

On the Subadditivity of Tail-Value at Risk: An Investigation with Copulas

S. Desmedt^{*‡}, J.F. Walhin^{¶§}

* Secura, Avenue des Nerviens 9-31, B-1040 Brussels, Belgium

‡ Corresponding author, Tel: +32 2 5048228, e-mail: stijn.desmedt@secura-re.com

¶ Fortis, Montagne du Parc 3 (1MA3F), B-1000 Brussels, Belgium

§ Institut de Sciences Actuarielles, Université Catholique de Louvain,
rue des Wallons 6, B-1348 Louvain-la-Neuve, Belgium

Abstract

In this paper, we compare the point of view of the regulator and the investors about the required solvency level of an insurance company. We assume that the required solvency level is determined using the Tail-Value at Risk and analyze the diversification benefit, both on the required capital and on the residual risk, when merging risks. To describe the dependence structure, we use a range of various copulas. This allows to judge whether or not the Tail-Value at Risk is too subadditive under a wide range of conditions. Furthermore, we discuss the effect of different copulas on the diversification possibilities.

Keywords

Required solvency level, Tail-Value at Risk, Diversification benefit, Stochastic dependence, Copulas, Tail dependence.

1 Introduction

Assume that the loss incurred by an insurance company is given by the realization of a random variable X , defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. To protect the insured, regulators demand that the insurance company holds an amount of money large enough to be able to compensate the policyholders with a high probability. Obviously, a fraction of that amount is provided by the premiums paid by the policyholders. The missing amount is provided by the shareholders who put money at risk in the insurance company. They demand a certain return on this capital.

In this paper, we will concentrate on measures to calculate the required solvency level. Because this has obvious connections with the right tail of the random variable representing the loss, a risk measure based on quantiles seems to be adequate. Quantiles have been called Value-at-Risk (VaR) by bankers for a long time and we will follow this terminology here. The VaR at the level p is given by:

$$\text{VaR}_p[X] = \inf\{x \in \mathbb{R} | F_X(x) \geq p\}, 0 < p < 1,$$

where $F_X(x) = \mathbb{P}[X \leq x]$ is the cumulative density function of X . More generally, we will resort to risk measures to determine the required solvency level. Therefore, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Γ be a non-empty set of \mathcal{F} -measurable random variables. A risk measure ρ is a functional:

$$\rho : \Gamma \rightarrow \mathbb{R} \cup \{\infty\}.$$

Let us now analyze the situation when we merge two risks X_1 and X_2 . The regulator wants to minimize the shortfall risk:

$$(X - \rho[X])_+ = \max(0, X - \rho[X]).$$

For a merger, the following inequality holds with probability one (see Dhaene et al.):

$$(X_1 + X_2 - \rho[X_1] - \rho[X_2])_+ \leq (X_1 - \rho[X_1])_+ + (X_2 - \rho[X_2])_+. \quad (1)$$

Therefore, from the viewpoint of avoiding shortfall, the aggregation of risk is to be preferred in the sense that the shortfall decreases. The underlying reason is that within the merger, the shortfall of one of the entities can be compensated by potentially better results for the other.

However, when investors have an amount of capital $\rho[X_1] + \rho[X_2]$, they will prefer investing in two separate companies because the following inequality holds with probability one:

$$(\rho[X_1] + \rho[X_2] - X_1 - X_2)_+ \leq (\rho[X_1] - X_1)_+ + (\rho[X_2] - X_2)_+.$$

Investors will get a higher return by investing in two separate companies due to the fire walls that exist between X_1 and X_2 . Indeed, if for one of the separate companies $X_1 - \rho[X_1] > 0$, this will not affect company 2. For a merger however, if $\rho[X_2] - X_2 > 0$ and $X_1 - \rho[X_1] > 0$, a part of the capital invested in company 2 will be used to compensate the bad results for company 1. Investors may have incentives to invest in the merger once:

$$\rho[X_1 + X_2] \leq \rho[X_1] + \rho[X_2]. \quad (2)$$

A risk measure ρ that conforms to (2) for all $X_1, X_2 \in \Gamma$ is said to be subadditive. On the other hand, a risk measure is superadditive when for all $X_1, X_2 \in \Gamma$:

$$\rho[X_1 + X_2] \geq \rho[X_1] + \rho[X_2].$$

It is well-known that the VaR is not subadditive. Therefore, we will look for other risk measures that are subadditive. For such a risk measure, we do not necessarily have that

$$(X_1 + X_2 - \rho[X_1 + X_2])_+ \leq (X_1 - \rho[X_1])_+ + (X_2 - \rho[X_2])_+ \quad (3)$$

for all outcomes of X_1 and X_2 . Of course, when ρ is superadditive, (3) is fulfilled for all outcomes of X_1 and X_2 . However, for the reasons given above, superadditive risk measures will not motivate investors.

As mentioned in Dhaene et al., if for a given random couple (X_1, X_2) , we have that $\mathbb{P}[X_1 > \rho[X_1], X_2 > \rho[X_2]] > 0$ and that equation (3) is satisfied for all outcomes of X_1 and X_2 , then we need to have that $\rho[X_1 + X_2] \geq \rho[X_1] + \rho[X_2]$. Hence, a subadditive risk measure satisfying (3) for every outcome of all random couples (X_1, X_2) needs to be additive for all random couples for which $\mathbb{P}[X_1 > \rho[X_1], X_2 > \rho[X_2]] > 0$. Only for random couples with $\mathbb{P}[X_1 > \rho[X_1], X_2 > \rho[X_2]] = 0$, a credit could be given for the capital

requirement of the merger. Hence, condition (3) limits the range of possible risk measures considerably.

Dhaene et al. analyzed the effect of weakening the condition that ρ should satisfy inequality (3) for any outcome of all random couples (X_1, X_2) to the requirement that on average:

$$\mathbb{E}(X_1 + X_2 - \rho[X_1 + X_2])_+ \leq \mathbb{E}(X_1 - \rho[X_1])_+ + \mathbb{E}(X_2 - \rho[X_2])_+ \quad (4)$$

for all random couples (X_1, X_2) . They showed that all translation invariant and positively homogeneous risk measures satisfy condition (4) for every bivariate normal distribution and more generally, for every bivariate elliptical distribution. A risk measure is said to be translation invariant if for all $b \in \mathbb{R}$ and for each random variable $X \in \Gamma$ we have that $\rho[X + b] = \rho[X] + b$. A positively homogeneous risk measure satisfies $\rho[aX] = a\rho[X]$ for all $a > 0$ and $X \in \Gamma$. Now suppose a risk measure is translation invariant, positively homogeneous and subadditive. If it also satisfies the property that for all $X_1, X_2 \in \Gamma$ with $\mathbb{P}[X_1 \leq X_2] = 1$ we have that $\rho[X_1] \leq \rho[X_2]$ (monotonicity), it is said to be coherent in the sense of Artzner et al. (1999).

Although there exist several coherent risk measures, we will focus in the present paper on the TVaR only. This is undoubtedly the most popular coherent risk measure in practice. The TVaR of a random variable X is defined as:

$$\text{TVaR}_p[X] = \frac{1}{1-p} \int_p^1 \text{VaR}_q[X] dq, 0 < p < 1,$$

where p is a given confidence level. TVaR at a level p is equal to the average of all quantiles of X above the p -quantile. This gives it, just like the VaR, a nice intuitive interpretation. The TVaR as we define it is related to the expected shortfall as defined in Acerbi and Tasche (2002). These authors see losses as a negative outcome of a random variable and hence look at the left-hand side of the distribution. For continuous random variables, the TVaR is equal to the conditional tail expectation (CTE), which is defined as:

$$\text{CTE}_p[X] = \mathbb{E}[X|X > \text{VaR}_p[X]], 0 < p < 1.$$

The CTE is not subadditive. As shown in Dhaene et al., examples can be constructed for which (4) does not hold for the TVaR. Hence, TVaR can be too subadditive in the sense of condition (4).

