European cash options are also quoted in percentage points and tenths of point, but each point equals \$200 cash value. The CBOT lists PCS options both as “small cap” contracts, which limit the amount of aggregate industry losses that can be included under the contract to \$20 billion, and as “large cap” contracts, which track losses from \$20 billion to \$50 billion. Based on the PCS loss index, the cash value of the PCS call spread at maturity

T , $C_L(T)$, is:

$$C_L(T) = \begin{cases} (K - A), & K \leq L(T) < \infty \\ (L(T) - A), & A \leq L(T) < K \\ 0, & 0 \leq L(T) < A \end{cases}$$

$$= (L(T) - A)1_{\{L(T) > A\}} - (L(T) - K)1_{\{L(T) > K\}}$$

where $L(T)$ presents the relevant PCS loss index, K and A represent the cap point and strike point of the call, with $K = 200$ and $0 < A \leq 200$ for a small cap and $K = 500$ and $200 \leq A \leq 500$ for a large cap. $1_{\{\cdot\}}$ denotes the indicator function.

Pricing of European PCS Catastrophe Call Spread

Under the risk-neutral measure Q , the price of the PCS catastrophe call spread at time t is the discounted expected value of $C_L(T)$. For simplicity, we assume the interest rate is constant. Thus the price of the European PCS catastrophe call spread at time t , $C_L(t)$, can be given as follows:

$$C_L(t) = e^{-r(T-t)} E^Q \left[(L(T) - A)1_{\{L(T) > A\}} - (L(T) - K)1_{\{L(T) > K\}} \right],$$

where r represents the risk-free interest rate.

To derive the pricing formula of European PCS catastrophe call spread, we follow Chang, Chang, and Yu (1996) to assume that the size of the n -th loss, Y_n , follows a lognormal distribution with mean θ and variance δ^2 and then apply the classical martingale method under risk neutral measure. The pricing process is similar to the European future call option thus the formula of the European PCS catastrophe call spread can be obtained as Theorem 2:

Theorem 2 : The value of European PCS catastrophe call spread is

$$C_L(t) = \sum_{m=0}^{\infty} Q(m, T-t) \left(\left[L(t)\phi(d_{1m}^L) - A \exp(-r_m(T-t))\phi(d_{2m}^L) \right] - \left[L(t)\phi(d_{3m}^L) - K \exp(-r_m(T-t))\phi(d_{4m}^L) \right] \right) \quad (3.4)$$

where

$$(I \times I) - \text{matrix } d_{1,2m}^L = \frac{\ln[L(t)/A] + (r \pm \frac{1}{2}\sigma_L^2)(T-t) + m\theta - \Lambda\kappa(T-t)}{\sqrt{\sigma_L^2(T-t) + m\delta^2}},$$

$$(I \times I) - \text{matrix } d_{3,4m}^L = \frac{\ln[L(t)/K] + (r \pm \frac{1}{2} \sigma_L^2)(T-t) + m\theta - \Lambda\kappa(T-t)}{\sqrt{\sigma_L^2(T-t) + m\delta^2}}.$$

Note that when $\lambda_1 = \lambda_2 = \dots = \lambda_I = \lambda$, the Markov modulated Poisson reduces to the Poisson process with constant intensity λ . Hence, the equation (2.3) could reduce to the model of Cummins and Geman (1993) and Geman and Yor (1997) and the pricing formula equation (3.4) can be rewritten as below:

$$C_L(t) = \sum_{m=0}^{\infty} \frac{e^{-\lambda(\kappa+1)(T-t)} (\lambda(\kappa+1)(T-t))^m}{m!} \left(\left[L(t)\phi(d_{1m}^L) - A \exp(-r_m(T-t))\phi(d_{2m}^L) \right] - \left[L(t)\phi(d_{3m}^L) - K \exp(-r_m(T-t))\phi(d_{4m}^L) \right] \right), \quad (3.5)$$

where
$$d_{1,2m}^L = \frac{\ln[L(t)/A] + (r \pm \frac{1}{2} \sigma_L^2)(T-t) + m\theta - \lambda\kappa(T-t)}{\sqrt{\sigma_L^2(T-t) + m\delta^2}},$$

$$d_{3,4m}^L = \frac{\ln[L(t)/K] + (r \pm \frac{1}{2} \sigma_L^2)(T-t) + m\theta - \lambda\kappa(T-t)}{\sqrt{\sigma_L^2(T-t) + m\delta^2}}.$$

This pricing formula implies that the expected number of jumps of catastrophe events per time unit is driven by the Poisson process. In addition, this pricing formula is also similar to Chang, Chang, and Yu (1996) if the underlying is the catastrophe future price.

