



## 1. Introduction.

Our aim in this paper is to propose a new methodology for modelling operational risk, based on risk and ruin theory. This is in compliance with the commitment of the Basel Committee on Banking Supervision (2001) (see its consultative report on the New Basel Capital Accord (Basel II)) to improve stability in the financial sector by reducing market risk, credit risk and operational risk. The first pillar, under the three pillar approach of Basel II, considers Minimal Capital Requirements and this is where new quantitative modelling methods, based on sound mathematical, statistical and probabilistic methodology are expected to provide a practically applicable tool for quantitative risk management. The demand for such new methods, which relate to solvency issues within the insurance industry, is also recognized within the new EU Solvency II project and the working programme of the International Association of Insurance Supervisors (IAIS) (see e.g. Linder and Ronkainen 2004). Actuarial techniques for quantifying operational risk in general insurance have recently been summarized by Tripp et al. (2004).

There are three alternative groups of methods for mitigating operational risk, outlined in the Basel Committee on Banking Supervision (2004), the basic indicator approach (BIA), the standardized approach (TSA) and the advanced measurement approach (AMA). The latter focuses on using internal and external loss data, among other techniques, and is often referred to as the Loss Distribution Approach (LDA). Under the AMA modelling framework the role of insurance in mitigating operational risk is also recognized. There are several examples of works under the LDA approach and here we will mention the common Poisson shock models of Ebnöther et al. (2001, 2002) and of Brandts (2004) and the ruin probability based models considered by Embrechts and Samorodnitsky (2003) and Embrechts et al. (2004). A more recent paper, considering the effect of insurance on setting the capital charge for operational risk is that of Bazzarello et al. (2006). The LDA approach has recently been used by Dutta and Perry (2006), who have considered fitting appropriate loss distributions to operational loss data under the 2004 Loss Data Collection Exercise (LDCE) and the Quantitative Impact Study 4 (QIS-4). Thus, it is more and more evident that LDA methods are becoming important for internal risk modelling purposes and at Basel-defined business line and event type level modelling in order to improve the stability of the financial services industry. LDA methods are flexible and could be used within the whole financial industry sector, by central and commercial banks, insurance companies and supervisory bodies. No doubt, a great potential for developing such methods lies within the paradigm of ruin theory as has already been noted by Embrechts et al. (2004).

The classical ruin theory is over 100 years old and since the fundamental paper of Lundberg (1903), the number of publications (books, monographs and academic articles) in the probabilistic, statistical and actuarial literature is vast. Important contributions to the field have been made by Cramér (1930), Seal (1978), Gerber (1988), Shiu (1987), Dickson (1994), Waters

(1983), DeVilder (1999), Grandel (1990), Picard and Lefèvre (1997), Asmussen (2000), Ignatov and Kaishev (2000, 2004, 2006) to mention only a few. Ruin theory may be viewed as the theoretical foundation of insolvency risk modelling. Under the classical ruin theory model, the (premium) income to an (insurance) company is modelled by a straight line  $h(t) = u + c t$ , where  $u$  is the company's initial risk capital at time  $t = 0$  and  $c$  is the premium income per unit of time, received by the company. The outgoing flow of claims paid by the company is modelled by a stochastic process,

$$S(t) = \sum_{i=1}^{N(t)} W_i, \quad (1)$$

where,  $W_i$ ,  $i = 1, 2, \dots$  are assumed independent identically distributed (i.i.d.) random variables, modeling the amount of the consecutive individual losses, occurring at random moments in time. The stochastic process  $N(t)$ , usually assumed a homogeneous Poisson process with parameter  $\lambda$ , is counting the number of such losses up to time  $t$ . The risk (surplus) process of the company is then defined as

$$R(t) = u + c t - S(t)$$

and the probability  $P(T \leq \infty)$  that the aggregate amount of the loss payments,  $S(t)$ , will exceed the in-flowing premium income  $h(t) = u + c t$  at some future moment,  $T$ , is called the infinite-time probability of ruin of the company. In other words this is the probability that the risk process  $R(t)$  will become negative in some future moment, within an infinite time horizon.

The practical validity of model (1) for the aggregate operational losses under the LDA approach has been confirmed by Dutta and Perry (2006), who summarize the operational risk measuring experience of US banks under the QIS-4 submission.

Recently, Embrechts et al. (2004) proposed to take an actuarial point of view and directly apply the (classical) ruin probability model to the context of operational risk, under the LDA approach. Thus, the random variables  $W_i$ ,  $i = 1, 2, \dots$  in model (1) are viewed as representing operational risk losses and the aggregate loss amount,  $S(t)$ , due to different types of operational risk, is expressed as a superposition of the risk processes, corresponding to each type of risk. The rate  $c$  is seen "as a premium rate paid to an external insurer for taking (part of) the operational risk losses or as a rate paid to (or accounted for by) a bank internal office" (Embrechts et al., 2004). In order to reserve against operational risk, it is proposed to set the initial capital  $u$  and the income rate  $c$  in such a way that it satisfies the equation

$$P(T \leq x) = P\left(\inf_{0 \leq t \leq x} (u + c t - S(t)) < 0\right) = \epsilon \quad (2)$$

