Catastrophe Equity Put in Markov Jump Diffusion Model

(Topic of paper: Pricing Risk)

Lin, Shih-Kuei, Assistant Professor
Department of Finance
National University of Kaohsiung
No. 700, Kaohsiung University Rd.,
Nan Tzu Dist., 811, Kaohsiung, Taiwan, R.O.C.
E-mail: square@nuk.edu.tw
Tel: 886-7-5919496
Fax: 886-7-5919329

Chang, Chia-Chien, Ph.D Student
Department of Finance
National Sun Yat-sen University
No. 70, Lien-hai Rd, 804, Kaohsiung, Taiwan, R.O.C.
E-mail: u7501247@yahoo.com.tw
Tel: 886-7-5252000
Fax: 886-7-5254899

Abstract

Most prior studies (c.f. Louberge, Kellezi, and Gilli (1999), Lee and Yu (2002), Vaugirard (2003), Cox et al. (2004) and Jaimungal and Wang (2006)) assume that the loss claim arrival process follows Poisson process when pricing the catastrophe insurance products. However, for catastrophic events, the assumption that resulting claims occur in terms of the Poisson process is inadequate as it has constant intensity. To overcome this shortcoming, this paper proposes Markov Modulated Poisson process (MMPP) where the underlying state is governed by a homogenous Markov chain to model the arrival process for catastrophic events. Next, we propose a generalization of Radon-Nikodym processes that a changing measure corresponds to a change of drift for the underlying Brownian motion and a change of the stochastic intensity for the Markov jump diffusion model. We derive pricing formulae and dynamic hedging for Catastrophe equity put options and our pricing formulae could be the generalization of Cox, Fairchild, and Pedersen (2004) and Jaimungal and Wang (2006).

Keywords: Markov Modulated Poisson process, Radon-Nikodym processes, Markov jump diffusion model, Catastrophe equity put options.

I. Introduction

Insurance companies have traditionally used reinsurance contracts to hedge themselves against losses from catastrophic events. However, the reinsurance industry is also limit in size relative to the magnitude of these damages, creating large fluctuations in the price and availability of reinsurance during years as multiple catastrophes occur. Because of worldwide capacity shortage and increasing number of catastrophe in recent years, in the early 1990s, certain catastrophe derivatives products like catastrophe futures, catastrophe option, and catastrophe futures option which trade in the Chicago Board of Trade (CBOT) provide underwriters and risk managers an effective alterative to hedge and trade catastrophic losses, especially for catastrophe option. Catastrophe options are based on catastrophe loss indices provided daily by Property Claim Services (PCS) – a US industry authority which estimates catastrophic property damage since 1949.

Catastrophe equity put options (CatEPuts) is also another innovation derivative for catastrophe risk management. The first CatEPut was issued on behalf of RLI Corporation in 1996, giving RLI the right to issue up to $50 million of cumulative convertible preferred shares. CatEPuts are a form of option that stock insurers can buy from investors. Those options give an insurance company the right to sell a specified amount of its stock to investors at a predetermined price if catastrophe losses surpass a specified trigger. A major advantage of CatEPuts is that they make equity funds available at a predetermined price when an organization needs them the most: immediately following a catastrophe. If the insurance company suffers a loss of capital due to a catastrophe, its stock price is likely to fall, lowering the amount it would receive for newly issued stock. Catastrophe equity puts provide instant equity at a predetermined price to help an insurance company regain its capital following a catastrophe. A disadvantage of CatEPuts is that they dilute ownership in the insurance company following a loss. The amount of equity increases when the put option is exercised, thereby reducing the existing shareholders' percentage of ownership.

Certain articles focus on the pricing of catastrophe insurance products. For example, Cox and Schwebach (1992), Cummins and Geman (1995), and Chang, Chang, and Yu (1996) price the CAT futures and CAT call spreads under deterministic interest rate and PCS loss process. Litzenberger, Beaglehole, and Reynolds (1996), Zajdenweber (1998) and Louberge, Kellezi, and Gilli (1999) price a zero-coupon CAT bond with hypothetical catastrophe loss distribution. Lee and Yu (2002) focus on the computation of default-risky CAT under the consideration of default risk, basis risk, and moral hazard using Monte Carlo method.

Vaugirard (2003a, 2003b) adapts the jump-diffusion model of Merton (1976) to develop a valuation framework that allows for catastrophic events, interest rate uncertainty, and non-traded underlying state variables. Dassios and Jang (2003) use the Cox process to model the claim arrival process for catastrophic events and then apply the model to the pricing of catastrophe insurance derivatives under constant interest rate.

When pricing catastrophe linked financial options, it is prudent to develop a model which depicts the joint dynamics of the share value process and losses. Cox et al. (2004) are the first to investigate such a model for pricing catastrophe linked financial options. They assumed that the asset price process is driven by a geometric Brownian motion with additional downward jumps of a specific size in the event of a catastrophe. Besides, their model assumes that only the event of a catastrophe affects the share value price, while the size of the catastrophe is irrelevant. Jaimungal and Wang (2005) generalize the results of Cox et al. (2004) to analyze the pricing and hedging of catastrophe put options under stochastic interest rates with losses generated by a compound Poisson process. Furthermore, asset prices are modeled through a jump-diffusion process which is correlated to the loss process.

Therefore, most pricing models for catastrophe insurance products assume that the loss claim arrival process follows Poisson process. However, for catastrophic events, the assumption that resulting claims occur in terms of the Poisson process is inadequate as it has deterministic intensity. This study intends to contribute to the literature by providing an alternative point process needs to be used to generate the arrival process. We will propose a new model, a Markov jump diffusion model, which extends the Poisson process used in the jump diffusion model to be Markov modulated Poisson process. Markov modulated Poisson process stands for a doubly stochastic Poisson process where the underlying state is governed by a homogenous Markov chain (cf. Last and Brandt, 1995). More precisely, instead of constant average jump rate in the years under the jump diffusion model, under Markov modulated Poisson process, the arrival rates of new information, good or bad news, are different from the abnormal vibrations of the loss dependent on current situation. The Markov jump diffusion model with two states, the so-called switched jump diffusion model, depends on the status of the economy. In switched jump diffusion model, the jump rates are different in different status, more precisely; the jump rates are large in one state and small in other status. Figure 1 and 2 show PCS loss and the number of CAT in the US from 1950 to 2004, respectively [1]. Figure 1 seems to reveal the smaller jump rate of natural catastrophes before 1990s year and larger jump rate of natural catastrophes after 1990s year. If the arrival process (jump rate) of natural catastrophes stands for Poisson process, then it could appear constant

average jump rate in the years. However, Figure 2 exhibits significantly the different jump rates at different year, especially after 1980s year. Therefore, we could infer that the arrival process (jump rate) of natural catastrophes could be different at different status and the Markov modulated Poisson process could be more fit than Poisson process to capture the arrival rate of natural catastrophes.