If $\lambda_1 = \lambda_2 = \dots = \lambda_I = 0$, then European PCS catastrophe call spread can be given by:

$$C_L(t) = \left(\left[L(t)\phi(d_1^L) - A \exp(-r(T-t))\phi(d_2^L) \right] - \left[L(t)\phi(d_3^L) - K \exp(-r(T-t))\phi(d_4^L) \right] \right). \quad (3.6)$$

Hence, the pricing formula equation (3.6) will become similar to traditional Black and Scholes's pricing formula for European call spread.

Dynamic Hedging of European PCS catastrophe call spread

This section will examine how to hedge against moderate changes in the total insured loss. In complete markets, Delta-Gamma hedging techniques will be used to measure the sensitivity of the price of European PCS catastrophe call spread to total insured loss movements at the first and second order. Hence, by partial differential of the equation (3.3), we have:

$$\Delta_L(t) = \frac{\partial}{\partial L(t)} C_L(t) = \sum_{m=0}^{\infty} Q(m, T-t) \left[\phi(d_{1m}^L) - \phi(d_{3m}^L) \right],$$

$$\Gamma_L(t) = \frac{\partial^2}{\partial L(t)^2} C_L(t) = \sum_{m=0}^{\infty} Q(m, T-t) \left[\frac{\phi'(d_{1m}^L)}{L(t)\sigma_L\sqrt{(T-t)}} - \frac{\phi'(d_{3m}^L)}{L(t)\sigma_L\sqrt{(T-t)}} \right].$$

Since our model considers the status follows Markov chain and the arrival rate of catastrophe events depends on status, our hedging formulas are the sum of transition probability multiplied by hedging formula of Black and Scholes adding jump component. If no catastrophe occurs, our hedging formulas reduce to hedging formula of Black and Scholes.

3. Default-Free Catastrophe Bonds

The structure of Default-Free Catastrophe Bonds

CAT bond is a liability hedged instrument for insurance companies. This security provisions have debt-forgiveness triggers whose generic design grants the interest payment and/or the return of principal forgiveness. Meanwhile, the debt forgiveness can be triggered by the insurer's actual losses or a composite index of insurer's losses during the specific period. Hence, under this structure, the insurance company could transfer the catastrophe risk to increase the ability to provide insurance protection.

The CAT bondholder accepts losing interest payments or a fraction of the principal if the loss index or the aggregate loss of issuing firm standing for natural risk or associated insurance claims hits a pre-specified threshold K . More specifically, if the loss index or the aggregate loss of issuing firm does not reach the threshold during a risk exposure period, the bondholder is paid the face amount. Otherwise, he receives the portion of the principal needed to be paid to bondholders. Consider the CAT discount bond whose payoffs, $PO_{i,T}$, at maturity date T are as follows:

$$PO_{i,T} = \begin{cases} aL & \text{if } L_{i,T} \leq K \\ \delta aL & \text{if } L_{i,T} > K \end{cases}$$

where $L_{i,T}$ is the aggregate loss of issuing firm i at maturity T . K is the trigger level in the CAT bond provisions and δ is the portion of the principal needed to be paid to bondholders if the forgiveness trigger has been pulled. L is the face amount of the issuing firm's total debts. a is the ratio of CAT bond's face amount to total outstanding debts.

