ECONOMIC CAPITAL ALLOCATIONS FOR NON-NEGATIVE PORTFOLIOS OF DEPENDENT RISKS

Topic: Risk Management of an Insurance Enterprize

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Abstract. This paper explores the problem of economic capital allocations in the context of non-negative multivariate (insurance) risks possessing a dependence structure. We derive a general result and illustrate it by useful examples. One such example, for instance, develops explicit expressions for the discussed economic capital decomposition rule when the underlying portfolio consists of dependent compound Poisson risks.

Keywords and phrases: Economic capital, economic capital allocation, tail conditional expectation risk measure, multivariate non-negative dependent risks.

1. Introduction and motivation

In recent years, an increasing number of financial conglomerates have adopted economic (risk) capital frameworks. According to various authors (cf., e.g., Zaik *et al.*, 1996) two central elements of such frameworks are: 1) holding sufficient capital to cover risks, and 2) allocating economic capital to each operating division or department.

At the international level, the immense importance of the aforementioned subjects can be clearly seen, on the one hand, in the European Commission's plans to apply the Basel II Bank Capital Adequacy Accord to all banks and investment firms in the European Union, and on the other, in the Commission's targets for the risk-based supervision of EU insurance companies, known as the Solvency II project.

At the national level, regulators around the world are increasingly applying the principles of the risk-based approach to all financial institutions under their jurisdiction. In Israel, for instance, programs similar to the Basel II and Solvency II projects have been developed and are gradually being implemented.

Although the phenomena discussed above are becoming mandatory in nature in many countries, their practical utilization is far from straightforward. The complexity involved is in general explained by the following three main cornerstones of successful risk measurement (and therefore of the subsequent risk management):

1. The multivariate probabilistic model possessing a convenient dependence structure - to describe risks' behavior;
2. The choice of appropriate risk functional - to translate the implications of the model into risk parlance;
3. The (analytic) solutions for the latter in the framework of the former - to, actually, measure risk numerically.

In this paper we address all of these three issues. Namely, we consider portfolios of (insurance) risks following the multivariate probabilistic model which has recently been proposed by Furman and Landsman (2007). The aforementioned class of distributions is referred to as the multivariate Tweedie family (MTwF), and it seems to answer well such peculiarities of the insurance industry demands as non-negative support, unimodality, positive skewness, and tolerance for large risks. As special cases, MTwF contains the multivariate inverse Gaussian, multivariate gamma, multivariate stable, and multivariate compound Poisson distributions in the sense that their univariate marginals are inverse Gaussian, gamma, positive stable, and compound Poisson, respectively. Furthermore, MTwF possesses a dependence structure, which is reflected in its covariance structure and allows for efficient modeling of multivariate portfolios of dependent risks (cf. loc. cite or Section 3 below for more details).

Relating to the second cornerstone above, we build the economic capital analysis developed here on the popular *tail conditional expectation* (TCE) risk measure and the allocation rule based on it. Literally speaking, the former provides the necessary economic capital for the whole financial institution, whereas the latter resolves the problem of its subsequent allocation to various departments (operating divisions, sources).

More precisely, let us denote by F and $\overline{F} = 1 - F$ the cumulative distribution function (cdf) and the decumulative distribution function (ddf), respectively, of a non-negative random variable (rv) X representing the risk. The tail conditional expectation risk measure, which coincides with the *expected shortfall* (ES) and the *conditional Value-at-Risk* (CVaR) under the assumption of continuous distributions (cf., e.g., Hürlimann 2003; McNeil, Frei, Embrechts, 2005, Lemma 2.16), is then formulated as

$$TCE_q[X] = \mathbf{E}\left[X | X > VaR_q[X]\right] = \frac{1}{\overline{F}(VaR_q[X])} \int_{VaR_q[X]}^{\infty} x dF(x), \tag{1.1}$$

subject to $\overline{F}(VaR_q[X]) > 0$ and

$$VaR_q[X] = \inf\{x : F(x) \geq q\}. \tag{1.2}$$

Functional (1.1) possesses a number of appealing properties, which make it an attractive risk measure (cf. Artzner *et al.*, 1999; Acerbi & Tasche, 2002; Tasche 2002). Very briefly speaking, TCE is sub-additive, scale- and translation-invariant, and satisfies first and second stochastic dominances (cf. Kaas *et al.*, 2001).

An alternative way to interpret equation (1.1) is to consider it a premium calculation principle (pcp), where the safety loading is determined by $VaR_q[X]$ and is therefore proportional to the probability q. When one chooses this way of reasoning, one is in general interested in relatively small q values, as distinct from the more common situations in the banking sector, when TCE denotes a risk measure, and thus q is usually taken to be larger than 0.99 (cf. Furman & Landsman, 2006a).

Another useful observation about TCE, made by Denault (2001) and Panjer and Jia (2001), implies that it can provide a natural decomposition of the total economic capital to its various constituents. Indeed, due to the additivity property of the expectation operator, one obtains that the "risk contribution" of the k-th business line to the total risk $S = X_1 + X_2 + \cdots + X_n$ of the conglomerate is formulated as

$$TCE_q[X_k|S] = \mathbf{E}\left[X_k|S > VaR_q[S]\right]. \tag{1.3}$$

It should be noted that a significant number of risk measures other than TCE has been proposed, starting with the arguably oldest Value-at-Risk (cf. Leavens, 1945), and up to the *distorted* risk measures (cf. Denneberg, 1994; Wang, 1995, 1996; Wang *et al.*, 1997). In addition, Furman and Landsman (2006b) proposed some tail variance-based risk measures, which on the one hand generalize equation (1.1), and on the other provide a tail-based counterpart to the classical variance and standard deviation pcp's. In general, the debate on what risk measure to apply in a given situation at hand is far from being over yet; however, this rather designing issue is far beyond the purposes of the discussion presented here.