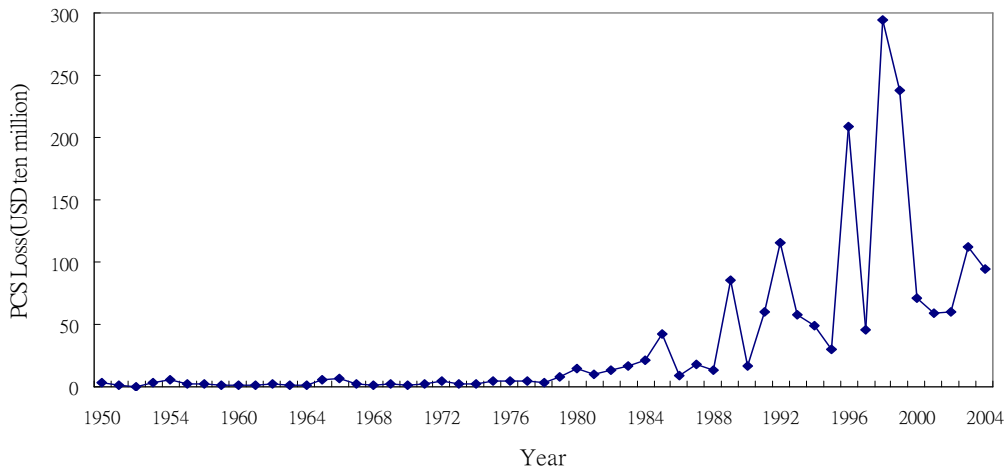

Figure 1: PCS loss in the United State during 1950 to 2004

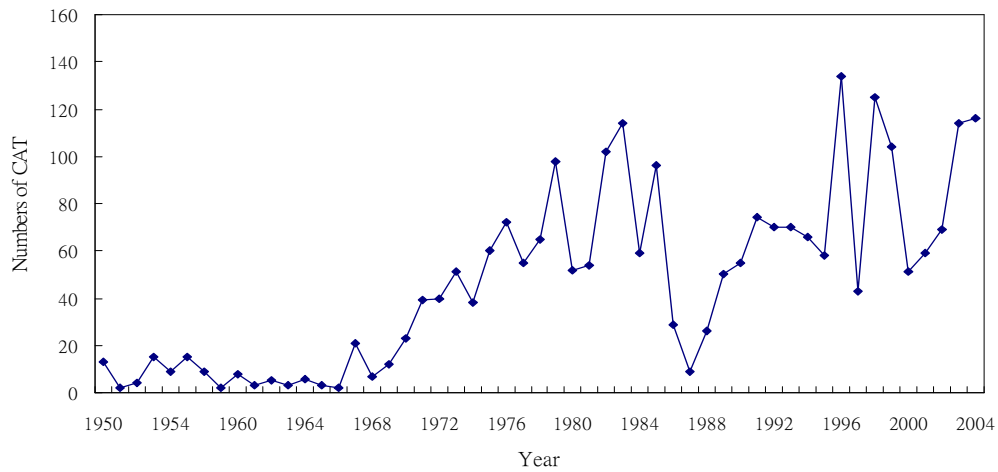

Figure 2: Number of CAT in the United State during 1950 to 2004

The contributions of this paper are follows: (1) To address the phenomenon that the arrival rate of natural catastrophes could be different at different status, we propose a more general Markov jump diffusion model to evaluate accurately the valuation of CatEPut with the constant interest rate assumption. (2) We propose a generalization of Radon-Nikodym processes that a changing measure corresponds to a change of drift for the underlying Brownian motion and a change of the stochastic intensity for the Markov jump diffusion

model. Though a change of measure from original probability measure to risk neutral probability measure, and we derive the close-form solutions of the CatEPut. When the total loss size follows specific function, our pricing formula can be reduced to the pricing formulas of Cox, Fairchild, and Pedersen (2004), and Jaimungal and Wang (2006). (3) When the jump size distribution follows the specific distribution, such as lognormal distribution, the pricing formulas and dynamic hedging of the CatEPut are proposed, respectively.

The remainder of the paper is organized as follows. Section 2 illustrations the model and then derive the pricing formula of CatEPut. Section 3 gives the dynamic hedging of the CatEPut. Section 4 is devoted to numerical analysis. Section 5 summarizes the article gives the conclusions. For ease of exposition, most proofs are in an appendix.

II. Pricing Catastrophe Equity Put Option (CatEPut)

1. The structure of Catastrophe Equity Put Option (CatEPut)

In general, the CatEPut option allows the owner to issue convertible (preferred) shares at a predetermined price, much like a regular put option; however, that right is only exercisable in the event that the accumulated losses, of the purchaser of protection, exceeds a critical coverage limit during the life time of the option. Hence, such a contract can be viewed as a special form of a double trigger put option, where the payoff of option is a function of underlying asset price and level of insured losses. More precisely, the CatEPut gives an insurer the right to sell a specified amount of its stock to investors at a predetermined price if catastrophe losses surpass a specified trigger. Thus, the CatEPut can provide insurers with additional equity capital precisely when they need funds to cover catastrophe losses. Let $P(T)$ be the payoff of CatEPut option at maturity time T and it is written mathematically as:

$$P(T) = 1_{\{L(T)>L+L(t_0)\}}\left(K - S(T)\right)1_{\{S(T)<K\}},$$

where $S(T)$ denotes the share value and $L(T)-L(t_0)$ denotes the total losses of the insured over the time period $[t_0,T)$. L is the specified level of losses above which the CatEPut becomes in-the-money, while K represents the strike price at which the issuer is obligated to purchase unit shares in the event that losses exceed L.