Pricing of Default-Free Catastrophe Bonds

According to the payoff of CAT bonds, the price of discount default-free CAT bonds at time t , $P_i(t)$, can be valued as follows:

$$P_i(t) = e^{-r(T-t)} E^Q \left[a L 1_{\{L_{i,T} \leq K\}} \right] + e^{-r(T-t)} E^Q \left[\eta a L 1_{\{L_{i,T} > K\}} \right] \quad (3.7)$$

This paper follows Vaugirard (2003 a, 2003b) to adapt the lognormal distribution with mean θ and variance δ^2 , then equation (3.7) can be rewritten as follows:

Theorem 3 : The value of discount default-free CAT bonds is

$$P_i(t) = e^{-r(T-t)} \left[\sum_{m=0}^{\infty} P(m, T-t) \phi(-d_{4m}^L) + \eta \left(1 - \sum_{m=0}^{\infty} P(m, T-t) \phi(-d_{4m}^L) \right) \right] a L \quad (3.8)$$

If $\lambda_1 = \lambda_2 = \dots \lambda_I = \lambda$, then new Markov modulated Poisson process $\Phi(t)$ simplifies to new Poisson process $N(t)$, i.e., total insured loss index could reduce to the model of Vaugirard (2003a, 2003b), which is driven by a Poisson jump diffusion process. Thus the equation (3.8) becomes:

$$P_i(t) = e^{-r(T-t)} \left[\sum_{m=0}^{\infty} \frac{e^{-\lambda(T-t)} (\lambda(T-t))^m}{m!} \phi(-d_{4m}^L) + \eta \left(1 - \sum_{m=0}^{\infty} \frac{e^{-\lambda(T-t)} (\lambda(T-t))^m}{m!} \phi(-d_{4m}^L) \right) \right] a L .$$

If $\lambda_1 = \lambda_2 = \dots \lambda_I = 0$, then the equation (3.8) reduces to

$$P_i(t) = e^{-r(T-t)} a L \left[\phi(-d_4^L) + \eta (1 - \phi(-d_4^L)) \right].$$

This pricing formula is similar to Louberge, Kellezi, and Gilli (1999) under the assumption that the catastrophe loss index or the aggregate loss of issuing firm stands for geometric Brownian motion process.

Dynamic Hedging of CAT bonds

Similar to European futures call and European PCS catastrophe call spread, we also provide the hedging strategy of CAT bonds. In complete markets, Delta–Gamma hedging techniques will be used to estimate the sensitivity of the CAT bond's price to total insured loss movements at the first and second order. Hence, by partial differential of the equation (3.8), we have:

$$\Delta_L(t) = \frac{\partial}{\partial L(t)} C_L(t) = e^{-r(T-t)} a L \sum_{m=0}^{\infty} P(m, T-t) \left[-\frac{\phi'(d_{4m}^L)}{L(t) \sigma_L \sqrt{(T-t)}} + \eta \frac{\phi'(d_{4m}^L)}{L(t) \sigma_L \sqrt{(T-t)}} \right],$$

$$\Gamma_L(t) = \frac{\partial^2}{\partial L(t)^2} C_L(t) = e^{-r(T-t)} a L \sum_{m=0}^{\infty} P(m, T-t) \left[\frac{\phi'(d_{4m}^L)}{L(t)^2 \sigma_L \sqrt{(T-t)}} - \eta \frac{\phi'(d_{4m}^L)}{L(t)^2 \sigma_L \sqrt{(T-t)}} \right].$$

IV. Numerical Analysis of Catastrophe Insurance Products

This section examines the sensitivity analysis for catastrophe insurance products, including European futures call, European PCS catastrophe call spread, and CAT bonds. The values of catastrophe insurance products depend on several parameters in a Markov jump diffusion model when the jump size is assumed to follow the lognormal distribution. We will establish a set of parameters and base value to evaluate catastrophe insurance products in Markov jump diffusion model and compare it to jump diffusion model formula.

To demonstrate how the European futures call is related to various parameters, we make some assumptions as follows: futures price, $F = 40$; exercise price, $K = 80$; interest rate, $r = 0.05$; catastrophe loss volatility, $\sigma_L = 0.4$; option term, $T = 0.25$ $T = 0.5$; truncation, $n = 8$; transition rate leaving state 1 and state 2, $\alpha_1 = 1$, $\alpha_2 = 1$. Table 1 report the value of European futures calls as a function of jump rate of catastrophe events, the standard deviation of jump size and mean of jump size under Markov jump diffusion model and jump diffusion model. For the parameters of jump rate of catastrophe events, we consider two jump rate: 1, and 3, to examine the jump rate effect. When jump rate of catastrophe events increases, this causes the volatility of European futures call increases, the Poisson probability increases, and exercise time early. Thus higher jump rate of catastrophe events, higher value of European futures call. In the other hand, compared with MJ(1,3), we find that Poi(1) underestimates MJ(1,3) and Poi(3) overestimates MJ(1,3). Hence, if different economy status has significant different jump arrival intensity in real economy, using jump diffusion model to evaluate the European futures calls could cause significant mispricing.