where the probability of ruin,  $P(T \leq x)$ , over a finite time interval,  $[0, x]$ ,  $0 < x \leq \infty$ , is set to a pre-assigned appropriate (small) value  $\epsilon > 0$ . As noted in Embrechts et al. (2004), if the time interval is of length  $x$  and  $c = 0$ , the risk capital  $u$  is equal to the operational value at risk at significance level,  $\alpha$ , i.e.,

$$u = \text{OR} - \text{VaR}_{1-\alpha}^x,$$

which is another popular risk measure considered in defining the capital charge for operational risk (see also Embrechts and Puccetti, 2006). Although Embrechts et al. (2004) refer to ruin probability results, obtained by them and by others (see e.g. Embrechts and Veraverbeke 1982, Asmussen 2000, Schmidli 1999), which extend the applicability of the classical ruin probability model, the following major limitations may still be outlined:

- The function  $h(t)$  is represented by a straight line, which is a simple but not a realistic assumption for the premium income.
- the losses,  $W_i$ ,  $i = 1, 2, \dots$ , are assumed independent and identically distributed which is also a restrictive assumption, not expected to hold for operational risk losses (see e.g. Panjer 2006 Chapter 8).
- the ruin probability estimates quoted and discussed by Embrechts et al. (2004) are asymptotic approximations, i.e., for ruin on infinity, and as mentioned by the authors, "are not fine enough for accurate numerical approximations" and their numerical properties are "far less satisfactory", since these estimates are in an integral form.

In what follows, we propose a methodology which aims at generalizing the discussed classical ruin probability framework and at making it a more practically applicable and useful approach for operational risk reserving. In particular, in our model, outlined in Section 2, we relax the above mentioned limitations and in Section 4, we consider a possible insurance coverage of the operational losses from a certain risk class (i.e., line of business or a BIS2 event type, as required by Basel Committee on Banking and Supervision 2004). Under the methodology proposed in Sections 3 and 4, it is possible to set not just a single value of the capital charge for operational risk, but to set a dynamic operational risk reserving strategy instead. This is briefly illustrated in Section 5 based on stylized numerical examples.

## 2. Ruin probabilities under a general model

Recently, a more general ruin probability model, relaxing the restrictive classical assumptions, has been considered by Ignatov and Kaishev (2000), where an explicit finite-time ruin probability formula was derived. Thus, the model considered by Ignatov and Kaishev (2000) assumes

- any non-decreasing (premium) income function  $h(t)$  as an alternative to the classical straight line case
- any joint distribution of the losses  $W_i$ ,  $i = 1, 2, \dots$ , allowing dependency between the loss amounts, as an alternative to the i.i.d. classical assumption

- finite time ruin probabilities, as an alternative to the asymptotic approximations of infinite ruin probabilities, suggested by Embrechts et al. (2004)

In a series of recent papers, (see Ignatov et al. 2001, 2004, Kaishev and Dimitrova 2006, and Ignatov and Kaishev 2004, 2006) the above mentioned, ruin probability model has been explored and extended further and the following new explicit non-ruin probability formulae have been derived. Claims are assumed to arrive at an insurance company with inter-arrival times  $\tau_1, \tau_2, \dots$ , identically, exponentially distributed r.v.s with parameter  $\lambda$ , i.e., the number of the claims  $N(t) = \#\{i : \tau_i + \dots + \tau_i \leq t\}$ , with  $\#$  denoting the number of elements in the set  $\{.\}$ , is a Poisson process with intensity  $\lambda$ . In the case of discrete claim severities, they are modeled by the integer valued r.v.s.  $W_1, W_2, \dots$  with joint distribution denoted by  $P(W_1 = w_1, \dots, W_i = w_i) = P_{w_1, \dots, w_i}$ , where  $w_1 \geq 1, w_2 \geq 1, \dots, w_i \geq 1, i = 1, 2, \dots$ . The r.v.s  $W_1, W_2, \dots$  are assumed to be independent of  $N(t)$ . Then, the risk process  $R(t)$ , at time  $t$  is given by

$$R(t) = h(t) - S(t), \quad (3)$$

where  $h(t)$  is a nonnegative, nondecreasing, real function, defined on  $R_+$ , representing the premium income of the insurance company and  $S(t)$  is the aggregate loss amount at time  $t$  defined as in (1) but assuming the losses have a joint distribution  $P_{w_1, \dots, w_i}$ .