The purpose of this paper is twofold. On the one hand, we want to show that the TVaR, is able to deal in an appropriate fashion with the diversification benefit of a merger under a wide range of dependence structures and margins. In our examples, we will observe that the TVaR only gives a credit for diversification when appropriate, hereby providing a framework for compromise between the expectations of the investors and those of the regulator. On the other hand, by taking a practical approach based on copulas to describe the dependence structure between the margins, we want to learn more about the behavior of different copulas with respect to the diversification benefit. Copulas have been gaining a lot of interest in insurance applications (see e.g. Frees and Valdez (1998), Venter (2001) and Blum et al. (2002)) and in other research areas. This has been the cause for some warnings and discussion lately (see Mikosch (2006) and the subsequent discussion papers).

The rest of the paper is organized as follows. In section 2, we define some measures to compare the residual risk of a conglomerate and of stand-alone companies. We illustrate the effect of merging independently and identically distributed exponential subsidiaries on these measures when TVaR is used as a risk measure. In section 3, we define the concept of a copula and the copulas which are used in this paper. Some well-known dependence measures are defined in section 4, where we also make a graphical analysis of some copulas. In section 5, we then analyze the residual risk of a conglomerate and a group of stand-alone companies based on a simulation study. We again use TVaR to determine the required solvency level. We conclude in section 6.

2 Residual Risk of Conglomerate and Stand-alones

Assume the risks X_i with $i \in \{1, \dots, K\}$. For the conglomerate $X = \sum_{i=1}^K X_i$, we will compute the mean, the variance, the skewness and the kurtosis of the residual risk $RR_X = (X - \rho[X])_+$:

$$\begin{aligned}\mathbb{E}[RR_X] &= \int_{\rho[X]}^{+\infty} (x - \rho[X])f_X(x)dx \\ \mathbb{V}ar[RR_X] &= \sigma^2[RR_X] = \int_0^{+\infty} ((x - \rho[X])_+ - \mathbb{E}[RR_X])^2 f_X(x)dx \\ \gamma[RR_X] &= \frac{1}{\mathbb{V}ar[RR_X]^{3/2}} \int_0^{+\infty} ((x - \rho[X])_+ - \mathbb{E}[RR_X])^3 f_X(x)dx \\ \kappa[RR_X] &= \frac{1}{\mathbb{V}ar[RR_X]^2} \int_0^{+\infty} ((x - \rho[X])_+ - \mathbb{E}[RR_X])^4 f_X(x)dx.\end{aligned}$$

For each of the entities X_i looked at as stand-alones, we assume we use the same risk measure ρ to determine the solvency level. Hence, we can write the mean, variance, skewness and kurtosis of the residual risk $RR_{X_i} = (X_i - \rho[X_i])_+$ as:

$$\begin{aligned}\mathbb{E}[RR_{X_i}] &= \int_{\rho[X_i]}^{+\infty} (x - \rho[X_i])f_{X_i}(x)dx \\ \mathbb{V}ar[RR_{X_i}] &= \sigma^2[RR_{X_i}] = \int_0^{+\infty} ((x - \rho[X_i])_+ - \mathbb{E}[RR_{X_i}])^2 f_{X_i}(x)dx \\ \gamma[RR_{X_i}] &= \frac{1}{\mathbb{V}ar[RR_{X_i}]^{3/2}} \int_0^{+\infty} ((x - \rho[X_i])_+ - \mathbb{E}[RR_{X_i}])^3 f_{X_i}(x)dx \\ \kappa[RR_{X_i}] &= \frac{1}{\mathbb{V}ar[RR_{X_i}]^2} \int_0^{+\infty} ((x - \rho[X_i])_+ - \mathbb{E}[RR_{X_i}])^4 f_{X_i}(x)dx.\end{aligned}$$

We denote:

$$\begin{aligned}\mu_3[RR_{X_i}] &= \int_0^{+\infty} ((x - \rho[X_i])_+ - \mathbb{E}[RR_{X_i}])^3 f_{X_i}(x)dx \text{ and} \\ \mu_4[RR_{X_i}] &= \int_0^{+\infty} ((x - \rho[X_i])_+ - \mathbb{E}[RR_{X_i}])^4 f_{X_i}(x)dx.\end{aligned}$$