The main concern of this paper is to attempt to evaluate (1.3) analytically, given the dependent portfolio of risks following the multivariate Tweedie distributions. We note that Panjer and Jia (2001), Landsman and Valdez (2003) and Cai and Li (2005) considered similar problems in the context of the multivariate normal, multivariate elliptical, and multivariate phase type distributions, correspondingly. In the present paper, we evaluate (1.1) and (1.3) in the general context of the multivariate Tweedie family, and we then illustrate our results by assuming multivariate dependent compound Poisson distributions, which is an important particular case of MTwF.

We further proceed as follows: Section 2 discusses in detail some existing general results for non-negative independent risks. In spite of the restricting assumption of independence, these results allow valuable insight into the more interesting dependent problems. Then, Section 3 briefly introduces the multivariate Tweedie distributions and the particular case of interest in this paper: the multivariate compound Poisson family. Section 4 develops general expressions for equations (1.1) and (1.3) in the context of MTwF, and Section 5 illustrates the results obtained with some examples. Section 6 concludes the paper and discusses its outcomes.

2. TCE and the economic capital allocation for non-negative independent risks

In this section we review some known results which are of interest. Unless otherwise stated, the results discussed appear in Furman and Landsman (2005).

Let the non-negative rv X have a finite expectation $\mathbf{E}[X] < \infty$. The TCE risk measure of X turns out to be proportional to the expectation of X, as can be seen from the following note.

Note 2.1. *Let $X^* \backsim F_{X^*}$ denote the associated with X rv with the cdf*

$$F_{X^*}(x) = \frac{\mathbf{E}[\mathbf{1}(X \le x)]}{\mathbf{E}[X]} = \frac{1}{\mathbf{E}[X]} \int_0^x t dF(t),$$

where $\mathbf{1}(\mathcal{A})$ is the indicator function of the set $\mathcal{A}$. Then, the tail conditional expectation risk measure of the risk X can be formulated as

$$TCE_q[X] = \mathbf{E}[X] \frac{\overline{F}_{X^*}(VaR_q[X])}{\overline{F}(VaR_q[X])}. \tag{2.1}$$

Representation (2.1) seems to be not only attractive but also useful. For instance, noting that X^* is stochastically greater than X, i.e., $P(X^* > x) \ge P(X > x)$ (cf., e.g., Patil & Rao, 1978), representation (2.1) immediately implies that $TCE_q[X] \ge \mathbf{E}[X]$, with equality only if $X^* \stackrel{d}{=} X$, where "$\stackrel{d}{=}$" stands for equality in distribution.

Another useful consequence of equation (2.1) is the appealing easiness of calculating TCE for such important rv's as gamma, Pareto, lognormal, Weibull, beta with continuous supports, and Poisson, binomial, negative binomial, logarithmic series, and hypergeometric with discrete supports. All these rv's are form-invariant with respect to their weighted counterparts (cf., e.g., Patil & Ord, 1976), and hence TCE straightforwardly follows using Tables 1,2 in loc. cite.

Let us further consider rv's X_j possessing cdf's F_{X_j}. Also, denote the aggregate risk by $S = X_1 + X_2 + \cdots + X_n \backsim F$, and its associated counterpart by S^* . Clearly, TCE of S is

$$TCE_q[S] = \mathbf{E}[S] \frac{\overline{F}_{S^*}(VaR_q[S])}{\overline{F}(VaR_q[S])}. \tag{2.2}$$

We are now in a position to consider the more general equation (1.3).

Lemma 2.1. *The risk contribution of X_j to S is formulated in terms of TCE as*

$$TCE_q[X_j|S] = \mathbf{E}[X_j]\frac{\overline{F}_{S-X_j+X_j^*}(VaR_q[S])}{\overline{F}(VaR_q[S])}, \tag{2.3}$$

where X_j^* is the associated counterpart of X_j.

In light of the latter expression, we are in general interested in the convolutions $S-X_j+X_j^*$. In some particular situations, these convolutions turn out to strongly simplify. Indeed, given some non-negative rv ξ_j independent on X_j, and if the associated counterpart of the latter rv can be rewritten as $X_j^* = X_j + \xi_j$, equation (2.3) reduces to

$$TCE_q[X_j|S] = \mathbf{E}[X_j]\frac{\overline{F}_{S+\xi_j}(VaR_q[S])}{\overline{F}(VaR_q[S])}. \tag{2.4}$$

To elucidate formula (2.4), we consider the two following examples.

Example 2.1. *Let $X_j,\ j=1,\dots,n$ be mutually independent rv's distributed gamma with the shape and rate parameters equal to γ_j and α_j, respectively, i.e., $X_j \backsim Ga(\gamma_j,\alpha_j)$. In such a case, for $\gamma=\gamma_1+\gamma_2+\cdots+\gamma_n$, $\overline{\alpha}=\max(\alpha_1,\alpha_2,\dots,\alpha_n)$ and setting without loss of generality $\alpha_j=\overline{\alpha}$, expression (2.3) can be rewritten as*

$$TCE_q[X_j|S] = \frac{\gamma_j}{\alpha_j}\frac{\overline{G}(VaR_q[S];\gamma+K+1,\overline{\alpha})}{\overline{G}(VaR_q[S];\gamma+K,\overline{\alpha})}, \tag{2.5}$$

where $\overline{G}(\cdot;\gamma,\alpha) = 1-G(\cdot;\gamma,\alpha)$ denotes the ddf of some gamma distributed rv with the shape and rate parameters equal to γ and α, respectively, and K is a specific discrete rv (cf. Furman & Landsman, 2005).

Formula (2.5) readily follows after noticing that in this case $X_j^* \backsim Ga(\gamma_j+1,\alpha_j)$, and therefore it can be represented by the convolution of two independent gamma rv's with common rate α_j and shapes equal to γ_j and 1. Therefore, representation (2.4) holds in this special case, where $\xi_j \backsim Ga(1,\alpha_j)$, and S is the convolution of n independent gamma rv's with shapes γ_j and rates α_j.

In our second example, equation (2.4) is satisfied again. Moreover, the rv ξ_j is equal to 1 over its whole domain, i.e., $P(\xi_j=1)=1$.