The function  $h(t)$  is such that  $\lim_{t \rightarrow \infty} h(t) = \infty$ . It may be continuous or discontinuous, in which case  $h^{-1}(y) = \inf\{z : h(z) \geq y\}$ . It will be convenient to denote the whole class of functions  $h(t)$ , by  $\mathcal{H}$ . We will denote also  $v_i = h^{-1}(i)$ , for  $i = 0, 1, 2, \dots$ , noting that  $0 = v_0 \leq v_1 \leq v_2 \dots$ . The time  $T$  of ruin is defined as

$$T := \inf\{t : t > 0, R_t < 0\} \quad (4)$$

and we will be concerned with the probability of non ruin  $P(T > x)$  in a finite time interval  $[0, x]$ ,  $x > 0$ . It has been shown by Ignatov and Kaishev (2000) that under this model the survival probability is given as

$$P(T > x) = e^{-x\lambda} \sum_{\substack{w_1 \geq 1 \\ \dots \\ w_n \geq 1}} P_{w_1, \dots, w_n} \sum_{j=0}^{k-1} (-1)^j b_j(z_1, \dots, z_j) \lambda^j \sum_{m=0}^{k-j-1} \frac{(x\lambda)^m}{m!}, \quad (5)$$

where  $n = [h(x)] + 1$ ,  $[h(x)]$  is the integer part of  $h(x)$ ,  $v_{n-1} \leq x < v_n$ ,  $k$  is such that  $w_1 + \dots + w_{k-1} \leq n - 1$ ,  $w_1 + \dots + w_k \geq n$ , ( $1 \leq k \leq n$ ),  $z_l = v_{w_1 + \dots + w_l}$ ,  $l = 1, 2, \dots$  and  $b_j(z_1, \dots, z_j)$  is defined recurrently as

$$b_j(z_1, \dots, z_j) = (-1)^{j+1} \frac{z_j^j}{j!} + (-1)^{j+2} \frac{z_j^{j-1}}{(j-1)!} b_1(z_1) + \dots + (-1)^{j+j} \frac{z_j^1}{1!} b_{j-1}(z_1, \dots, z_{j-1}),$$

with  $b_0 \equiv 1$ ,  $b_1(z_1) = z_1$ .

In Ignatov et al. (2001), formula (5) has been given the following exact, numerically convenient representation

$$P(T > x) = e^{-x\lambda} \sum_{k=1}^n \left( \sum_{\substack{w_1 \geq 1, \dots, w_{k-1} \geq 1 \\ w_1 + \dots + w_{k-1} \leq n-1}} P(W_1 = w_1, \dots, W_{k-1} = w_{k-1}; W_k \geq n - w_1 - \dots - w_{k-1}) \right. \\ \left. \sum_{j=0}^{k-1} (-1)^j b_j(z_1, \dots, z_j) \lambda^j \sum_{m=1}^{k-j-1} \frac{(x\lambda)^m}{m!} \right). \quad (6)$$

When claims have any continuous joint distribution, the probability of nonruin within a finite time  $x$  has recently been shown by Ignatov and Kaishev (2004) to admit the representation

$$P(T > x) = e^{-\lambda x} \left( 1 + \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} d y_1 \int_{y_1}^{h(x)} d y_2 \dots \int_{y_{k-1}}^{h(x)} A_k(x; h^{-1}(y_1), \dots, h^{-1}(y_k)) f(y_1, \dots, y_k) d y_k \right), \quad (7)$$

where  $y_0 \equiv 0$ ,  $A_k(x; v_1, \dots, v_k)$ ,  $k = 1, 2, \dots$  are the classical Appell polynomials  $A_k(x)$  of degree  $k$  with a coefficient in front of  $x^k$  equal to  $1/k!$ , defined by

$$A_0(x) = 1, \quad A'_k(x) = A_{k-1}(x), \quad A_k(v_k) = 0, \quad k = 1, 2, \dots$$

and  $f(y_1, \dots, y_k)$  is the joint density of the partial sums of consecutive claims  $Y_1 = W_1, Y_2 = W_1 + W_2, \dots, Y_i = W_1 + \dots + W_i$  for which  $P(Y_1 \leq Y_2 \leq \dots \leq Y_k \leq \dots) = 1$ . The density of the r.v.s  $Y_1, \dots, Y_i$  has the form

$$f(y_1, \dots, y_k) = \begin{cases} \varphi(y_1, \dots, y_k), & \text{if } 0 \leq y_1 \leq \dots \leq y_k \\ 0 & \text{otherwise} \end{cases}$$

where  $\varphi(y_1, \dots, y_k) \geq 0$  for  $0 \leq y_1 \leq \dots \leq y_k$  and

$$\int_{0 \leq y_1 \leq \dots \leq y_k} \dots \int \varphi(y_1, \dots, y_k) d y_1 \dots d y_k = 1.$$

Obviously the claim severities are related with the r.v.s  $Y_1, Y_2, \dots$  through the equalities  $W_1 = Y_1, W_2 = Y_2 - Y_1, W_3 = Y_3 - Y_2, \dots$ , i.e.,  $Y_1, Y_2, \dots$  and the joint density  $\psi(w_1, \dots, w_k)$  of the r.v.s  $W_1, W_2, \dots, W_k$  can be expressed as

$$\psi(w_1, \dots, w_k) = f(w_1, w_1 + w_2, \dots, w_1 + \dots + w_k) \text{ or}$$

$$f(y_1, y_2, \dots, y_k) = \psi(y_1, y_2 - y_1, \dots, y_k - y_{k-1}).$$

Both formulae (6) and (7) have been implemented numerically (see Ignatov et al. (2001) and Ignatov and Kaishev (2004)) and allow for the efficient computation of  $P(T > x)$  for any discrete or continuous joint distribution of the losses  $W_i$ ,  $i = 1, 2, \dots$ . The flexibility of these results makes them especially attractive in modelling operational risk capital allocation, which is considered in the next section.