In case the risks X_i are identically and independently distributed, the mean, variance, skewness and kurtosis of the sum of the residual risk of the separate entities, $RR_{X_{1:K}} =$

$\sum_{i=1}^K (X_i - \rho[X_i])_+$, can be written as follows:

$$\begin{aligned}\mathbb{E}[RR_{X_{1;K}}] &= \sum_{i=1}^K \mathbb{E}[RR_{X_i}] \\ \text{Var}[RR_{X_{1;K}}] &= \sigma^2[RR_{X_{1;K}}] = \sum_{i=1}^K \text{Var}[RR_{X_i}] \\ \gamma[RR_{X_{1;K}}] &= \frac{\sum_{i=1}^K \mu_3[RR_{X_i}]}{\text{Var}[RR_{X_{1;K}}]^{3/2}} \\ \kappa[RR_{X_{1;K}}] &= \frac{\sum_{i=1}^K \mu_4[RR_{X_i}]}{\text{Var}[RR_{X_{1;K}}]^2}.\end{aligned}$$

In the more general case, the variance, skewness and kurtosis of the sum of the residual risk of the separate entities and the distribution of the loss of the merger depend on the dependence structure and the marginal distributions of each of the entities. In such cases, we will use a simulation model. As a basis for comparison for the results generated through simulation, we first assume the subsidiaries are all identically and independently exponentially distributed since this allows to use explicit formulas and numerical approximation methods.

If $X_i \stackrel{\text{iid}}{\sim} \text{Expo}(\lambda)$ for $i \in \{1, 2\}$, i.e.:

$$F_{X_i}(x) = 1 - e^{-\lambda x}, \text{ for } x > 0 \text{ and } \lambda > 0,$$

it is well known that $\mathbb{E}[X_i] = \sigma[X_i] = \frac{1}{\lambda}$ and that $X = X_1 + X_2 \sim \text{Gamma}(\alpha = 2, \beta = \lambda)$, where the distribution function of a $\text{Gamma}(\alpha, \beta)$ -distributed random variable X is given by:

$$F_X(x) = \int_0^x \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} dy, \text{ for } x > 0, \alpha > 0 \text{ and } \beta > 0,$$

where $\Gamma(\alpha)$ denotes the Gamma-function. Because the α -parameter is integer, we in fact have the Erlang Distribution.

The VaR and TVaR for an exponential distribution are given by:

$$\begin{aligned}\text{VaR}_p[X_i] &= -\frac{\ln(1-p)}{\lambda}, \\ \text{TVaR}_p[X_i] &= \frac{1}{\lambda} + \text{VaR}_p[X_i] = \frac{1}{\lambda}(1 - \ln(1-p)).\end{aligned}$$

In general, the VaR for the gamma distribution has no closed form but good numerical approximations are available in a lot of statistical software packages. As shown in Landsman and Valdez (2004), if $X \sim \text{Gamma}(\alpha, \beta)$, we have that:

$$\text{TVaR}_p[X] = \frac{\alpha(1 - F_Y(\text{VaR}_p[X]))}{\beta(1-p)},$$

where $Y \sim \text{Gamma}(\alpha + 1, \beta)$.

Now assume that X_1 and X_2 are i.i.d. according to the exponential distribution with parameter $\lambda = \frac{1}{50}$. Then we have that $\text{TVaR}_{0.95}[X_i] = 200$, for $i \in \{1, 2\}$, and $\text{TVaR}_{0.95}[X] = 296$ and that $\text{TVaR}_{0.99}[X_i] = 280$, for $i \in \{1, 2\}$, and $\text{TVaR}_{0.99}[X] = 388$. For the residual risk, when taking the TVaR at a 95%- and 99%-level, we find the risk measures as summarized in table 1.