Example 2.2. *Let $X_j,\ j=1,\dots,n$ be mutually independent rv's distributed Poisson with λ_j, i.e., $X_j \backsim Poisson(\lambda_j)$. Then, it can be shown that $X_j^* \overset{d}{=} X_j+1$, and therefore, for*

$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, we have that

$$TCE_q[X_j|S] = \lambda_j \frac{\overline{Po}(VaR_q[S]-1;\lambda)}{\overline{Po}(VaR_q[S];\lambda)}, \tag{2.6}$$

where $\overline{Po}(\cdot;\lambda) = 1 - Po(\cdot;\lambda)$ denotes the ddf of a Poisson rv parametrized by λ.

It can be then noted that the contribution of the riskiness of X_j to S is stipulated in this special case by the expectation $\mathbf{E}[X_j] = \lambda_j$, only. Consequently, and due to equation (2.1), the relative contribution of the above mentioned riskiness is

$$\frac{TCE_q[X_j|S]}{TCE_q[S]} = \frac{\mathbf{E}[X_j]}{\mathbf{E}[S]}, \tag{2.7}$$

which is surprisingly very simple. We note that (2.7) does not hold in general (cf. Example 2.1 above).

To conclude, it should be noted that, although $X_1, \ldots, X_n$ are independent, this is not the case for the pair X_j and S which is certainly dependent. In light of this, Note 2.1 and Lemma 2.1 may prove to be of high practical importance. Other valuable applications of the aforementioned results are demonstrated in Section 4 below.

In the next section we abandon the assumption of independence, and we introduce the multivariate Tweedie family of distributions, along with its member which is of particular interest in this paper, the multivariate compound Poisson distribution.

3. Multivariate Tweedie distributions

Exponential dispersion models play a prominent role in statistics and actuarial science. This can be explained by the high level of generality that EDMs enable in the context of statistical inference for such widely popular distribution functions as normal, gamma, inverse Gaussian, stable, and many others. The specificity characterizing statistical modeling of actuarial subjects is that the underlying distributions mostly have non-negative supports and many EDMs possess this important phenomenon.

Although univariate EDMs are considerably rich and widely applied, in the multivariate context the case is very different. Unfortunately, the so-called natural multivariate EDMs are not as rich as the univariate ones. Namely, they do not include important multivariate distributions whose univariate marginals are, say, inverse Gaussian, gamma or compound Poisson. Moreover, the only valuable continuous member of such a natural multivariate extension of the univariate EDMs is the multivariate normal distribution (cf. Bildicar & Patil, 1968 for more details).

To overcome the aforementioned penury, Furman and Landsman (2007) proposed a new multivariate Tweedie[1] family of distributions. To follow their reasoning briefly, let $\mathbf{Y} = (Y_0, Y_1, \ldots, Y_n)^T$ be a random vector consisting of $(n+1)$ mutually independent additive Tweedie rv's, i.e., $Y_i \backsim Tw_p(\theta_i, \lambda_i)$, $i = 0, 1, \ldots, n$, possessing, under certain conditions, probability distribution functions (pdf's) or the probability mass functions (pmf's) of the form

$$f(x) = h(x; \lambda) \exp\left(\theta x - \lambda \kappa_p(\theta)\right), \tag{3.1}$$

where, for $\alpha = (p-2)/(p-1)$,

$$\kappa_p(\theta) = \begin{cases} e^{\theta}, & p = 1 \\ -\log(-\theta), & p = 2 \\ \frac{(\alpha-1)}{\alpha}\left(\frac{\theta}{\alpha-1}\right)^{\alpha}, & p \neq 1, 2 \end{cases}. \tag{3.2}$$

Also, let

$$A = \begin{pmatrix} \frac{\theta_0}{\theta_1} & 1 & 0 & 0 & \cdots & 0 \\ \frac{\theta_0}{\theta_2} & 0 & 1 & 0 & \cdots & 0 \\ \frac{\theta_0}{\theta_3} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\theta_0}{\theta_n} & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{3.3}$$

Then

Theorem 3.1. *(cf. loc. cite) Let $\mathbf{X} = A\mathbf{Y}$. Then, the j-th univariate marginal of $\mathbf{X}$ is*

$$\frac{\theta_0}{\theta_j} Y_0 + Y_j = X_j \backsim \begin{cases} Tw_1(\theta, \lambda_0 + \lambda_j), & p = 1 \\ Tw_2(\theta_j, \lambda_0 + \lambda_j), & p = 2 \\ Tw_p\left(\theta_j, \lambda_0 \left(\frac{\theta_0}{\theta_j}\right)^{\alpha} + \lambda_j\right), & p \neq 1, 2 \end{cases},$$

where $\alpha = (p-2)/(p-1)$.

The theorem ensures that MTwF possesses univariate marginals belonging to univariate Tweedie EDMs, and it establishes these marginal distributions explicitly. The multivariate Tweedie family of distributions can now be naturally defined.

[1]EDMs are classified by their unit variance functions $V(\cdot)$. EDMs having power form variance function $V(\mu) = \mu^p, p \in (-\infty, 0] \cup [1, \infty)$ are called Tweedie EDMs (cf., e.g., Jorgensen, 1997).

Definition 3.1. *The joint distribution of* $\mathbf{X} = A\mathbf{Y}$*, denoted by* $\mathbf{X} \backsim Tw_{n,p}\left(\boldsymbol{\theta}, \widetilde{\boldsymbol{\lambda}}\right)$*, is the* n*-variate additive Tweedie distribution. Here,* $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subset \mathbf{R}^n$ *is the* n*-variate vector of canonical parameters where* $\boldsymbol{\Theta}$ *is the Cartesian product of the domains* $\Theta \subset \mathbf{R}$*.*

It can be shown that MTwF is obtained as a solution of a generalized Cauchy's functional equation (cf. *loc. cite*). In view of the above, the multivariate Tweedie family is the only possible multivariate extension of EDMs, given matrix (3.3) and Definition 3.1.

Choosing an appropriate p parameter, we instantly arrive at, say, the multivariate Poisson ($p = 1$), multivariate gamma ($p = 2$), multivariate inverse Gaussian ($p = 3$), and multivariate compound Poisson ($p \in (1, 2)$) distributions. It should be emphasized that even more flexibility in modeling (insurance) risks can be obtained by considering multivariate Tweedie distributions corresponding to the non-integer p parameters.