### 3. Capital assessment under the general ruin probability model

The (non) ruin probability formulae (6) and (7), are flexible and can be directly applied under the LDA approach to operational risk modelling and capital assessment, assuming ruin probability is selected as an operational risk measure. To see this, note that taking into account the general ruin probability model outlined in Section 2, equation (2) can be rewritten as

$$P(T > x) = 1 - P\left(\inf_{0 \leq t \leq x} (h(t) - S(t)) < 0\right) = 1 - \epsilon \quad (8)$$

where the nonruin probability on the left-hand side can be directly expressed by formula (6) if loss severities  $W_i$ ,  $i = 1, 2, \dots$  are assumed discrete or by (7) if they are assumed continuous. Operational risk capital allocation, can now be formulated as "selecting" an appropriate "capital accumulation" function  $h(t) \in \mathcal{H}$ , such that equation (8) is satisfied for a sufficiently small preassigned value  $\epsilon > 0$ . It has to be noted that there may be infinitely many solutions to the functional equation (8), since the class  $\mathcal{H}$  is rather general. In particular the functions  $h(t) \in \mathcal{H}$  need not be continuous and thus, may incorporate jump discontinuities at some points in time. Moreover,  $\mathcal{H}$  need not necessarily be strictly increasing which means that step-wise constant functions  $h(t)$  may also be considered.

Somewhat surprisingly, the flexibility of the class  $\mathcal{H}$  leads to the possibility of selecting a function  $h(t)$ , which maximizes the probability of nonruin,  $P(T > x)$  over an appropriate subclass of  $\mathcal{H}$ . In other words, the bank has the flexibility of selecting different capital accumulation strategies,  $h(t)$ , for reserving against operational risk, so as to maximize its chances of survival from operational losses. For example, if the appropriate subclass is the class of all piecewise linear functions on  $[0, x]$ , with one jump of size  $J$ , at some instant  $t_J \in [0, x]$  the bank may put aside less amount,  $u$ , of initial capital at time  $t = 0$  and top up this capital by an amount  $J$  at some (optimal) later moment  $t_J$ . This point is illustrated numerically in Section 5 (see Fig. 2) where it is demonstrated that one and the same high nonruin probability  $1 - \epsilon$  can be achieved by different alternative choices of capital accumulations,  $h(t)$ , whose values at the terminal time point,  $x$ , coincide.

In general, to distinguish between different choices of the reserving capital accumulation function  $h(t)$ , and thus to facilitate the solution of (8), these choices can be attached a different utility which may for instance be related to the cost of borrowing capital from the bank.

For example, the bank may find it preferable to set less initial reserve  $u$  and top up its reserves at a later instant. In order to illustrate this point, assume that preference is measured by the Expected Present Value (EPV) of the continuous cash flow  $h'(t)$ , where  $h'(t)$  is the derivative of  $h(t)$ . Then, from two different solutions of (8), which provide equal probabilities of survival, the bank will choose the solution with lower EPV. Since our purpose here is to introduce the major concepts and discuss model (8) we will restrain from going into greater details with respect to this utility modelling aspect. Further, considerations are a subject of an extended version of this paper.

A second point which deserves to be made in connection with setting operational reserves according to (8) is that the joint distribution of the operational losses  $W_1, W_2, \dots$  can be any joint distribution, continuous or discrete. This is possible since formulae (6) and (7) are general and are valid for any i.i.d or dependent losses. Thus, they can easily accommodate any of the widely advocated one dimensional heavy tailed operational loss distributions, such as the Generalized Pareto Distribution (GPD), the Lognormal distribution or the less popular g-and-h and Generalized Beta Distribution of Second Kind (GB2) distributions, recently put forward by Dutta and Perry (2006).

It is a common argument in the operational risk modelling literature (see e.g. Embrechts and Puccetti 2006) that operational losses do in general exhibit dependence in their severity. Taking account of this dependence may require significantly higher capital reserves on aggregate as illustrated in Section 5, (see Fig. 3), based on ruin probability as operational risk measure. Thus, allowing for modelling dependence is an important feature of our proposed methodology. Dependence can be incorporated in the loss distribution using any of the available dependence modelling techniques, for example copulas or any of the existing multivariate distributions. It has to be noted that there are very few examples in the literature of dependent multivariate distributions which have been used to model dependent severities of consecutive insurance claims and operational losses. Illustrations of how this can be done are to be found in Ignatov, Kaishev and Krachunov (2001, 2004) for multivariate discrete distributions and Ignatov and Kaishev (2004) for continuous distributions.

We believe there is a great potential in exploring the applicability of appropriate classes of multivariate distributions in modelling dependence of operational losses and insurance claims. Alternatively, copulas can serve the same purpose and for details of how this can be done we refer to Kaishev and Dimitrova (2006). Although copulas have recently gained popularity, there are a number of practical difficulties in their use, related to their multivariate versions of dimension greater than 2, the estimation of their parameters, based on data, and their appropriate parameterization. In particular there are only a few families of (multivariate) copulas which involve sufficiently many parameters, so as to be flexible enough and capture real world multivariate loss dependences. One such popular example is the family of Elliptical copulas to which Gaussian and t-copulas belong. For information on copulas we refer to the popular monograph by Nelsen (2006).