Risk Measure	$TVaR_{0.95}$		$TVaR_{0.99}$	
	$X_{1;2}$	$X_1 + X_2$	$X_{1;2}$	$X_1 + X_2$
$\mathbb{E}[RR]$	1.839	1.065	0.368	0.206
$\sigma[RR]$	13.450	10.902	6.060	4.765
$\gamma[RR]$	11.011	15.156	24.708	34.335
$\kappa[RR]$	260.252	306.018	815.487	1563.420
$\mathbb{P}[RR = 0]$	0.964	0.981	0.993	0.996

Table 1: Risk measures of residual risk for 2 independent exponential risks.

Both at the 95% and 99% level, we observe that even though the TVaR for the conglomerate is considerably lower than the sum of the solvency levels of the two separate entities, the mean and the standard deviation of the residual risk of the conglomerate are considerably lower than for the sum of the separate entities. The skewness and the kurtosis of the residual risk for the merger are larger than for the two separate entities. The probability of default for the merger is considerably lower. This illustrates that due to the diversification benefit given by equation (1) and allowing for the subadditivity implicit to the TVaR, the conglomerate performs better both with respect to the mean and the standard deviation of the residual risk.

We make the same exercise at a 99% confidence level for a merger of 5 and of 10 independent risks with margins with $Expo(\frac{1}{50})$ -distribution. We then have $TVaR_{0.99}[\sum_{i=1}^5 X_i] = 650$ and $TVaR_{0.99}[\sum_{i=1}^{10} X_i] = 1024$. The results are given in table 2.

Risk Measure	$TVaR_{0.99}$		$TVaR_{0.99}$		10D vs. 5D	
	$X_{1;5}$	$X = \sum_{i=1}^5 X_i$	$X_{1;10}$	$X = \sum_{i=1}^{10} X_i$	Separate	Merger
$\mathbb{E}[RR]$	0.920	0.252	1.839	0.305	200%	121%
$\sigma[RR]$	9.581	5.758	13.550	6.913	141%	120%
$\gamma[RR]$	15.627	33.550	11.050	33.013	71%	98%
$\kappa[RR]$	326.195	1478.030	163.097	1420.910	50%	96%
$\mathbb{P}[RR = 0]$	0.982	0.996	0.964	0.996	98%	100%

Table 2: Risk measures of residual risk for 5 and 10 independent exponential risks.

In table 2, we observe that for the conglomerate, the expectation and standard deviation of the residual risk increase with about 20% if the number of subsidiaries is increased from 5 to 10. For the separate entities however, these measures increase with 100% and 41% respectively. This shows the interest of merging risks and that the TVaR is not too subadditive in this example, if we are interested in these measures of the residual risk. Of course, the probability that at least one of the subsidiaries goes default increases when the number of subsidiaries increases. For the conglomerate however, this probability remains (nearly) constant, showing that the subadditivity of the TVaR does not increase the default probability for the merger in this example. The skewness and kurtosis for the merger decrease only slowly when moving from 5 to 10 dimensions. These risk measures are importantly larger than when the companies are separated. This is however a simple consequence of the fact that the distribution of the residual risk for the merger has a probability of being zero which is a lot more important. Therefore, this should not be a reason to conclude that the TVaR is too subadditive.

In what follows, we will use the average residual risk and the probability that the residual risk is zero to assess if the subadditivity of the TVaR is acceptable. We will also assess what happens with the standard deviation of the residual risk for the merger and the sum of the stand-alones.

3 Copulas

3.1 Definition and Existence

The notion of copula was introduced by Sklar (1956). We define a d -dimensional copula.

Definition 1 (Multivariate Copula) *A d -dimensional copula C is a non-decreasing right-continuous function from the unit cube $[0, 1]^d$ to the unit interval $[0, 1]$ which satisfies the following properties:*