In the event of a catastrophe, the share value of insurance company, which experiences a loss, will experience a downward jump. Hence, it is prudent to develop a model which jointly describes the dynamics of the share value process and losses. Cox et al. (2004) assume that

the asset price process is driven by a geometric Brownian motion with additional downward jumps of a specified size in the event of a catastrophe. Furthermore, given assuming constant interest rate, their model assumes that only the event of a catastrophe decreases the share value price of the insurance company, while the size of the catastrophe is irrelevant. Jaimungal and Wang (2006) generalize the results of Cox et al. (2004) to consider the stochastic interest rate. Moreover, they propose that the loss sizes themselves should affects the share value price, and then assume that the losses follow a compound Poisson process.

We consider that the sample path for the share value price will be continuous expects at finite in time, where jumps occur with the catastrophe information and the arrival rate of catastrophe events depends on the status of the economy. Hence, our model setting extends the previous articles by Cox et al. (2004) and Jaimungal and Wang (2006), in which case $\Phi(t) = N(t)$ is a Poisson process. Under the original probability measure P, the joint share value and loss dynamics can be written as:

$$S(t) = S(0)\exp\left\{(\mu - \frac{1}{2}\sigma_S^2)t + \sigma_S^2 W t + L(t) - \Lambda\kappa t\right\}, \tag{1}$$

$$L(t) = \sum_{n=1}^{\Phi(t)} \ln Y_n, \tag{2}$$

where $S(0)$ is the initial price, the drift μ is the instantaneous return of $S(t)$ at time t and σ_S is the constant volatility of the asset. $\{W(t): t > 0\}$ is a standard Brownian motion and $\{L(t): t > 0\}$ denotes the total loss process of the insured. $\{Y_n : n = 1, 2, ...\}$ are independent for the sequence and identically distributed nonnegative random variables representing the size of the $n\text{-}th$ loss with the density $f_Y(y)$, The term $\Lambda\kappa t$ is included in equation (1) to compensate for the presence of the downward jumps in share value price due to losses.

$\{\Phi(t): t > 0\}$ is the Markov modulated Poisson process with arrival rate of catastrophe events Λ and transition probability P(m,t) under the original probability measure P. Λ denotes $I \times I$ diagonal matrix with diagonal elements λ_i and $\mathrm{P(m,t)} := P_{ij}(m,t)$ denotes the transition probability at jump times m from state $X(0) = i$ to state $X(t) = j$. We consider a finite state space $\mathbf{X} = \{1,2...I\}$ and let $\{X, P_i : i \in \mathbf{X}\}$ be a Markov jump process on a state space $\mathbf{X}$, with transition rate $\Psi(i, j)$ denoted as:

$$\Psi(i,j)=\begin{cases}\nu(i,j), & i\neq j,\\ -\sum_{j\neq i}\nu(i,j), & otherwise.\end{cases}$$

where $i,j\in \mathbf{X}$.

We assume that $\kappa\equiv E(Y_n-1)<\infty$, which implies that the means of the jump sizes are finite for the total losses of the insured. And we suppose $\sum_{m=0}^{\infty}(\kappa+1)^m \mathrm{P(m,t)}<\infty$, which guarantee that the means of the jump sizes with the Markov modulated Poisson process are finite. Furthermore, all three sources of randomness, $W(t)$ standard Brownian motion, $\Phi(t)$ Markov modulated Poisson process and Y_n the jump size, are assumed to be independent.

When pricing catastrophe insurance products, prior studies almost assume that the arrival process of catastrophe event follow Poisson process with constant arrival rate, however, figure 1 and 2 displays that the arrival rate of catastrophe events could be significant difference at different situation. To capture this phenomenon, this article assumes that the arrival process of catastrophe event follows Markov modulated Poisson process where the underlying state is governed by a homogeneous Markov chain. The share value price $S(t)$ follows the geometric Brownian motion during the time period $(0,t]$ given that no information of catastrophe event arrivals during the time period. When the information of catastrophe event arrivals at time t, the price changes instantaneously from $S(t\_)$ to $S(t)=e^{L(t)}S(t\_)$. Since the total loss sizes could result in the downward jump of share value price, the total loss sizes can be a function of α, Y_n, and $\Phi(t)$ i.e., $L(t)=f(\alpha,Y_n,\Phi(t))$, where the factor α is the percentage drop in the share value price per unit of loss.

Proposition 2.1: Let $\eta(t)$ denote the Radon-Nikodym process

$$\eta(t)=\frac{dQ}{dP}\cdot\frac{dQ(m,t)}{dP(m,t)}\cdot\frac{dQ(Y_1^Q,Y_2^Q,...,Y_m^Q)}{dP(Y_1,Y_2,...,Y_m)}=\exp\left[-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t+\left(\frac{\mu-r}{\sigma}\right)W^Q t\right]\prod_{n=1}^{m}Y_n\exp(-\Lambda\kappa t)$$

where $W=W^Q+\left(\frac{\mu-r}{\sigma}\right)t$.

The proof detailed is sketched in Appendix A. Note that owing to the Markovian structure of the diffusion and $\Phi(t)$, under the risk neutral probability measure Q, $\Phi(t)$ is still a Markov modulated Poisson process with transition probability $\mathrm{Q(m,t)}$. Besides, the

jump size distribution and the transition rate of the Markov modulated Poisson process are unaffected by the measure change and then under the risk neutral probability measure Q, hence the joint share value and loss dynamics can be written as:

$$S(t) = S(0)\exp\left\{(r - \frac{1}{2}\sigma_S^2)t + \sigma_S^2 W^Q t + L(t) - \Lambda\kappa t\right\}, \tag{3}$$

$$L(t) = \sum_{n=1}^{\Phi(t)} \ln Y_n, \tag{4}$$

where under the risk neutral probability measure Q, the new density of $\{Y_n : n = 1,2,...\}$ becomes $f_Y^Q(y_n) = \frac{y_n f_Y(y)}{(\kappa+1)}$, and $\{\Phi(t) : t > 0\}$ is a new Markov modulated Poisson process with arrival rate of catastrophe events $\Lambda(\kappa+1)$ and transition probability $\mathrm{Q(m,t)} = (\kappa+1)^m \exp(-\Lambda\kappa t)\mathrm{P(m,t)}$.