Further aspects of the methodology outlined in this section are discussed and illustrated numerically in Section 5.

#### 4. Capital assessment under insurance on operational losses

Another important aspect of modelling operational risk capital assessment, recognized under the AMA approach, is the effect on it of insurance on operational losses. The latter has been considered recently by Brandts (2004) and Bazzarello et al. (2006) where it is assumed that individual operational losses are insured with an external insurer under an excess of loss (XL) contract. Under this model there is a deductible  $d$  and a policy limit  $m$  and what is covered by the insurer is

$$W_i^r = \min(\max(W_i - d, 0), m), \quad i = 1, 2, \dots$$

whereas the net loss covered by the bank internal operational risk management (ORM) office is

$$W_i^c = W_i - W_i^r = \min(W_i, d) + \max(0, W_i - (d + m)), \quad i = 1, 2, \dots \quad (9)$$

Thus, under such an arrangement, there are two parties providing the operational loss cover, the bank (i.e., its ORM office) which plays the role of an internal direct insurer and the external insurer, which could be viewed as a reinsurer. The role of the latter party is essential and the probability of it defaulting has been considered by Brandts(2004) and by Bazzarello et al. (2006).

Here, we take a different approach, motivated by the observation that both parties share the operational risk they jointly cover, and hence in defining the total risk capital, allocated overall and split by the two parties, it is meaningful to consider their joint chances of not defaulting, i.e., to consider the probability of their joint survival. To follow details of this approach we will introduce some further notation.

Denote by  $Y_1^c = W_1^c$ ,  $Y_2^c = W_1^c + W_2^c$ , ... and by  $Y_1^r = W_1^r$ ,  $Y_2^r = W_1^r + W_2^r$ , ... the consecutive partial sums of operational losses to the bank ORM office and to the external insurer, respectively. Obviously in view of (9) we have that  $Y_i^c + Y_i^r = Y_i$ ,  $i = 1, 2, \dots$ , i.e., operational losses are shared. Under this XL reinsurance model, the total capital,  $h(t)$ , accumulated by the bank ORM office is also divided between the two parties so that  $h(t) = h_c(t) + h_r(t)$ , where  $h_c(t)$ , is the ORM office's capital accumulation function and  $h_r(t)$  models premium income of the external insurer, assumed also non-negative, non-decreasing functions on  $\mathbb{R}_+$ . As a result, the risk process,  $R(t)$ , can be represented as a superposition of two risk processes, that of the ORM office

$$R_i^c = h_c(t) - Y_{N_i}^c \quad (10)$$

and of the insurer

$$R_t^r = h_r(t) - Y_{N_t}^r \quad (11)$$

i.e.,  $R(t) = R_t^c + R_t^r$ .

Denote by  $P(T^c > x, T^r > x)$ , the probability of joint survival of the bank ORM office and the external insurer up to time  $x$ , where  $T^c$  and  $T^r$ , denoting the moments of ruin of the two parties, are defined as in (4), replacing  $R(t)$  with  $R_t^c$  and  $R_t^r$  respectively. Clearly, the two events  $(T^c > x)$  and  $(T^r > x)$ , of survival of the bank ORM office and the insurer are dependent since the two risk processes  $R_t^c$  and  $R_t^r$  are dependent through the common loss arrivals and the loss severities  $W_i$ ,  $i = 1, 2, \dots$  as seen from (10) and (11). This motivates us to consider the probability of joint survival,  $P(T^c > x, T^r > x)$ , as a joint measure of operational risk when operational losses are insured. The following risk capital allocation problem can then be formulated, which takes into account the fact that the two parties share the risk and the total capital accumulated.

**Problem 1.** For fixed deductible  $d$  and policy limit  $m$ , find capital accumulation functions  $h(t)$  and  $h_c(t)$  such that  $h(t) = h_c(t) + h_r(t)$  and

$$P(T^c > x, T^r > x) = 1 - \epsilon. \quad (12)$$

In order to solve this problem, the explicit expression for the probability of joint survival up to a finite time  $x$ , recently derived by Kaishev and Dimitrova (2006) can be used. According to Kaishev and Dimitrova (2006)

$$P(T^c > x, T^r > x) = e^{-\lambda x} \left( 1 + \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \dots \int_0^{h(x)-w_1-\dots-w_{k-1}} A_k(x; \tilde{v}_1, \dots, \tilde{v}_k) \psi(w_1, \dots, w_k) dw_k \dots dw_2 dw_1 \right) \quad (13)$$

where

$$\tilde{v}_j = \min(\tilde{z}_j, x), \tilde{z}_j = \max(h_c^{-1}(y_j^c), h_r^{-1}(y_j^r)), y_j^c = \sum_{i=1}^j w_i^c, y_j^r = \sum_{i=1}^j w_i^r, j = 1, \dots, k,$$

$$w_i^c = \min(w_i, d) + \max(0, w_i - (d + m)), w_i^r = \min(m, \max(0, w_i - d)), \text{ and}$$