For the concern of parameters of the standard deviation of jump size and mean of jump size, due to same reason with jump rate of catastrophe events, this table also reveals that higher the standard deviation of jump size and mean of jump size due to catastrophe occurrences result in higher the European Futures Call price.

Next, we examine the sensitivity analysis for European PCS catastrophe small cap, thus

the cap point (K) is 200. We compare numerically European PCS catastrophe call spread values obtained from equation (3.4) to equation (3.5) in jump diffusion model. We assume the PCS loss index (L) is 40 and the riskless rate (r) is 0.05. Option term (T) is 0.5 and truncation (n) is 8. The mean of jump size (θ) is 0.0001 and the strike point (A) is 20, 40, and 80, respectively, to represent deep-in the-money, at-the-money, and deep-out-of-the-money, respectively. Assume the jump rate of catastrophe loss at state 1 or state 2 are set to be 1 and 3, respectively, to reflect the frequencies of catastrophe events per year. The transition rate of these two states is 1 and 1, respectively, to capture the leaving length for jump rate at different state. We also report volatility of jump size ranges from 0.1 year to 0.2 in order to examine the volatility effect.

Table 2 reports the value of European PCS catastrophe call spread as a function of moneyness, jump rate, and standard deviation of jump size. For out-the-money and at-the-money options, the results are also consistent with the prediction that the European PCS catastrophe call spread experiences as single call option. Hence, when jump rate of catastrophe events increases or standard deviation of jump size increases, the value of European PCS catastrophe call spread increases. However, for in-the-money options, because catastrophe event happens a lot or makes huge loss, when jump rate of catastrophe events increases, the probability that PCS loss index exceed cap point higher and thus European PCS catastrophe call spread decreases. By the same taken, for in-the-money options, we could find that higher standard deviation of jump size, lower European PCS catastrophe call spread. In addition, compared with MJ(1,3), the results are also consistent with the prediction that no matter what case of moneyness, Poi(1) underprices MJ(1,3) and Poi(3) overprices MJ(1,3).

Finally, for the numerical analysis of CAT bond, assume that the total amount of the issuing firm's debts, which include CAT bonds, has a face value of \$100. The strike value (K) is set to be 100, and the loss amount ($L_{i,T}$) is set at 100. The interest rate (r) is 0.05 and the mean of jump size (θ) is 1. Truncation (n) is set at 8. The portion of principal needed to be repaid, (η) is set at 0.5 when debt forgiveness has been triggered. The ratio of the amount of CAT bonds to the insurer's total debt (a) is set to be 0.1. We report standard deviation of jump size ranges from 0.2 to 0.5 to demonstrate the volatility effect. We also report maturity ranges from 0.5 year to 1 year in order to examine the maturity effect. In Markov jump diffusion model, assume the jump rate of catastrophe loss at state 1 or state 2 are set to be 1 and 3, respectively, and the transition rate of these two states is 1 and 1, respectively.

Table3 reports the CAT bond prices under alternative sets of occurrence jump rate, maturity, and standard deviation of jump size. As jump rate rises up, the probability that loss amount exceed strike value increase, then according to the payoff of CAT bond, we can obtain CAT bond price decreasing. In the other hand, for volatility effect, due to the similar phenomenon with jump rate, we also find that higher standard deviation of jump size due to catastrophe occurrences results in lower CAT bond price. For maturity effect, longer maturity leads to higher discount rate, and thereby lower CAT bond price.

V. Conclusions

We propose a Markov jump diffusion model, which extends the Poisson process used in the jump diffusion model to be Markov modulated Poisson process. Markov modulated Poisson process stands for a doubly stochastic Poisson process where the underlying state is governed by a homogenous Markov chain to model the arrival process for catastrophic events. It is also the generalization of Cummins and Geman (1993), Chang, Chang, and Yu (1996) and Geman and Yor (1997). Further, we apply Markov jump diffusion model to derive pricing formulas and hedging strategy of catastrophe insurance products: European futures call option, catastrophe PCS call spread, and CAT bond.