More apparently useful properties of $\mathbf{X}$, such as higher order moments and their products, multivariate pdf's, multivariate additivity characteristics, and some Chebyshev's type inequalities, can be found in the *loc. cite.*

Figures 3.1, 3.2, 3.3 and 3.4 compare some bivariate members of MTwF with independent and dependent univariate marginal distributions. It can be clearly observed that in the framework of the proposed family dependent risks are riskier than the independent ones.

3.1. **The multivariate compound Poisson distributions.** We now consider $Tw_{n,p}(\boldsymbol{\theta}, \tilde{\boldsymbol{\lambda}})$ when p is in the interval $(1, 2)$ (in what follows, it is denoted by $Tw_{n,(1,2)}(\boldsymbol{\theta}, \tilde{\boldsymbol{\lambda}})$). This case relates to the multivariate compound Poisson distributions with gamma severities, and it is very important in, say, the insurance industry due to the fact that it allows for randomness both in claims' frequencies and amounts.

Let $N_0, N_1, \ldots, N_n$ denote a sequence of mutually independent Poisson rv's, such that, for $i = 0, 1, \ldots, n$, $N_i \backsim Poisson(\lambda_i \kappa_p(\theta))$. Also, for $k = 0, 1, \ldots$, let $Y_{i,k} \backsim Ga(\theta, -\alpha)$, be mutually independent and identically distributed gamma rv's independent of N_i, where $\theta, \alpha < 0$ and $\alpha = (p-2)/(p-1)$.

Denote by

$$Y_i = \begin{cases} \sum_{k=1}^{N_i} Y_{i,k}, & N_i > 0 \\ 0, & N_i = 0 \end{cases},$$

the compound Poisson distribution with gamma claim severities, which can also be written as $Y_i \backsim Tw_{(1,2)}(\theta, \lambda_i)$, and is therefore referred to as the Tweedie compound Poisson distribution. Then, the pdf of Y_i can be formulated as

$$f_{Y_i}(y) = h(y;\lambda)\exp\left(\theta y - \lambda_i \kappa_p(\theta)\right),$$

with

$$h(y;\lambda) = \sum_{n=1}^{\infty} \frac{\left(\lambda\kappa_p\left(-\frac{1}{y}\right)\right)^n}{y\Gamma(-n\alpha)\, n!}$$

for $y > 0$, and

$$f_{Y_i}(0) = P(Y_i = 0) = P(N = 0) = \exp\left(-\lambda_i \kappa_p(\theta)\right),$$

otherwise.

We now have the necessary random vector $\mathbf{Y} = (Y_0, Y_1, \ldots, Y_n)^T$ and, considering common θ parameters and applying Definition 3.1, we arrive at the resulting multivariate compound Poisson distribution, that is $\mathbf{X} \backsim Tw_{n,(1,2)}(\boldsymbol{\theta}, \tilde{\boldsymbol{\lambda}})$, where $\boldsymbol{\theta}$ is a vector of θ's and $\tilde{\boldsymbol{\lambda}} = (\lambda_0 + \lambda_1, \lambda_0 + \lambda_2, \ldots, \lambda_0 + \lambda_n)^T$ (cf. Theorem 3.1).

Also, we have that the aggregate risk is given by

$$S = \sum_{j=1}^{n} X_j = n\sum_{k=1}^{N_0} Y_{0,k} + \sum_{j=1}^{n}\sum_{k=1}^{N_j} Y_{j,k}, \tag{3.4}$$

and thus, the distribution of S is not a Tweedie compound Poisson. Indeed, 1) after setting $\theta_0/\theta_j = n$ and $\lambda_j = 0$ in Theorem 3.1, we obtain that

$$nY_0 = n\sum_{k=1}^{N_0} Y_{0,k} \backsim Tw_{(1,2)}(\theta/n, \lambda_0 n^{\alpha}),$$

and 2) the distribution of the double sum in equation (3.4) is

$$\sum_{k=1}^{N_j} Y_{j,k} = \sum_{j=1}^{n}\sum_{k=1}^{N_j} Y_{j,k} \backsim Tw_{(1,2)}(\theta, \lambda),$$

where $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. In light of this, the distribution of S is a compound Poisson with Poisson parameter $\Lambda = \lambda_0 n^{\alpha} + \lambda$ and the corresponding risk distribution possessing the following cdf

$$F(s) = \frac{1}{\Lambda}\left(\lambda_0 n^{\alpha} G(s;\theta/n, -\alpha) + \lambda G(s;\theta,-\alpha)\right).$$

We conclude this section with other useful properties of Tweedie compound Poisson rv's:

- The expectation of X_j is $\mathbf{E}[X_j] = (\lambda_0 + \lambda_j)\kappa_p(\theta)\alpha\theta^{-1}$;
- The variance of X_j is $\mathbf{Var}[X_j] = (\lambda_0 + \lambda_j)\kappa_p(\theta)(\alpha - 1)\alpha\theta^{-2}$;
- The correlation between X_i and X_j, $i \neq j$ is $\mathbf{Corr}[X_i, X_j] = \frac{\lambda_0}{\sqrt{(\lambda_0+\lambda_j)(\lambda_0+\lambda_i)}}$.

4. TCE and based on it capital-at-risk decomposition rule

We have so far considered two of three pillars of the so-called advanced risk management, i.e., the multivariate probabilistic model and the risk functional to measure the degree of riskiness it implies. In the two following theorems we evaluate equations (1.1) and (1.3) in the general framework of MTwF. The former theorem derives the economic capital for the whole financial conglomerate, whereas the latter establishes the capital to be set aside by the j-th operating division given the existence of the conglomerate along with its dependence structure.

Recall, that we denote by $S = X_1 + X_2 + \cdots + X_n$ the aggregate risk of the financial conglomerate, which, according to Definition 3.1, is generally written as

$$S = \eta Y_0 + \sum_{j=1}^{n} Y_j, \tag{4.1}$$

subject to $\eta = \sum_{j=1}^{n} \frac{\theta_0}{\theta_j}$.