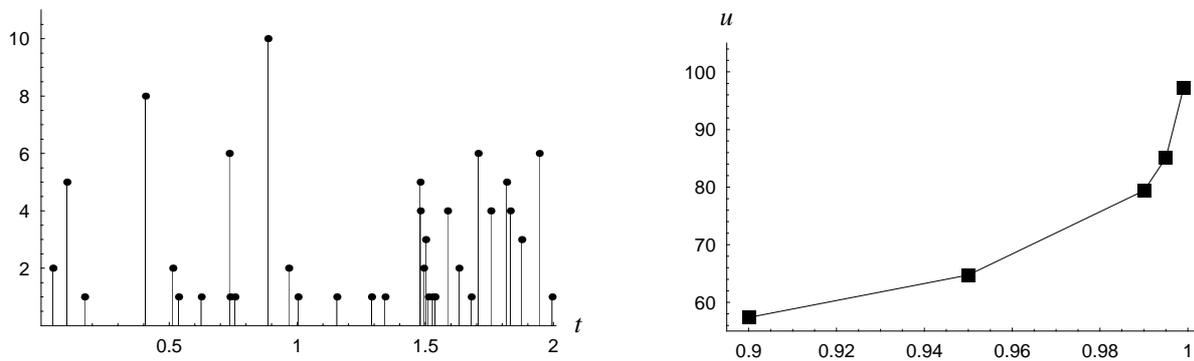
$A_k(x; \tilde{v}_1, \dots, \tilde{v}_k)$ ,  $k = 1, 2, \dots$  are the classical Appell polynomials  $A_k(x)$  of degree  $k$ , defined as in (7).

Let us note that expression (13) is a generalization of formula (7) which follows from (13) in the special case of  $m = 0$ . Formula (13) has been implemented using the *Mathematica* system and to follow its numerical performance (also in solving optimal reinsurance problems) we refer to Kaishev and Dimitrova (2006). Thus formula (13) can be successfully applied to represent the left-hand side of equation (12) and solve Problem 1. Numerical illustrations of

its solution are a subject of an extended version of this paper. Next we provide some numerical illustrations of the methodology described in Section 3

## 5. Numerical illustrations

In order to illustrate the methodology outlined in Section 3, we consider three alternative distributions of the consecutive losses. In our first example, operational risk losses are assumed i.i.d. with a discrete, logarithmic distribution, i.e.  $W_i \sim \text{Log}(\alpha)$  with a generic p.m.f.  $P(W = i) = -\alpha^i / (i \ln(1 - \alpha))$ . We have calibrated this distribution against operational risk loss data by taking  $\alpha = 0.73$  which allows us to approximately match its mean and variance to the Lognormal distribution fitted by Brandts (2004), (see Table 5 therein) to the aggregate losses from the 2002 LDCE data file. Of course, the logarithmic distribution we use, has lighter tail than the Lognormal one, but it suits our illustrative purposes here. A set of operational losses arriving in the interval  $[0, 2]$  with inter-arrival times distributed as  $\text{Exp}(20)$ , i.e.,  $\lambda = 20$  and with severities simulated from the  $\text{Log}(0.73)$  distribution are presented in the left panel of Fig. 1. In the right panel of Fig. 1, for  $h(t) = u + ct$ , we have presented values of the initial capital  $u$  for different choices of the probability of survival  $P(T > 2)$ , for fixed value of the rate  $c = 25$ .



**Fig. 1.** Left panel: Simulated operational loss data,  $W_i \sim \text{Log}(0.73)$ . Right panel: initial capital  $u$ , for choices of  $P(T > 2)$  equal to 90%, 95%, 99%, 99.5% and 99.9%,  $h(t) = u + 25t$ ,  $\tau_i \sim \text{Exp}(20)$ .

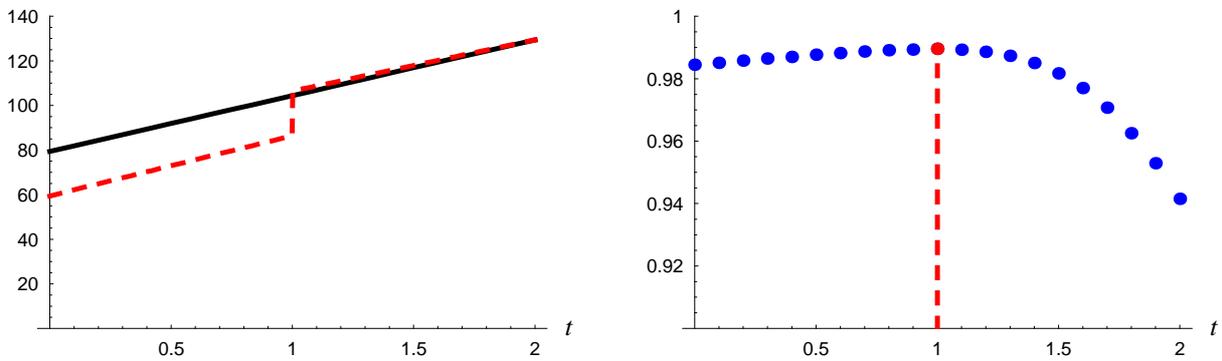
As can be seen, the capital charge  $u$  increases nonlinearly with the increase of the probability of survival, at a much higher rate as  $P(T > 2)$  approaches one. These calculations have been performed in *Mathematica*, solving (8) with  $P(T > 2)$  expressed by (6), applying the Newton algorithm. In particular,  $P(T > 2) = 0.99$  is achieved for  $h(t) = 79.4 + 25t$ . To illustrate the fact that the same probability 0.99 can be achieved by alternative choices of the capital accumulation function  $h(t)$ , we have next assumed that it belongs to the subclass of all piecewise linear functions on  $[0, x]$ , with one jump of size  $J$ , at some instant  $t_J \in [0, x]$ , i.e.,