2. Pricing of Catastrophe Equity Put Option (CatEPut)

Under the risk-neutral measure Q, the value of CatEPut contracts can be obtained by discounted expectations. Let $P(t;t_0)$ represent the value of the option at time t, which was signed at time $t_0 < t$ and matures at time T, then we have,

$$P(t;t_0) = e^{-rT} E^Q\left[1_{\{L(T)>L+L(t_0)\}}\left(K - S(T)\right)1_{\{S(T)<K\}}\right].$$

where r represents the risk-free interest rate.

A detailed proof is sketched in Appendix B, thus the formula of CatEPut can be obtained as Theorem 1:

Theorem 1：The value of CatEPut contracts admits the following representation through the Radon-Nikodym processes for Brownian motion and Markov modulated Poisson process:

$$P(t;t_0) = \sum_{m=1}^{\infty} \mathrm{Q(m,T\text{-}t)}\left[\int_{\tilde{L}}^{\infty} f_P^m(y^m) K \frac{\exp(\Lambda\kappa - r)(T-t)}{(\kappa+1)^m}\phi(-d_{2m}^S(y^m))dy - \int_{\tilde{L}}^{\infty} f_Q^m(y_Q^m) S(t)\phi(-d_{1m}^S(y_Q^m))dy\right] \tag{5}$$

where y^m means the total losses of the insured m large claims under the original probability measure P, and y_Q^m represents the total losses of the insured m large claims under the risk-neutral probability measure Q, $\phi(\cdot)$ denotes the cumulative distribution function for a standard normal random variable,

$$\tilde{L} = \max(L + L(t_0) - L(t), 0) ,$$

$$(I \times I) - \text{matrix} \quad d_{1m}^{S} = \frac{\ln\left(S(t)/K\right) + (r + \frac{1}{2}\sigma_S^2)(T - t) + y_Q^m - \Lambda\kappa(T - t)}{\sigma_S\sqrt{T - t}} ,$$

$$(I \times I) - \text{matrix} \quad d_{2m}^{S} = \frac{\ln\left(S(t)/K\right) + (r - \frac{1}{2}\sigma_S^2)(T - t) + y^m - \Lambda\kappa(T - t)}{\sigma_S\sqrt{T - t}} .$$

Corollary 1: The value of CatEPut contracts, through the Radon-Nikodym process for Brownian motion, is given as:

$$P(t;t_0) = \sum_{m=1}^{\infty} \text{P(m,T-t)} \int_{\tilde{L}}^{\infty} f_P^m(y^m)\left[K\,\phi(-d_{2m}^S(y^m)) - S(t)\exp(y^m - \Lambda\kappa(T - t))\phi(-d_{1m}^S(y^m))\right]dy$$

$$= \sum_{m=1}^{\infty} \text{P(m,T-t)} \int_{\tilde{L}}^{\infty} f_P^m(y^m)\, P\left(S(t)\exp(y^m - \Lambda\kappa(T - t)), K, T - t, r, \sigma_S\right)dy$$

If $\lambda_1 = \lambda_2 = \ldots \lambda_I = \lambda$, then Markov modulated Poisson process $\Phi(t)$ simplifies to Poisson process $N(t)$ with arrival rate of catastrophe events λ. Furthermore, let $L(t) = f(\alpha, Y_n, N(t)) = -\alpha\sum_{n=1}^{N(t)} Y_n$, thus the equation (3) reduces to

$$S(\text{t}) = S(0)\exp\left\{(r - \frac{1}{2}\sigma_S^2)t + \sigma_S^2 W^Q t - \alpha\sum_{n=1}^{N(t)} Y_n + \alpha\lambda\kappa t\right\} . \tag{6}$$

which implies that the drop in asset price depends on the total loss level, rather than on only the total number of losses and loss of per unit will result in the percentage α drop in the share value price. This equation is also the dynamic process setting for the model of Jaimungal and Wang (2006) under the assumption of constant interest rate and the pricing formula of CatEPut is given by

$$P(t;t_0) = \sum_{m=1}^{\infty} \frac{e^{-\lambda(T-t)}\left[\lambda(T - t)\right]^m}{m!} \int_{\tilde{L}}^{\infty} f_P^m(y^m) P\left(S(t)\exp\left(-\alpha\left(y^m - \lambda\kappa(T - t)\right)\right), K, T - t, r, \sigma_S\right)dy$$

If $L(t) = f(\alpha, Y_n, \Phi(t)) = -\alpha 1_{\{\Phi(t) > n\}}$, then the equation (3) reduces to

$$S(t) = S(0)\exp\left\{(r - \frac{1}{2}\sigma_S^2)t + \sigma_S^2 W^Q t - \alpha\Phi(t) + \Lambda\kappa t\right\} ,$$

which implies that only the event of a catastrophe decreases the share value price, yet the size of the catastrophe is irrelevant. The catastrophe event is defined in term of a Markov

modulated Poisson process $\Phi(t)$. And it defines a catastrophe to be $\Phi(T) \geq n$ and thus value of $L(t)$ will be $-\alpha$ only if the number of catastrophe occurring exceeds n. The pricing formula of CatEPut can be the following transition probability sum of the occurrence of n large claims of put option under Markov jump diffusion model:

$$P(t;t_0) = \sum_{m=n}^{\infty} \text{P(m,T-t)}\, P\left(S(t)\exp\left(-\alpha m + \Lambda\kappa(T-t)\right), K, T-t, r, \sigma_S\right). \tag{7}$$