Theorem 4.1. *Let $\mathbf{X} \backsim Tw_{n,p}\left(\boldsymbol{\theta}, \tilde{\boldsymbol{\lambda}}\right)$ be an MTwF random vector. The tail conditional expectation risk measure for S is then formulated as*

$$TCE_q\left[VaR_q[S]\right] = \mathbf{E}\left[\eta Y_0\right] \frac{\overline{F}_{S-\eta Y_0+(\eta Y_0)^*}\left(VaR_q[S]\right)}{\overline{F}\left(VaR_q[S]\right)} + \mathbf{E}\left[Y\right] \frac{\overline{F}_{S-Y+Y^*}\left(VaR_q[S]\right)}{\overline{F}\left(VaR_q[S]\right)}, \tag{4.2}$$

where $Y = \sum_{j=1}^{n} Y_j$, and Y^* denotes the associated counterpart of Y.

Proof. Follows from the additive property of the expectation operator and Lemma 2.1. □

Theorem 4.2. *Let $\mathbf{X} \backsim Tw_{n,p}\left(\boldsymbol{\theta}, \tilde{\boldsymbol{\lambda}}\right)$ be an MTwF random vector. Then the economic capital allocation based on TCE is*

$$TCE_q[X_j|S] = \frac{\theta_0}{\theta_j}\mathbf{E}\left[Y_0\right] \frac{\overline{F}_{S-\eta Y_0+(\eta Y_0)^*}\left(VaR_q[S]\right)}{\overline{F}\left(VaR_q[S]\right)} + \mathbf{E}\left[Y_j\right] \frac{\overline{F}_{S-Y_j+Y_j^*}\left(VaR_q[S]\right)}{\overline{F}\left(VaR_q[S]\right)}. \tag{4.3}$$

Proof. First note that

$$TCE_q[X_j|S]\left[VaR_q[S]\right] = \frac{\theta_0}{\theta_j}\mathbf{E}\left[Y_0|S > VaR_q[S]\right] + \mathbf{E}\left[Y_j|S > VaR_q[S]\right]$$

according to the additive property of conditional expectations.

Though the expression for the first term of the right hand side of the above equation does not follow from Lemma 2.1 directly, it can be derived in a similar fashion, i.e.,

$$\begin{aligned}\mathbf{E}\left[Y_0|S > Var_q[S]\right] &= \mathbf{E}[Y_0] - \mathbf{E}[Y_0 \mathbf{1}(S \le VaR_q[S])] \\ &= \frac{\mathbf{E}\left[Y_0\right] - \int_0^{VaR_q[S]/\eta} u f_{Y_0}\left(u\right) F_{S-\eta Y_0}\left(VaR_q[S] - \eta u\right) du}{\overline{F}\left(VaR_q[S]\right)},\end{aligned}$$

which after the change of variables $t = \eta u$ and noticing that $f_{Y_0}\left(u\right) = \eta f_{\eta Y_0}\left(t\right)$, yields

$$\begin{aligned}\mathbf{E}\left[Y_0|S > VaR_q[S]\right] &= \frac{\mathbf{E}\left[Y_0\right] - \int_0^{VaR_q[S]} \frac{1}{\eta} t f_{\eta Y_0}\left(t\right) F_{S-\eta Y_0}\left(VaR_q[S] - t\right) dt}{\overline{F}\left(VaR_q[S]\right)} \\ &= \frac{\mathbf{E}\left[Y_0\right] - \mathbf{E}\left[\eta Y_0\right] \int_0^{VaR_q[S]} \frac{1}{\eta} f_{(\eta Y_0)^*}\left(t\right) F_{S-\eta Y_0}\left(VaR_q[S] - t\right) dt}{\overline{F}\left(VaR_q[S]\right)} \\ &= \frac{\mathbf{E}\left[Y_0\right]\left(1 - \int_0^{VaR_q[S]} F_{S-\eta Y_0}\left(VaR_q[S] - t\right) dF_{(\eta Y_0)^*}\left(t\right)\right)}{\overline{F}\left(VaR_q[S]\right)} \\ &= \mathbf{E}\left[Y_0\right] \frac{\overline{F}_{S-\eta Y_0 + (\eta Y_0)^*}\left(VaR_q[S]\right)}{\overline{F}\left(VaR_q[S]\right)}. \qquad (4.4)\end{aligned}$$

The expression for $\mathbf{E}\left[Y_j|S > VaR_q[S]\right]$ follows directly from Lemma 2.1 and can be formulated as

$$\mathbf{E}\left[Y_j|S > VaR_q[S]\right] = \mathbf{E}\left[Y_j\right] \frac{\overline{F}_{S-Y_j+Y_j^*}\left(VaR_q[S]\right)}{\overline{F}\left(VaR_q[S]\right)}, \qquad (4.5)$$

which completes the proof. □

In what follows we shall illustrate Theorems 4.1 and 4.2 by an example where the underlying random vector follows Tweedie compound Poisson distributions.

5. Economic capitals and multivariate dependent compound Poisson distributions

In the previous section and in Section 2, we pointed out that closure under the *associated transform* essentially simplifies calculations of TCE and the economic capital decomposition rule based on it. We further show that, although the aforementioned closure does not hold in the context of Tweedie compound Poisson rv's, a convenient representation of the associated counterpart can still be found.

Let $S \backsim Tw_{(1,2)}(\theta,\lambda)$ be the univariate Tweedie compound Poisson rv. We establish the second part of the above statement as Lemma 5.1, while the first part follows from

$$f_{S^*}(s)=\frac{sf\left(s\right)}{\mathbf{E}\left[S\right]}=\exp\left(\theta s-\lambda\kappa_p\left(\theta\right)\right)\sum_{k=1}^{\infty}\frac{\left(-\theta s\right)^{-k\alpha+1}}{s\Gamma\left(-k\alpha+1\right)}\frac{\left(\lambda\kappa_p\left(\theta\right)\right)^{k-1}}{\left(k-1\right)!},$$

which is of the form

$$h(s;\lambda,\theta)\exp\left(\theta x-\lambda\kappa_p\left(\theta\right)\right),$$

and therefore S^* is not a Tweedie compound Poisson. In fact, it is not an EDM either, since $h(\cdot)$ is dependent on θ.