For numerical analysis, we evaluate catastrophe insurance products in Markov jump diffusion model and compare it to jump diffusion model formula. Compared with MJ(1,3), we find that MJ(1,3) is between Poi(1) and Poi(3). Hence, if different economy status has significant different jump arrival intensity in real economy, using jump diffusion model to evaluate the European futures calls could cause significant mispricing.

Table 1 Value of European Futures Call

θ	Model	δ	
		0.2	0.5
0.01	Poi(1)	1.1772	13.4221
	Poi(3)	2.1449	19.0494
	MJ(1,3)	1.6636	16.2337
0.05	Poi(1)	1.5842	13.9698
	Poi(3)	2.7353	19.4478
	MJ(1,3)	2.1672	16.7014

This table reports the value of European Futures Call as a function of transition rate, jump rate and mean of jump size. The parameters are $K = 80$, $F = 40$, $r = 0.05$, $\sigma_L = 0.4$, $T = 0.25$, $n = 8$. Poi(1) and Poi(3) denote the value with annual jump rate 1 and 3, respectively in jump diffusion model. MJ(1,3) denote the value with annual jump rate is 1 at state 1 and 3 at state 2 in Markov jump diffusion model.

Table 2 Value of European PCS Call Spread

L/A	Model	δ	
		0.1	0.2
0.5	Poi(1)	1.4691	4.9064
	Poi(3)	2.1122	7.4517
	MJ(1,3)	1.7922	6.1994
1	Poi(1)	16.2490	20.5582
	Poi(3)	19.7601	25.6922
	MJ(1,3)	18.0115	23.1417
2	Poi(1)	40.9767	38.6298
	Poi(3)	34.5255	32.6823
	MJ(1,3)	37.7589	35.6645

This table reports the value of European PCS Call Spread as a function of transition rate of two states, jump rate and mean of jump size. The parameters are $K = 200$, $L = 40$, $r = 0.05$, $\sigma_L = 0.4$, $T = 0.5$, $n = 8$. Poi(1) and Poi(3) denote the value with annual jump rate 1 and 3, respectively in jump diffusion model. MJ(1,3) denote the value with annual jump rate is 1 at state 1 and 3 at state 2 in Markov jump diffusion model.

Table 3 Value of Catastrophe Bonds

T	Model	δ	
		0.2	0.5
0.5	Poi(1)	95.5964	94.9359
	Poi(3)	94.8436	94.1772
	MJ(1,3)	95.3860	94.7410
1	Poi(1)	92.7376	92.0991
	Poi(3)	91.9497	91.4653
	MJ(1,3)	92.8158	92.2112

This table reports the value of Catastrophe Bonds as a function of transition rate, jump rate and mean of jump size. The parameters are $K = 100, L_{i,T} = 100, r = 0.05, \sigma_L = 0.4, n = 8, L = 100, a = 0.1, \eta = 0.5$. Poi(1) and Poi(3) denote the value with annual jump rate 1 and 3, respectively in jump diffusion model. MJ(1,3) denote the value with annual jump rate is 1 at state 1 and 3 at state 2 in Markov jump diffusion model.

Endnotes

¹ The quarterly data comes from the Insurance Service Office (ISO). The PCS track insured catastrophe loss estimates on national, regional, and state basis in the US from information obtained by PCS. The term “natural catastrophe” includes hurricanes, storms, floods, waves, and earthquakes. The event of “natural catastrophe” denotes a natural disaster that affects many insurers and when claims are expected to reach a certain dollar threshold. Initially the threshold was set to \$5 million. In 1997, ISO increased its dollar threshold to \$25 million.

Appendix A

The payoff of call option on a future contract under risk neutral measure Q is

$$\begin{aligned} C(t) &= e^{-r(T-t)} E^Q \left[(F(T) - K)^+ \right] \\ &= e^{-r(T-t)} E^Q \left[(L(T) - K) 1_{\{L(T) > K\}} \right] \\ &= e^{-r(T-t)} \left(E^Q \left[L(T) 1_{\{L(T) > K\}} \right] - E^Q \left[K 1_{\{L(T) > K\}} \right] \right) = e^{-r(T-t)} (A - B) \end{aligned}$$

where

$$\begin{aligned} A &= E^Q \left[L(T) 1_{\{L(T) > K\}} \right] \\ &= E^Q \left[L(t) \exp \left\{ \left(r - \frac{1}{2} \sigma_L^2 \right) (T-t) + \sigma_L W^Q(T-t) \right\} \prod_{n=0}^{\Phi(T-t)} Y_n e^{-\Lambda \kappa (T-t)} 1_{\{L(T) > K\}} \right] \end{aligned} \quad (\text{A.1})$$