$$h(t) = \begin{cases} u + c_1 t & , & 0 \leq t < t_J \\ u + c_1 t_J + J + s_1(t - t_J) & , & t_J \leq t \leq x \end{cases} ,$$

In the left panel of Fig. 2., two choices of  $h(t)$  are plotted,  $h_1(t) = 79.4 + 25 t$  and

$$h_2(t) = \begin{cases} 59.4 + 27 t & , & 0 \leq t < 1 \\ 59.4 + 27 + 20 + 23 (t - 1) & , & 1 \leq t \leq 2 \end{cases} .$$

As illustrated in the right panel of Fig. 2, moving the location  $t_J$  of the jump  $J = 20$  from  $t_J = 0$  to  $t_J = 2$ , while keeping the rest of the parameters fixed, we can see that a maximum of  $P(T > 2) = 0.99$  is achieved for  $t_J = 1$ . Indeed, both functions  $h_1(t)$  and  $h_2(t)$  provide equal chances of survival, 99% and also, accumulate equal risk capital at the end of the time interval,  $x = 2$ , i.e.  $h_1(2) = h_2(2) = 129.4$ . But obviously the choice  $h_2(t)$  is preferable since it requires less capital,  $u = 59.4$ , to be put aside initially, compared to  $u = 79.4$  for the choice  $h_1(t)$ .



**Fig. 2.** Left panel: Two choices of the capital accumulation function,  $h_1(t) = 79.4 + 25 t$  and  $h_2(t) = (59.4 + 27 t) \mathbb{1}_{\{0 \leq t < 1\}} + (59.4 + 27 + 20 + 23 (t - 1)) \mathbb{1}_{\{1 \leq t \leq 2\}}$ . Right panel:  $P(T > 2)$  as a function of the location  $t_J$  of the jump in  $h_2(t)$  of size  $J = 20$ .

In order to demonstrate how the methodology works under the assumption that losses have a continuous multivariate distribution, we consider two alternatives. In the first case, the severities of the consecutive risk losses  $W_i$ ,  $i = 1, 2, \dots$  are assumed independent, identically  $\text{Exp}(0.5)$  distributed, so that their mean matches the mean of the 2002 LDCE data. A simulation from the joint distribution of the severities of two i.i.d. risk losses,  $W_i \sim \text{Exp}(0.5)$ ,  $i = 1, 2$  is given in the upper left panel of Fig. 3. In the second case,  $W_i$ ,  $i = 1, 2, \dots$  are assumed dependent, with joint distribution function given by the Rotated Clayton copula,  $C^{\text{RCI}}(u_1, \dots, u_k; \theta)$  and  $\text{Exp}(0.5)$  marginals. Considering these two cases allows us to study the effect of dependence on the risk capital allocation, in particular on the size of the initial capital charge  $u$ .

The Rotated Clayton copula,  $C^{\text{RCI}}(u_1, \dots, u_k; \theta)$ , is defined as