If $\lambda_1 = \lambda_2 = \ldots \lambda_I = \lambda$ and $L(t) = f(\alpha, Y_n, N(t)) = -\alpha 1_{\{N(t)>n\}}$, then the equation (3) reduces to

$$S(t) = S(0)\exp\left\{(r - \frac{1}{2}\sigma_S^2)t + \sigma_S^2 W^Q t - \alpha N(t) + \lambda\kappa t\right\},$$

which also implies that only the event of a catastrophe decreases the share value price, yet the size of the catastrophe is irrelevant. The catastrophe event is defined in term of a Poisson process $N(t)$. The value of $L(t)$ will be $-\alpha$ only if the number of catastrophe occurring exceeds n. More precisely, the share value price $S(\text{t})$ follows the geometric Brownian motion during the time period $(0,t]$ given that no catastrophe event occurs during the time period. Once the catastrophe event occurs at time t, the price changes instantaneously from $S(t\_)$ to $\text{e}^{-\alpha} S(t\_)$. Besides, this equation is similar to the dynamic process setting for the model of Cox et al. (2004) and the pricing formula of CatEPut can be the following put option under jump diffusion model multiplied by the Poisson probability of the occurrence of n large claims:

$$P(t;t_0) = \sum_{m=n}^{\infty} \frac{e^{-\lambda(T-t)}\left[\lambda(T-t)\right]^m}{m!} P\left(S(t)\exp\left(-\alpha + \lambda\kappa(T-t)\right), K, T-t, r, \sigma_S\right). \tag{8}$$

In other words, by assuming that the loss size is deterministic and the probability density function of loss sizes is a Dirac density, the model of Jaimungal and Wang (2006), the equation (4), can also reduce to the equation (5).

Note that if $\lambda_1 = \lambda_2 = \ldots \lambda_I = \lambda$ and the occurrence of catastrophe events has no impact on the insurance company's share value price, $\alpha = 0$, the pricing formula could reduce to the Poisson probability sum of Black-Scholes put option.

Although the deterministic loss case could provide the behavior of the CatEPut, such the simplification could not capture the phenomena of uncertainty in the loss size. Various statistical distributions are used in actuarial models of insurance claims process to describe the loss size. Most popular prior articles, such as Louberge, Kellezi, and Gilli (1999) and Lee and Yu (2002), assume that loss size follows mutually independent, identical, and lognormal

distribution. Thus we investigate the case in which loss size are drawn from the lognormal distribution and then give the following Corollary:

Corollary 2: If the jump size Y_n follows a lognormal distribution with mean θ and variance δ^2, then equation (5) can be rewritten as follows:

$$P(t;t_0)=\sum_{m=1}^{\infty}\text{Q(m,T-t)}\left[K\exp\left(-r_m(T-t)\right)\phi(-d_{2m}^L)\phi(-d_{2m}^S)-S(t)\phi(-d_{1m}^L)\phi(-d_{1m}^S)\right]\cdot \tag{9}$$

where

$$\mathrm{r}_m=(r-\Lambda\kappa)+\frac{m\theta+\frac{1}{2}m\delta^2}{(T-t)},\quad d_{2m}^L=\frac{\tilde{L}-m\theta}{\sqrt{m\delta^2}},\quad \tilde{L}=\max(L+L(t_0)-L(t),0),$$

$$d_{1,2m}^S=\frac{\ln\left[S(t)/K\right]+r_m(T-t)\pm\frac{1}{2}\sigma_m^2(T-t)}{\sqrt{\sigma_m^2(T-t)}},\quad \sigma_m^2=\sigma_S^2+\frac{m\delta^2}{(T-t)},\quad d_{1m}^L=\frac{\tilde{L}}{\sqrt{m\delta^2}}.$$

The proof is sketched in Appendix C.

III. Dynamically Hedging the Catastrophe Equity Put

When the reinsurance company provides a CatEPut option to an insurance company, they will simultaneously hedge their position to avoid taking on huge losses. Hence, this section will illustrate how to hedge against moderate changes in the share value price level. In complete markets, Delta–Gamma hedging techniques will be used to measure the sensitivity of the option's price to share value price movements at the first and second order. We consider that loss size follows fixed loss size and lognormal loss size, respectively.

Consider a CatEPut issued at time t_0 with fixed loss sizes. The pricing equation for such a CatEPut is given in (7). Then the Delta and Gamma of the CatEPut with deterministic loss sizes are given by the following expressions:

$$\Delta_a(t;t_0)=\frac{\partial}{\partial S(t)}P(t;t_0)=\sum_{m=n}^{\infty}\frac{e^{-\lambda(T-t)}\left[\lambda(T-t)\right]^m}{m!}\left[\mathrm{e}^{-\alpha+\lambda\kappa(T-t)}(\phi(d_{1m}^S)-1)\right],$$

$$\Gamma_a(t;t_0)=\frac{\partial^2}{\partial S^2(t)}P(t;t_0)=\sum_{m=n}^{\infty}\frac{e^{-\lambda(T-t)}\left[\lambda(T-t)\right]^m}{m!}\left[e^{-\alpha+\lambda\kappa(T-t)}\frac{\phi'(d_{1m}^S)}{S(t)\sigma_S\sqrt{(T-t)}}\right].$$

In addition, consider a CatEPut issued at time t_0 with lognormal loss sizes. The pricing equation for such a CatEPut is given in (8). Then the Delta and Gamma of the CatEPut with lognormal loss sizes are given by the following expressions:

$$\Delta_b(t;t_0)=\frac{\partial}{\partial S(t)}P(t;t_0)=\sum_{m=n}^{\infty}\text{Q(m,T-t)}\left[(\phi(d_{1m}^{L})\phi(d_{1m}^{S})\text{-1})\right],$$

$$\Gamma_b(t;t_0)=\frac{\partial^2}{\partial S^2(t)}P(t;t_0)=\sum_{m=n}^{\infty}\text{Q(m,T-t)}\left[\frac{\phi(d_{1m}^{L})\phi'(d_{1m}^{S})}{S(t)\sigma_S\sqrt{(T-t)}}\right].$$

IV. Numerical Analysis of Catastrophe Equity Put

This section will consider a sensitivity analysis for CatEPut price under various parameters change using the equation (7). In order to evaluate the value of CatEPut, we need to make some assumptions as follows: equity price of insurance company, $S=25$; exercise price, $K=80$; interest rate, $r=0.05$; equity volatility, $\sigma_S=0.2$; option term, $T=1$; truncation, $n=5$. Assume the occurrence jump rate of catastrophe loss at state 1 or state 2 are set to be 0.2 and 0.5, respectively, to reflect the frequencies of catastrophe events per year. The transition rate of these two states is 1 and 10, respectively, to capture the leaving length for jump rate at different state.

We assume that the jump rate under different status is unknown. To demonstrate how the CatEPut is related to transition rate and jump rate under different underlying status, we consider the hidden switch Poisson process.