By Girsanov's theory,

$$W^R = W^Q - \int_t^T \sigma_L dt,$$

equation (A.1) can be rewritten as:

$$\begin{aligned} A &= F(t) E^R \left[\exp \left(\sum_{n=0}^{\Phi(T-t)} \ln Y_n - \Lambda \kappa (T-t) \right) 1_{\{L(T) > K\}} \right] \\ &= \sum_{m=0}^{\infty} P(m, T-t) F(t) E^R \left[\exp \left(\sum_{n=0}^m \ln Y_n - \Lambda \kappa (T-t) \right) 1_{\{L(T) > K\}} \middle| \Phi(T-t) = m \right] \end{aligned} \quad (\text{A.2})$$

By Radon-Nikodym derivative of transition probability and jump size,

$$\frac{dQ(m, t)}{dP(m, t)} = (\kappa + 1)^m \exp(-\Lambda \kappa t),$$

$$dQ(Y_1^Q, Y_2^Q, \dots, Y_m^Q) = \frac{y^m f_L^m(y)}{(\kappa + 1)^m},$$

the equation (A.2) can be obtained:

$$\sum_{m=0}^{\infty} Q(m, T-t) F(t) E^R \left[\mathbf{1}_{\left\{ L(t) \exp\left\{ \frac{1}{2} \sigma_L^2 (T-t) + \sigma_L W^R(T-t) \right\} \prod_{n=0}^m Y_n^Q e^{-\Lambda \kappa (T-t)} > K \right\}} \middle| \Phi(T-t) = m \right] \quad (\text{A.3})$$

Assume that Y_n stands for lognormal distribution with mean θ and variance δ^2 , i.e.,

$\ln Y_n \sim N(\theta, \delta^2)$ and then $\kappa + 1 = E(Y_n) = e^{\theta + \frac{1}{2} \delta^2}$. Since $dQ(Y_n^Q) = \frac{y f_Y(y)}{(\kappa + 1)}$, we have:

$$dQ(Y_n^Q) = \frac{1}{y \sqrt{2\pi\delta}} \exp\left[-\frac{(\ln y - (\theta + \delta^2))^2}{2\delta^2}\right]$$

Since Y_n is independent identically distributed nonnegative random variables, the new jump size under Q is

$$\sum_{n=1}^m \ln Y_n^Q = \sum_{n=1}^m \ln Y_n - m(\theta + \delta^2).$$

Hence, we have:

$$\sum_{m=0}^{\infty} Q(m, T-t) F(t) E^R \left[\mathbf{1}_{\left\{ F(t) \exp\left\{ \frac{1}{2} \sigma_L^2 (T-t) + \sigma_L W^R(T-t) + \sum_{n=1}^m \ln y_n^Q + m(\theta + \delta^2) - \Lambda \kappa (T-t) \right\} > K \right\}} \middle| \Phi(T-t) = m \right] \quad (\text{A.4})$$

Let $W^R(T-t) = \sqrt{T-t} Z$, where $Z \sim N(0,1)$. By the same taken, set $\sum_{n=1}^m \ln Y_n^Q = \delta \sqrt{m} Z_m^Q$,

hence, equation (A.4) can be described as:

$$\begin{aligned} A &= \sum_{m=0}^{\infty} Q(m, T-t) F(t) P^R \left[\sigma_L \sqrt{T-t} Z + \delta \sqrt{m} Z_m^Q > \ln\left(\frac{K}{F(t)}\right) - \frac{1}{2} \sigma_L^2 (T-t) \right. \\ &\quad \left. + \Lambda \kappa (T-t) - m\theta - m\delta^2 \middle| \Phi(T-t) = m \right] \\ &= \sum_{m=0}^{\infty} Q(m, T-t) F(t) P^R \left[Z < \frac{\ln\left(\frac{F(t)}{K}\right) + \frac{1}{2} \sigma_L^2 (T-t) - \Lambda \kappa (T-t) + m\theta + m\delta^2}{\sqrt{\sigma_L^2 (T-t) + \delta^2 m}} \middle| \Phi(T-t) = m \right] \\ &= \sum_{m=0}^{\infty} Q(m, T-t) F(t) P^R \left[Z < \frac{\ln\left(\frac{F(t)}{K}\right) + \left(-\Lambda \kappa + (m\theta + \frac{1}{2} m\delta^2) / (T-t)\right) (T-t) + \frac{1}{2} \left(\sigma_L^2 + \delta^2 m / T-t\right) (T-t)}{\sqrt{\left(\sigma_L^2 + \delta^2 m / T-t\right) T-t}} \middle| \Phi(T-t) = m \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} Q(m, T-t) F(t) P^R \left[Z < \frac{\ln\left(\frac{F(t)}{K}\right) + \eta_m(T-t) + \frac{1}{2}\sigma_m^2(T-t)}{\sqrt{\sigma_m^2(T-t)}} \middle| \Phi(T-t) = m \right] \\
&= \sum_{m=0}^{\infty} Q(m, T-t) F(t) N(d_{1m})
\end{aligned}$$

where

$$(I \times I) - \text{matrix } d_{1m} = \frac{\ln\left(\frac{F(t)}{K}\right) + \eta_m(T-t) + \frac{1}{2}\sigma_m^2(T-t)}{\sqrt{\sigma_m^2(T-t)}}, \quad \eta_m = -\Lambda\kappa + \frac{m\theta + \frac{1}{2}m\delta^2}{(T-t)},$$

$$\sigma_m^2 = \sigma_L^2 + \delta^2 m / (T-t).$$

Similarly,

$$\begin{aligned}
B &= E^Q \left[K 1_{\{L(T) > K\}} \right] \\
&= KP^Q \left[L(t) \exp \left\{ \left(r - \frac{1}{2}\sigma_L^2 \right) (T-t) + \sigma_L W^Q(T-t) \right\} \prod_{n=0}^{\Phi(t)} Y_n e^{-\Lambda\kappa(T-t)} > K \right] \\
&= \sum_{m=0}^{\infty} P(m, T-t) K e^{-rT} P^Q \left[L(t) \exp \left\{ \left(r - \frac{1}{2}\sigma_L^2 \right) (T-t) + \sigma_L W^Q(T-t) \right. \right. \\
&\quad \left. \left. + \sum_{n=0}^m \ln Y_n - \Lambda\kappa(T-t) \right\} > K \middle| \Phi(T-t) = m \right] \\
&= \sum_{m=0}^{\infty} P(m, T-t) K P^Q \left[\sigma_L \sqrt{T-t} Z + \delta \sqrt{m} Z_m > \ln \left(\frac{K}{F(t)} \right) + \frac{1}{2}\sigma_L^2(T-t) + \Lambda\kappa(T-t) - m\theta \middle| \Phi(T-t) = m \right] \\
&= \sum_{m=0}^{\infty} P(m, T-t) K P^Q \left[Z < \frac{\ln[F(t)/K] - \frac{1}{2}\sigma_L^2(T-t) - \Lambda\kappa(T-t) - m\theta}{\sqrt{\sigma_L^2(T-t) + \delta^2 m}} \middle| \Phi(T-t) = m \right] \tag{A.5}
\end{aligned}$$

Due to $Q(m, T-t) = (\kappa+1)^m e^{-\Lambda\kappa(T-t)} P(m, T-t)$, and $(\kappa+1)^m = [E(Y_n)]^m = \left[\exp\left(\theta + \frac{1}{2}\delta^2\right) \right]^m$,

thus equation (A.5) becomes

$$B = \sum_{m=0}^{\infty} K Q(m, T) \frac{\exp(-\Lambda\kappa)(T-t)}{\left[\exp\left(\theta + \frac{1}{2}\delta^2\right) \right]^m} P^Q \left[Z < \frac{\ln[F(t)/K] + \eta_m(T-t) - \frac{1}{2}\sigma_m^2(T-t)}{\sqrt{\sigma_m^2(T-t)}} \middle| \Phi(T-t) = m \right]$$