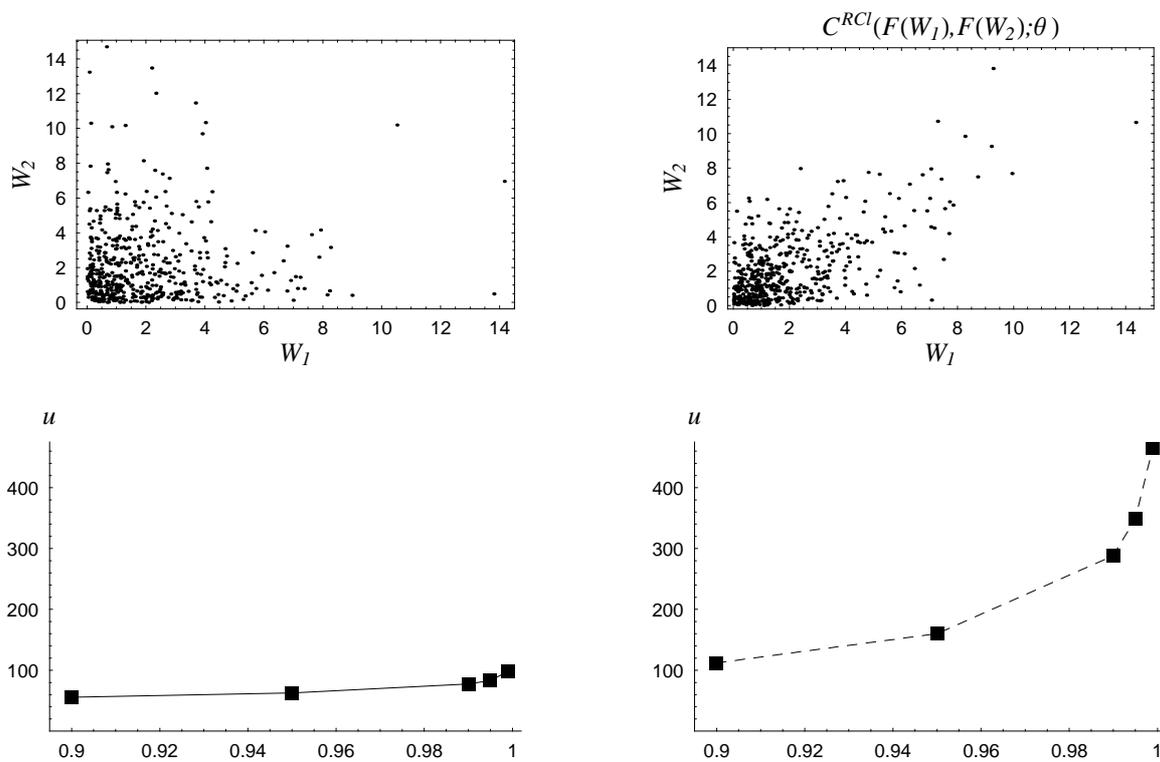
$$C^{\text{RCI}}(u_1, \dots, u_k; \theta) = \sum_{i=1}^k u_i - k + 1 + \left( \sum_{i=1}^k (1 - u_i)^{-\theta} - k + 1 \right)^{-1/\theta} , \quad (14)$$

with density  $c^{\text{RCI}}(u_1, \dots, u_k; \theta) = c^{\text{Cl}}(1 - u_1, \dots, 1 - u_k; \theta)$  and parameter  $\theta \in (0, \infty)$ . The value  $\theta = 0$  corresponds to independence. The Rotated Clayton copula has upper tail depen-

dence with coefficient  $\lambda_U = 2^{-1/\theta}$  and is suitable for modeling dependence between extreme operational losses.

Losses with dependence according to a Rotated Clayton copula with parameter  $\theta = 1$  and identical  $\text{Exp}(0.5)$  marginals, are illustrated in the upper right panel of Fig. 3 through a random sample of 500 data points. The presence of positive dependence, determined by  $\theta = 1$ , and of upper tail dependence,  $\lambda_U = 2^{-1}$ , are clearly visible.

We refer the reader to Kaishev and Dimitrova (2006) for further applications of this copula in modelling dependence of insurance claim severities combined with other (heavy-tailed) marginal distributions.



**Fig. 3.** Upper left panel: Simulated i.i.d.  $\text{Exp}(0.5)$  losses. Upper right panel: Simulated dependent losses following  $C^{\text{RCI}}(u_1, u_2; 1)$  with  $\text{Exp}(0.5)$  marginals. Lower left panel: initial capital  $u$ , for choices of  $P(T > 2)$  equal to 90%, 95%, 99%, 99.5% and 99.9% in the case of i.i.d.  $\text{Exp}(0.5)$  losses,  $h(t) = u + 25t$ ,  $\tau_i \sim \text{Exp}(20)$ . Lower right panel: initial capital  $u$ , for choices of  $P(T > 2)$  equal to 90%, 95%, 99%, 99.5% and 99.9% in the case of dependent losses following  $C^{\text{RCI}}(u_1, u_2; 1)$  with  $\text{Exp}(0.5)$  marginals,  $h(t) = u + 25t$ ,  $\tau_i \sim \text{Exp}(20)$ .

The lower left and right panels of Fig. 3 illustrate the heavy impact of dependence between loss severities on the value of the initial capital charge  $u$ , given  $h(t) = u + 25t$ ,  $x = 2$  and Poisson inter-arrival times  $\tau_i \sim \text{Exp}(20)$ . As can be seen, in order to achieve survival probability  $P(T > 2) = 0.90$  the capital charge  $u = 55.7$  in the case of i.i.d.  $W_i \sim \text{Exp}(0.5)$  and  $u = 112$  assuming dependence. Furthermore, if a probability of  $P(T > 2) = 0.999$  is to be achieved the corresponding values are  $u = 98.3$  for i.i.d. losses and  $u = 446$ , for dependent

losses, which is 4.54 times higher. The values of the capital charge  $u$  have been calculated solving (8), with  $P(T > 2)$  given by formula (7).

Of course, the choice of the Rotated Clayton copula with parameter  $\theta = 1$ , leads to Kendall's  $\tau = 0.33$  and upper tail dependence  $\lambda_U = 0.5$ , which tailors a reasonably strong dependence, so the result could be a bit extreme, but convincingly illustrates the importance of considering dependence when setting operational risk capital charge. In conclusion, we have demonstrated that the proposed methodology which is based on solving (8) and (12) using the explicit formulae (6), (7) and (13) is a promising modelling tool for (dynamic) operational risk capital allocation.

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