Let
$$\Psi=\begin{bmatrix}-\alpha_1 & \alpha_1\\ \alpha_2 & -\alpha_2\end{bmatrix}$$

and
$$\Lambda=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$$

By using Equation (A.3) in Appendix A with $z=1$, we have $P^*(1,t)=e^{\Psi(t)}$. Hence,

$$P^*(1,t)=\frac{1}{\alpha_1+\alpha_2}\begin{bmatrix}\alpha_2+\alpha_1 e^{-(\alpha_1+\alpha_2)t} & \alpha_1-\alpha_1 e^{-(\alpha_1+\alpha_2)t}\\ \alpha_2-\alpha_2 e^{-(\alpha_1+\alpha_2)t} & \alpha_1+\alpha_2 e^{-(\alpha_1+\alpha_2)t}\end{bmatrix}.$$

It is easy to obtain the limiting distribution as

$$\pi_1=\lim_{t\to\infty}P_{i1}(1,t)=\frac{\alpha_2}{\alpha_1+\alpha_2},$$

$$\pi_2=\lim_{t\to\infty}P_{i2}(1,t)=\frac{\alpha_1}{\alpha_1+\alpha_2},$$

where π_1 and π_2 represent the probability that the jump rate stay at state 1 and state 2, respectively.

Note that in Table 1, if α_1 increases then the Markov chain will leave state 1 rapidly,

hence the decreasing (increasing) of jump rate at state 2 makes the CatEPut price decreasing (increasing). Similarly, if α_2 increases then the Markov chain will leave state 2 rapidly, hence the increasing (decreasing) of jump rate at state 1 makes the CatEPut price increasing (decreasing). For the concern of the parameters λ_1 and λ_2. If one increases while the other one is fixed, this causes the volatility of CatEPut price increases, the transition probability increases. Thus the increasing of the jump rate causes the CatEPut price increasing. For the concern of parameter of α, we find that higher the percentage drop in share value due to catastrophe events results in higher the CatEPut price.

Table 1 The Value of CatEPut

(α_1,α_2)	α	(λ_1,λ_2)			
		(0.2,0.2)	(0.2,0.5)	(0.5,0.2)	(0.5,0.5)
(1,1)	0.2	18.6876	29.9585	29.9585	40.5115
	0.5	21.8767	34.8083	34.8083	46.8028
(1,10)	0.2	18.6876	20.9064	38.7492	40.5115
	0.5	21.8767	24.4408	44.8144	46.8028
(10,1)	0.2	18.6876	38.7492	20.9064	40.5115
	0.5	21.8767	44.8144	24.4408	46.8028
(10,10)	0.2	18.6876	30.3471	30.3471	40.5115
	0.5	21.8767	35.2763	35.2763	46.8028

The parameters of base valuation are $S=25$, $K=80$, $r=0.05$, $\sigma_S=0.2$, $T=1$, $n=5$.

V. Conclusions

This paper proposes Markov Modulated Poisson process, which the underlying state is governed by a homogenous Markov chain to model the arrival process for catastrophic events. Further, we propose a generalization of Radon-Nikodym processes that a changing measure corresponds to a change of drift for the underlying Brownian motion and a change of the stochastic intensity for the Markov jump diffusion model. By the change of measure, the pricing formula and dynamic hedging for CatEPut are derived and our pricing formula could reduce to Cox, Fairchild, and Pedersen (2004) and Jaimungal and Wang (2006).

We assume that the jump rate under different status is unknown and use hidden switch Poisson process to report numerical analysis. Numerical results show that when the transition rate increases, then the decreasing of jump rate makes the CatEPut price decreasing. In addition, the higher jump rate, the higher CatEPut price. For the concern of parameter of a, we find that higher the percentage drop in share value due to catastrophe events results in higher the CatEPut price.

Endnotes

[1] The data comes from the Insurance Service Office (ISO). The PCS indices track insured catastrophe loss estimates on national, regional, and state basis in the US from information obtained by PCS. The term "natural catastrophe" includes hurricanes, storms, floods, waves, and earthquakes. The event of "natural catastrophe" denotes a natural disaster that affects many insurers and when claims are expected to reach a certain dollar threshold. Initially the threshold was set to $5 million. In 1997, ISO increased its dollar threshold to $25 million.

Appendix A : Change of measure

$$\frac{dQ}{dP}\cdot\frac{dQ(m,t)}{dP(m,t)}\cdot\frac{dQ(Y_1^Q,Y_2^Q,...,Y_m^Q)}{dP(Y_1\ ,Y_2\ ,...,Y_m)}=\exp\left[-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t+\left(\frac{\mu-r}{\sigma}\right)W^Q t\right]\prod_{n=1}^{m}Y_n\exp(-\Lambda\kappa t)\quad\text{(A.1)}$$

Equation (A.1) can be rewritten as:

$$dQ\,dQ(m,t)\,dQ(Y_1^Q,Y_2^Q,...,Y_m^Q)$$
$$=\exp\left[-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t+\left(\frac{\mu-r}{\sigma}\right)W^Q t\right]dP\prod_{n=1}^{m}Y_n\exp(-\Lambda\kappa t)\,dP(m,t)\,dP(Y_1\ ,Y_2\ ,...,Y_m)$$
$$=\frac{1}{\sqrt{2\pi}}\exp-\left[\frac{\left(W-\left(\frac{\mu-r}{\sigma}\right)t\right)^2}{2t}\right]\prod_{n=1}^{m}Y_n\exp(-\Lambda\kappa t)\,dP(m,t)\,dP(Y_1\ ,Y_2\ ,...,Y_m)$$

Thus the new Brownian is $W^{\mathrm{Q}}=W-\left(\frac{\mu-r}{\sigma}\right)t$, and integrate the Brownian motion, we have**:**

$$dQ(m,t)dQ(Y_1^Q,Y_2^Q,...,Y_m^Q)=(\kappa+1)^m\exp(-\Lambda\kappa t)\,dP(m,t)\prod_{n=1}^{m}Y_n\frac{1}{(\kappa+1)^m}dP(Y_1\ ,Y_2\ ,...,Y_m)$$

Further, we investigate the transition probability $P(m,t)$, for $0\le z\le 1$, define

$$P^*(z,t)=\sum_{m=0}^{\infty}P(m,t)\,z^m\qquad\text{(A.2)}$$

with $P(m,0)=1_{\{m=0\}}D_{ij}$, where $D_{ij}=1,\ if\ i=j;0$, otherwise.