Lemma 5.1. *Let $S \backsim Tw_{(1,2)}(\theta,\lambda)$ and S^* be its associated counterpart. Then $S^* \overset{d}{=} S+\xi$, where $\xi \backsim Tw_2(\theta,-\alpha-1)$.*

Proof. Let $tw_2(\cdot;\theta,-\alpha)$ denote the pdf of Tweedie rv with $p=2$ (gamma distribution), and N be Poisson rv having pdf $tw_1(\cdot;\lambda)$. Then, we can rewrite the density of S^* as

$$\begin{aligned}
f_{S^*}\left(s\right) &= \sum_{k=1}^{\infty}e^{-\lambda\kappa_p(\theta)}\frac{\left(\lambda\kappa_p\left(\theta\right)\right)^{k-1}}{\left(k-1\right)!}e^{\theta s}\frac{s^{-k\alpha}\left(-\theta\right)^{-k\alpha+1}}{\Gamma\left(-k\alpha+1\right)}\\
&= \sum_{k=1}^{\infty}tw_1\left(k-1;\lambda\kappa_p(\theta)\right)\cdot tw_2\left(s;\theta,-k\alpha+1\right)\\
&= \sum_{k=1}^{\infty}tw_1\left(k-1;\lambda\kappa_p(\theta)\right)\cdot\left(tw_2\left(s;\theta,-\left(k-1\right)\alpha\right)\circ tw_2\left(s;\theta,-\alpha+1\right)\right),
\end{aligned}$$

where $\circ$ stands for convolution.

Further, conditioning on $N+1=k$, we arrive at

$$f_{S^*}\left(s\right)=\sum_{k=1}^{\infty}f_{N+1}\left(k\right)\cdot f_{(S+\xi)|N+1=k}\left(s\right)=f_{S+\xi}\left(s\right),$$

which completes the proof. □

We can now formulate the TCE risk measure of S as follows

Theorem 5.1. *Let $S \backsim Tw_{(1,2)}(\theta,\lambda)$, then we have that*

$$TCE_S\left[VaR_q[S]\right]=\lambda\kappa_p\left(\theta\right)\frac{\alpha}{\theta}\frac{\overline{F}_{S+\xi}\left(VaR_q[S]\right)}{\overline{F}\left(VaR_q[S]\right)}.$$

Proof. Follows from Lemmas 2.1 and 5.1. □

Another way to prove Theorem 5.1 is by the law of total (iterated) expectations. More precisely

$$TCE_q\left[VaR_q[S]\right] = \mathbf{E}\left[S|S > VaR_q[S]\right] = \mathbf{E}\left[\mathbf{E}\left[S|S > VaR_q[S]|N\right]\right].$$

Then, the right-most side of the above equation is written as

$$\mathbf{E}\left[\mathbf{E}\left[S|S > VaR_q[S]|N = n\right]\right] = \frac{n\alpha}{\theta}\frac{\overline{G}\left(VaR_q[S]; -n\alpha+1, \theta\right)}{1-q},$$

and consequently

$$\begin{aligned} TCE_q\left[VaR_q[S]\right] &= \frac{1}{1-q}\sum_{k=0}^{\infty}\frac{k\alpha}{\theta}\overline{G}\left(VaR_q[S]; -k\alpha+1, \theta\right)e^{-\lambda\kappa_p(\theta)}\frac{\left(\lambda\kappa_p\left(\theta\right)\right)^k}{k!} \\ &= \frac{\lambda\kappa_p\left(\theta\right)}{1-q}\frac{\alpha}{\theta}\int_{VaR_q[S]}^{\infty}\sum_{k=0}^{\infty}e^{\theta x}\frac{x^{-k\alpha}\left(-\theta\right)^{-k\alpha+1}}{\Gamma\left(-k\alpha+1\right)}e^{-\lambda\kappa_p(\theta)}\frac{\left(\lambda\kappa_p\left(\theta\right)\right)^{k-1}}{(k-1)!}dx \\ &= \frac{\alpha\lambda\kappa_p\left(\theta\right)}{\theta}\frac{\overline{F}_{S^*}\left(VaR_q[S]\right)}{1-q} = \frac{\alpha\lambda\kappa_p\left(\theta\right)}{\theta}\frac{\overline{F}_{S+\xi}\left(VaR_q[S]\right)}{1-q}, \end{aligned}$$

that coincides with Theorem 5.1.

We further consider multivariate dependent Tweedie compound Poisson distributions.

Corollary 5.1. *Let* $\mathbf{X} \backsim Tw_{n,(1,2)}(\boldsymbol{\theta}, \tilde{\boldsymbol{\lambda}})$ *be a Tweedie compound Poisson random vector, and* S *be the sum of its univariate marginal constituents. Then, the tail conditional expectation risk measure and the economic capital allocation derived from it are formulated correspondingly as*

$$TCE_q\left[S\right] = \frac{n\lambda_0\kappa_{(1,2)}\left(\theta\right)\alpha}{\theta}\frac{\overline{F}_{S+n\xi}\left(VaR_q[S]\right)}{\overline{F}\left(VaR_q[S]\right)} + \frac{\lambda\kappa_{(1,2)}\left(\theta\right)\alpha}{\theta}\frac{\overline{F}_{S+\xi}\left(VaR_q[S]\right)}{\overline{F}\left(VaR_q[S]\right)}$$

and

$$TCE_q\left[X_j|S\right] = \frac{\lambda_0\kappa_{(1,2)}\left(\theta\right)\alpha}{\theta}\frac{\overline{F}_{S+n\xi}\left(VaR_q[S]\right)}{\overline{F}\left(VaR_q[S]\right)} + \frac{\lambda_j\kappa_{(1,2)}\left(\theta\right)\alpha}{\theta}\frac{\overline{F}_{S+\xi}\left(VaR_q[S]\right)}{\overline{F}\left(VaR_q[S]\right)},$$

where $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ *and* $\xi \backsim Tw_2(\theta, -\alpha - 1)$.