$$= \sum_{m=0}^{\infty} KQ(m, T) \frac{\exp(-\Lambda \kappa)(T-t)}{\left[\exp\left(\theta + \frac{1}{2} \delta^2\right) \right]^m} N(d_{2m})$$

where

$$(I \times I) \text{-matrix } d_{2m} = \frac{\ln[F(t)/K] + \eta_m(T-t) - \frac{1}{2} \sigma_m^2(T-t)}{\sqrt{\sigma_m^2(T-t)}},$$

$$\sigma_m^2 = \sigma_L^2 + \frac{\delta^2 m}{(T-t)}.$$

Therefore, the option price on future contract can be described as:

$$C(t) = \exp(-r_m(T-t)) \left[\sum_{m=0}^{\infty} Q(m, T-t) F^F(t) N(d_{1m}^F) - \sum_{m=0}^{\infty} KQ(m, T) N(d_{1m}^F) \right].$$

References

1. Aase, K., 1995, Catastrophe Insurance Futures Contracts. Norwegian School of Economics and Business Administration, Institute of Finance and Management Science, Working Paper.
2. Aase, K., 1999, An Equilibrium Model of Catastrophe Insurance Futures and Spreads. *Geneva Papers on Risk and Insurance Theory*, 24, 69-96.
3. Canter, M., J. B. Cole, and R. L. Sandor, 1997, Insurance derivatives: A New Asset Class for the Capital Markets and A New Hedging Tool for the Insurance Industry. *Journal of Applied Corporate Finance*, 10(3), 69–83.
4. Chang, C., J. Chang, and M.-T. Yu, 1996, Pricing Catastrophe Insurance Futures Call Spreads: A Randomized Operational Time Approach, *Journal of Risk and Insurance*, 63, 599-617.
5. Cox, S. H., and R. G. Schwebach, 1992, Insurance Futures and Hedging Insurance Price Risk, *Journal of Risk and Insurance*, 59, 628-644.
6. Cummins, J. D., and H. Geman, 1993, An Asian Option to the Valuation of Insurance Futures Contracts, Wharton School Center for Financial Institutions, Working Paper.
7. Cummins, J. D., and H. Geman, 1995, Pricing Catastrophe Futures and Call Spreads: An arbitrage Approach, *Journal of Fixed Income*, 4, 46-57.
8. Dassios, A. and J.-W. Jang, 2003, Pricing of Catastrophe Reinsurance and Derivatives Using the Cox Process with Shot Noise Intensity, *Financial Stochastic*, 7, 73-95.
9. Geman, H. and M. Yor, 1997, Stochastic time changes in catastrophe option pricing, *Insurance: Mathematics and Economics*, 21, 185-193,
10. Hoyt, R. E. and K. A. McCullough, 1999, Catastrophe Insurance Options: Are They Zero-Beta Assets ? *The Journal of Insurance Issues*, 22(2), 147–163.
11. Last, G., and A. Brandt, 1995, Marked Point Processes on the Real Line: The Dynamic Approach. Springer-Verlag, New York.
12. Lee, J.-P. and M.-T. Yu, 2002, Pricing Default-Risky CAT Bonds with Moral Hazard and Basis Risk, *Journal of Risk and Insurance*, 69, 25-44.
13. Litzenberger, R. H., D. R. Beaglehole, and C. E. Reynolds, 1996, Assessing Catastrophe Reinsurance-Linked Securities as a New Asset Class, *Journal of Portfolio Management*, 23, 76-86.
14. Louberge, H., E. Kellezi, and M. Gilli, 1999, Using Catastrophe-Linked Securities to Diversify Insurance Risk: A Financial Analysis of CAT Bonds, *Journal of Risk and*

- Insurance*, 22, 125-146.
15. Merton, R. C., 1976, Option Pricing when Underlying Stock Returns are Discontinuous, *Journal of Financial Economics*, 3, 125–44.
 16. Vaugirard V., 2003a, Valuing Catastrophe Bonds by Monte Carlo Simulations, *Applied Mathematical Finance*, 10, 75–90.
 17. Vaugirard, V., 2003b, Pricing Catastrophe Bonds by an Arbitrage. *The Quarterly Review of Economics and Finance Approach*, 43, 119-132.
 18. Zajdenweber, D., 1998, The Valuation of Catastrophe-Reinsurance-Linked Securities, Paper presented at American Risk and Insurance Association Meeting.