By using Kolmogorov's forward equation, the derivative of $P(m,t)$ becomes

$$\frac{d}{dt}P(m,t)=(\Psi-\Lambda)P(m,t)+1_{\{m\ge 1\}}\Lambda\,P(m-1,t),$$

thus it's unique solution is

$$P^*(z,t)=e^{[\Psi-(1-z)\Lambda]t},\qquad\text{(A.3)}$$

where $e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}$, for any $(I \times I)$ matrix A.

Finally, by using Laplace inverse transform (A.2) and the unique solution (A.3), we obtain the joint distribution of X and $\Phi(t)$ at time t when

$$P(m,t) = \frac{\partial^m}{\partial z^m m!} P^*(z,t)\big|_{z=0} .$$

Let $Q(m,t) = (\kappa+1)^m \exp(-\Lambda\kappa t)\, P(m,t)$,

Thus $\mathrm{Q}^*(z,t) = \sum_{m=0}^{\infty} \mathrm{Q}(m,t)\, z^m = \sum_{m=0}^{\infty} (\kappa+1)^m \exp(-\Lambda\kappa t) P(m,t) z^m$

$$= \sum_{m=0}^{\infty} (z(\kappa+1))^m \exp(-\Lambda\kappa t) P(m,t)$$

and the unique solution of $Q^*(z,t)$ is

$$Q^*(z,t) = e^{\left[\Psi - (1-z(\kappa+1))\Lambda\right]T} e^{-\Lambda\kappa T} = e^{\left[\Psi - ((1-z)(\kappa+1)\Lambda)\right]T} .$$

Consequently, under the Markov modulated Poisson process $\Phi(t)$, the original transition probability is $P(m,t)$, with transition rate Ψ and $I \times I$ diagonal matrix Λ with diagonal elements λ_i. Through the change of measure, the risk neutral transition probability is $Q(m,t)$ with transition rate Ψ and $I \times I$ diagonal matrix $(\kappa+1)\Lambda$ with diagonal elements λ_i. Therefore, the Radon-Nikodym derivative of the transition probability can be considered as:

$$\frac{dQ(m,t)}{dP(m,t)} = (\kappa+1)^m \exp(-\Lambda\kappa t) .$$

Finally, we look into the jump size, where $(Y_1, Y_2, ..., Y_m)$ are independent identically distribution random variables. Hence, the Radon-Nikodym derivative of the jump size can be:

$$dQ(Y_1^Q, Y_2^Q, ..., Y_m^Q) = \frac{y_1 f_Y(y_1)}{(\kappa+1)} \frac{y_2 f_Y(y_2)}{(\kappa+1)} .... \frac{y_m f_Y(y_m)}{(\kappa+1)} .$$

That is the new density function of the jump size is $f_Y^Q(y) = \frac{y\, f_Y(y)}{(\kappa+1)}$ under risk neutral measure Q.

Appendix B

$$P\left(t;t_0\right)=e^{-rT}E^{Q}\left[1_{\{L(T)>L+L(t_0)\}}\left(K-S\left(T\right)\right)^{+}\left|F_t\right.\right] \tag{B.1}$$

Using the law of expected iteration, equation (B.1) can be rewritten as

$$e^{-r(T-t)}E^{Q}\left[1_{\{L(T)>L+L(t_0)\}}E^{Q}\left(\left(K-S\left(T\right)\right)^{+}\left|L(T)\right.\right)\left|F_t\right.\right]=\text{A-B}$$

where

$$\text{A}=e^{-r(T-t)}E^{Q}\left[1_{\{L(T)>L+L(t_0)\}}E^{Q}\left(K\,1_{\{S(T)<K\}}\left|L(T)\right.\right)\left|F_t\right.\right]$$

$$=e^{-r(T-t)}K\ \mathrm{E}^{\mathrm{Q}}\left\{1_{\{L(T)>L+L(t_0)\}}P^{Q}\left(S(t)\exp(r-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S W_S^{Q}(T-t)\right.\right.$$

$$\left.\left.+L(T)-L(t)-\Lambda\kappa(T-t)<K\right)\left|F_t\right.\right\}$$

$$=\sum_{m=0}^{\infty}P(m,T)\ \mathrm{E}^{\mathrm{Q}}\left\{\left[1_{\{L(T)>L+L(t_0)\}}e^{-r(T-t)}K\,N(-d_{2m}^{S}(L(T)-L(t)))\left|\Phi(T-t)=m\right.\right]\left|F_t\right.\right\} \tag{B.2}$$

Through the change of measure, we know $(\kappa+1)^m P(m,T-t)e^{-\Lambda\kappa(T-t)}=Q(m,T-t)$, and then obtain $P(m,T-t)=\dfrac{Q(m,T-t)e^{\Lambda\kappa(T-t)}}{(\kappa+1)^m}$, where $(\kappa+1)^m=\left[E(Y_n)\right]^m$. Thus equation (B.2) becomes

$$\sum_{m=1}^{\infty}Q(m,T-t)\int_{\tilde{L}}^{\infty}f_P^m(y^m)\,K\frac{\exp(\Lambda\kappa-r)(T-t)}{(\kappa+1)^m}\phi(-d_{2m}^{S}(y^m))dy,$$

where

$$(I\times I)-\text{matrix}\quad d_{2m}^{S}=\frac{\ln\left(\dfrac{S(t)}{K}\right)+(r-\frac{1}{2}\sigma_S^2)(T-t)+y^m-\Lambda\kappa(T-t)}{\sigma_S\sqrt{T-t}}.$$

$$\text{B}=e^{-r(T-t)}E^{Q}\left[1_{\{L(T)>L+L(t_0)\}}E^{Q}\left(S\left(T\right)1_{\{S(T)<K\}}\left|L(T)\right.\right)\left|F_t\right.\right]$$

$$=e^{-rT}E^{Q}\left[1_{\{L(T)>L+L(t_0)\}}S(t)\exp\left\{(r-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S W_S^{Q}(T-t)\right.\right.$$

$$\left.\left.+L(T)-L(t)-\Lambda\kappa(T-t)\right\}1_{\{S(T)<K\}}\left|F_t\right.\right] \tag{B.3}$$