Proof. Follows from Theorems 4.1 and 4.2 and Lemma 5.1. □

We also note that in this case, the additivity of the allocation rule can be easily observed from the two latter formulae.

6. Closing comments

Multivariate dependent probabilistic models having convenient dependence structures, suitable risk functionals, and the ability to evaluate the latter in the framework of the former are three main pillars of successful risk measurement, which is a natural precursor of the even more demanding process of managing financial risks.

It must be emphasized that every one of the above mentioned concepts bears its own specific challenge making the whole process far from straightforward. For instance, there generally exists a considerable number of methods of multivariate modeling univariate margins which are followed by adding a dependence structure. However, the insurance and finance industries dictate specific laws that must be obeyed. Namely, mostly only multivariate models defined on $\mathbf{R}_+^n$, preserving unimodality and positive skewness, can serve as appropriate candidates for model insurance losses. These peculiarities discard, for instance, the elliptical family of multivariate distributions, although it is very useful in general finance. Also, there is a trade-off between, on the one hand, the approximation level provided by the model and on the other, its analytical complexity. The present popular *copula* multivariate structures, for example, lead to some essential analytic complications in both inference and risk measurement. Consequently, one has to impose an additional restriction of tractability on the choice of the multivariate cumulative distribution function and its dependence structure and, thus, to reject even more models, although they might have described insurance losses well.

The choice of a suitable risk measure is not obvious either. As already stressed earlier, a significant number of risk measures of various kinds exist nowadays, the earliest of which would seem to be the Value-at-Risk and the latest the distorted risk measures and the measures based on the tail variance. Several axiomatic approaches to risk measurement have also been developed. The debate over what risk measure to apply is still far from being over.

Needless to say, the necessary evaluation of the chosen risk measure in the context of the multivariate probability distribution describing risks is also rather challenging. Indeed, abandoning, say, the independence assumption makes the model more realistic but at the same time much less tractable analytically. The advanced nature of today's most popular risk measures along with the formally unpleasant probability distribution functions of the model also lead to a highly challenging problem.

In this paper we have made an attempt to solve the above problems, when the underlying probabilistic model is the multivariate Tweedie family, and the risk measure is the well known tail conditional expectation. Moreover, the more complicated issue of the consequent allocation of the economic capital to its various constituents has also been considered in the general framework of the aforementioned set up.

References

Acerbi, C. and Tasche, D. (2002). On the coherence of expected shortfall. *Journal of Banking & Finance* **26**(7), 1487-1503.

Artzner, P., Delbaen, F. and Eber, J.M., & Heath, D. (1999). Coherent measures of risk. *Mathematical Finance* **9**, 203-228.

Bildikar, S. and Patil, G.P. (1968). Multivariate exponential type distributions. *Annals of Statistics* **39**(4), 1316-1326.

Cai, J. and Li, H. (2005). Conditional tail expectations for multivariate phase-type distributions. *Journal of Applied Probability* **42**(3), 810-825.

Denault, M. (2001). Coherent allocation of risk capital. *Journal of Risk* **4**(1), 7-21.

Denneberg, D. (1994). *Non-additive measure and integral.* Kluwer, Dordrecht.

Furman, E. and Landsman, Z. (2005). Risk capital decomposition for a multivariate dependent gamma portfolio. *Insurance: Mathematics and Economics* **37**, 635-649.

Furman, E. and Landsman, Z. (2006a). On some risk-adjusted tail-based premium calculation principles. *Journal of Actuarial Practice* **13**, 175-191.

Furman, E. and Landsman, Z. (2006b). Tail variance premium with applications for elliptical portfolio of risks. *ASTIN Bulletin* **36**(2), 433-462.

Furman, E. and Landsman, Z. (2007). Multivariate Tweedie distributions and some related Capital-at-Risk analysis. Technical report N 06-12-1, Actuarial Research Center, University of Haifa, Israel.

Hürlimann, W. (2003). Conditional Value-at-Risk bounds for compound Poison risks and a normal approximation. *Journal of Applied Mathematics* **3**, 141-153.

Jorgensen, B. (1997). *The theory of dispersion models.* Chapman & Hall, London.

Kaas, R., Goovaerts, M.J., Dhaene, J. and Denault, M. (2001). *Modern actuarial risk theory.* Kluwer Academic Publishers.

Landsman, Z. and Valdez, E. (2003). Tail conditional expectation for elliptical distributions. *North American Actuarial Journal,* **7**(4), 55–71.

Leavens, D.H. (1945). Diversification of investments. *Trusts and Estates* **80**, 469-473.

McNeil, A.J., Frey, R. and Embrechts, P. (2005). *Quantitative risk management*, Princeton University Press, Princeton and Oxford.

Panjer, H. H. and Jia, J. (2001). Solvency and capital allocation. Institute of Insurance and Pension Research, University of Waterloo, Research Report N 01-14.

Patil, G.P. and Ord, J.K. (1976). On size-biased sampling and related form-invariant weighted distributions, *Sankhya,* Ser. B **38**, 48-61.

Patil, G.P. and Rao, C.R. (1978). Weighted distributions and size-biased sampling with applications to wildlife populations and human families. *Biometrics* **34**, 179-189.

Tasche, D. (2002). Expected shortfall and beyond. *Journal of Banking and Finance* **26**(7), 1519-1533.

Wang, S. S. (1995). Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics* **17**, 43-54.

Wang, S. S. (1996). Premium calculation by transforming the layer premium density. *ASTIN Bulletin* **26**, 71-92.

Wang, S., Young, V. R. and Panjer, H. H. (1997). Axiomatic characterization of Insurance prices. *Insurance: Mathematics and Economics* **21**, 173-183.

Zaik, E., Walter, J., Kelling, G. and James, C. (1996). RAROC at bank of America: from theory to practice. *Journal of Applied Corporate Finance* **9**(2), 83–93.