Let $\eta_1(t)$ denote the Radon-Nikodym process for Brownian motion given by the formula:

$$\eta_1(t) \equiv \left(\frac{dR}{dQ}\right)_t = \exp\left\{ \int_0^t \sigma_S dW_S^Q(u) - \frac{1}{2}\int_0^t \sigma_S^2\, du \right\},$$

and $\eta_2(t)$, $\eta_3(t)$ represent the Radon -Nikodym processes for transition probability and jump size, respectively as follows:

$$\eta_2(t) \equiv \frac{dQ(m,t)}{dP(m,t)} = (\kappa+1)^m \exp(-\Lambda\kappa t),$$

$$\eta_3(t) \equiv d\, y_Q^m(Y_1^Q, Y_2^Q, ..., Y_m^Q) = \frac{y^m f_Y(y^m)}{(\kappa+1)^m},$$

hence, the equation (B.3) becomes

$$\sum_{m=1}^{\infty} Q(m,T-t)\, E^Q\left[1_{\{y_Q^m > \max(L+L(t_0)-L(t),0)\}}\, S(t)\, N(-d_{1m}^S(y_Q^m)) \middle| \Phi(T-t) = m \middle| F_t \right]$$

$$= \sum_{m=1}^{\infty} Q(m,T-t) \int_{\tilde{L}}^{\infty} f_Q^m(y_Q^m)\, S(t) \phi(-d_{1m}^S(y_Q^m))dy$$

where

$$(I \times I) - \text{matrix} \quad d_{1m}^S = \frac{\ln\left(\frac{S(t)}{K}\right) + (r + \frac{1}{2}\sigma_S^2)(T-t) + y_Q^m - \Lambda\kappa(T-t)}{\sigma_S\sqrt{T-t}}.$$

Therefore, the value of CatEPut at time t is given as:

$$P(t;t_0) = \sum_{m=1}^{\infty} Q(m,T-t)\left[\int_{\tilde{L}}^{\infty} f_P^m(y^m)\, K \frac{\exp(\Lambda\kappa - r)(T-t)}{(\kappa+1)^m} \phi(-d_{2m}^S(y^m))dy - \int_{\tilde{L}}^{\infty} f_Q^m(y_Q^m)\, S(t) \phi(-d_{1m}^S(y_Q^m))dy \right]$$

Appendix C

Since that Y_n follows a lognormal distribution with mean θ and variance δ^2, then $\kappa+1=E(Y_n)=\mathrm{e}^{\theta+\frac{1}{2}\delta^2}$. In addition, $y^m=\sum_{n=0}^{m}\ln Y_n=m\theta+\sqrt{m}\delta Z_m$, we have

$$\sum_{m=1}^{\infty}Q(m,T\text{-}t)\,E^Q\left\{\left[1_{\{y^m>\max(L+L(t_0)\text{-}L(t),0)\}}\,K\frac{\exp(\Lambda\kappa-r)(T\text{-}t)}{(\kappa+1)^m}N(\text{-}d_{2m}^S(y^m))\middle|\Phi(T\text{-}t)=m\right]\middle|F_t\right\}$$

$$=\sum_{m=1}^{\infty}Q(m,T\text{-}t)\,K\exp\left(-r_m(T-t)\right)\phi(\text{-}d_{2m}^L)\phi(\text{-}d_{2m}^S)$$

where

$$\mathrm{r}_m=\frac{(r-\Lambda\kappa)(T-t)+m\theta+\frac{1}{2}m\delta^2}{(T\text{-}t)},\quad d_{2m}^L=\frac{\tilde{L}\text{-}m\theta}{\sqrt{m\delta^2}},\quad \tilde{L}=\max(L+L(t_0)\text{-}L(t),0),$$

$$d_{2m}^S=\frac{\ln\left[S(t)/K\right]+r_m(T\text{-}t)-\frac{1}{2}\sigma_m^2(T\text{-}t)}{\sqrt{\sigma_m^2(T\text{-}t)}},\quad \sigma_m^2=\sigma_S^2+\frac{m\delta^2}{(T\text{-}t)}.$$

Furthermore, due to $dQ(Y_n^Q)=\frac{yf_Y(y)}{(\kappa+1)}$, we have:

$$dQ(Y_{\mathrm{n}}^Q)=\frac{1}{\sqrt{2\pi\delta}}\exp-\left[\frac{(\ln\mathrm{y}-(\theta+\delta^2))^2}{2\delta^2}\right]$$

Since Y_n is independent identically distributed nonnegative random variables, the new jump size under Q is

$$\sum_{n=1}^{m}\ln Y_n^Q=\sum_{n=1}^{m}\ln Y_n-m(\theta+\delta^2).$$

Hence, we have:

$$\sum_{m=1}^{\infty}Q(m,T\text{-}t)\,E^Q\left[1_{\{y_Q^m>\max(L+L(t_0)\text{-}L(t),0)\}}\,S(t)\,N(\text{-}d_{1m}^S(y_Q^m))\middle|\Phi(T\text{-}t)=m\middle|F_t\right]$$

$$=\sum_{m=1}^{\infty}Q(m,T\text{-}t)\,S(t)\,\phi(\text{-}d_{1m}^L)\phi(\text{-}d_{1m}^S)$$

where

$$d_{1m}^L=\frac{\tilde{L}}{\sqrt{m\delta^2}},\quad d_{1m}^S=\frac{\ln\left[S(t)/K\right]+r_m(T\text{-}t)+\frac{1}{2}\sigma_m^2(T\text{-}t)}{\sqrt{\sigma_m^2(T\text{-}t)}}.$$

Therefore, the option price can be described as

$$P\left(t;t_0\right)=\sum_{m=1}^{\infty}Q(m,T\text{-}t)\left[\ K\exp\left(-r_m\,(T-t)\right)\phi(\text{-}d_{2m}^{L})\phi(\text{-}d_{2m}^{S})-S(t)\,\phi(\text{-}d_{1m}^{L})\phi(\text{-}d_{1m}^{S})\right]\cdot$$

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