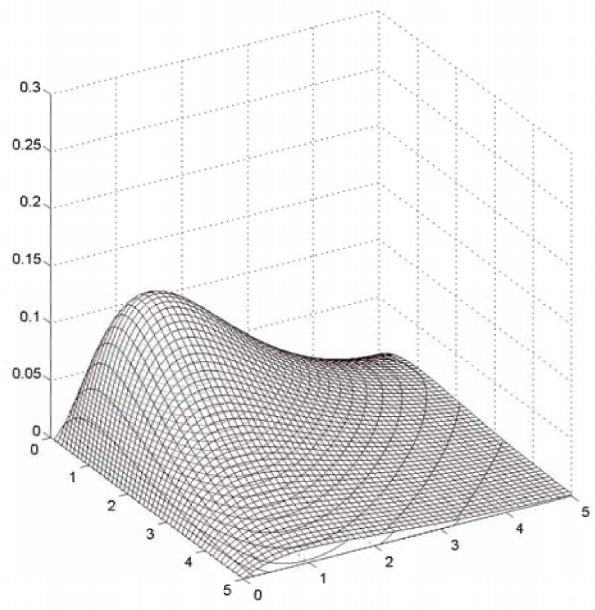

FIGURE 3.1. Bivariate gamma with independent univariate marginal distributions.

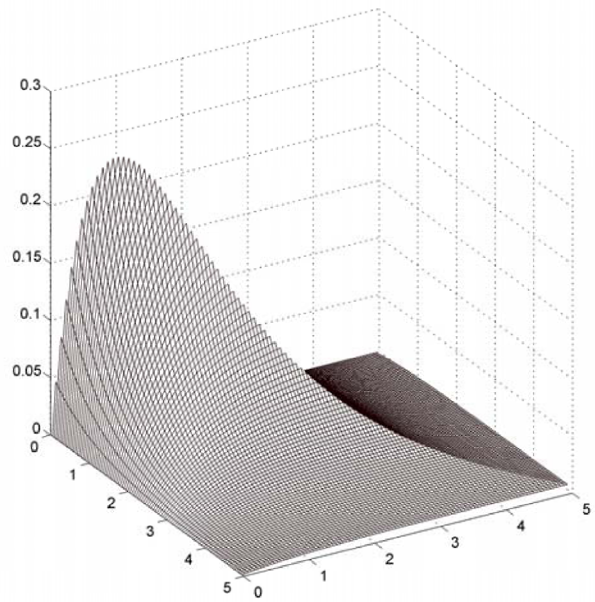

FIGURE 3.2. Bivariate gamma with dependent univariate marginal distributions.

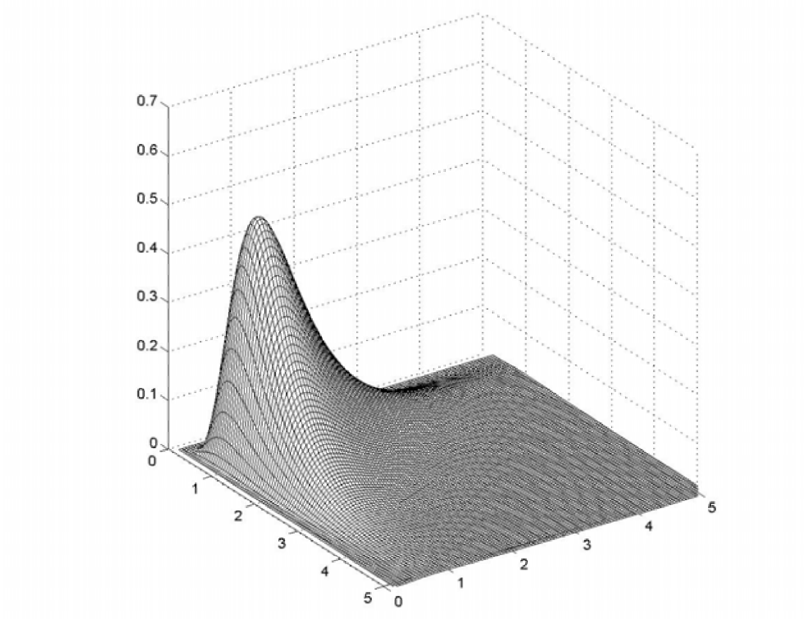

FIGURE 3.3. Bivariate inverse Gaussian with independent univariate marginal distributions.

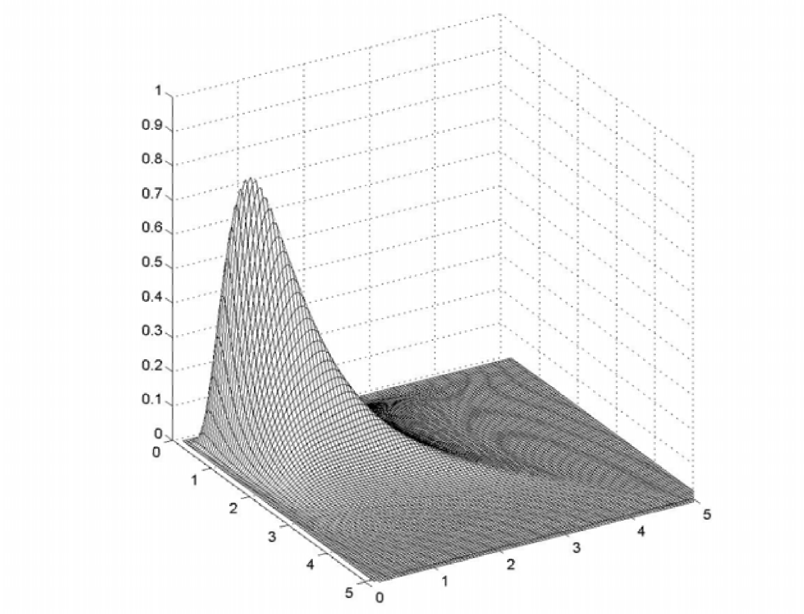

FIGURE 3.4. Bivariate inverse Gaussian with dependent univariate marginal distributions.