1. $C(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_d) = 0$ for $i \in \{1, \dots, d\}$,
2. $C(1, \dots, 1, u_i, 1, \dots, 1) = u_i$ for $i \in \{1, \dots, d\}$,
3. For all (a_1, \dots, a_d) and (b_1, \dots, b_d) in $[0, 1]^d$ with $a_i < b_i$ for $i \in \{1, \dots, d\}$:

$$\Delta_{a_1, b_1} \dots \Delta_{a_d, b_d} C(u_1, \dots, u_d) \geq 0 \text{ for all } (u_1, \dots, u_d) \in [0, 1]^d, \quad (5)$$

where

$$\Delta_{a_i, b_i} C(u_1, \dots, u_d) = C(u_1, \dots, u_{i-1}, b_i, u_{i+1}, \dots, u_d) - C(u_1, \dots, u_{i-1}, a_i, u_{i+1}, \dots, u_d).$$

A copula can be interpreted as the joint distribution function of a random vector on the unit square. Note that condition (5) in definition (1) ensures that

$$\mathbb{P}[a_1 \leq U_1 \leq b_1, \dots, a_d \leq U_d \leq b_d] \geq 0$$

for all (a_1, \dots, a_d) and (b_1, \dots, b_d) in $[0, 1]^d$ with $a_i < b_i$ for $i \in \{1, \dots, d\}$ and where (U_1, \dots, U_d) denotes a d -dimensional uniform random vector with copula C .

It follows from the next theorem that the joint distribution function of a continuous random vector can be written as a function of its margins and a unique copula.

Theorem 1 (Sklar's Theorem in d -dimensions) *Let F be a d -dimensional distribution function with marginal distribution functions F_1, \dots, F_d . Then there is a d -dimensional copula C such that for all $x \in \mathbb{R}^d$:*

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)). \quad (6)$$

If F_1, \dots, F_d are all continuous, then C is unique. Conversely, if C is a d -dimensional copula, and F_1, \dots, F_d are distribution functions, then F defined by (6) is a d -dimensional distribution with margins F_1, \dots, F_d .

See Nelsen (1999) for a proof.

3.2 Examples of Copulas

Every d -dimensional copula C satisfies the inequality:

$$\max \left\{ 0, \sum_{i=1}^d u_i - (n-1) \right\} \leq C(u_1, \dots, u_d) \leq \min\{u_1, \dots, u_d\}, \text{ for all } (u_1, \dots, u_d) \in [0, 1]^d. \quad (7)$$

For $d \geq 3$, the left-hand side of (7) does not satisfy the condition for being a copula (see Denuit et al. (2005) for an explanation). In dimension 2, the left hand side of (7) is called the Fréchet lower bound copula, which we denote with C_L . Random couples with this copula are said to be countermonotonic. The right hand side of (7) is the d -dimensional Fréchet upper bound copula, which we denote with C_U . Random vectors with this copula are said to be comonotonic. Comonotonicity is the strongest possible dependence.

Below, we define some other well-known copulas:

- **The independence copula:**

For d independent random variables X_1, \dots, X_d with respective distribution functions F_1, \dots, F_d , the joint distribution function is equal to $\prod_{i=1}^d F_i(x_i)$. Therefore, the copula underlying independent risks is given by:

$$C_I(u_1, \dots, u_d) = \prod_{i=1}^d u_i, \text{ for all } (u_1, \dots, u_d) \in [0, 1]^d.$$

- **The survival copula or flipped copula:**

Let C be a d -dimensional copula and let (U_1, \dots, U_d) be a uniform random vector on $[0, 1]^d$ with copula C . The survival function is defined and denoted with:

$$C_S(u_1, \dots, u_d) = \mathbb{P}[U_1 > u_1, \dots, U_d > u_d] \text{ for all } (u_1, \dots, u_d) \in [0, 1]^d.$$

C_S is not a copula since $C_S(0, \dots, 0) = 1$. However,

$$\bar{C}(u_1, \dots, u_d) = C_S(1 - u_1, \dots, 1 - u_d) \text{ for all } (u_1, \dots, u_d) \in [0, 1]^d$$

is a copula, which we call the survival copula or the flipped copula of the copula C . If (U_1, \dots, U_d) is a uniform random vector on $[0, 1]^d$ with copula C , then $(1 - U_1, \dots, 1 - U_d)$ is a uniform random vector on $[0, 1]^d$ with copula \bar{C} .

- **The normal copula:**

The d -dimensional random vector $X = (X_1, \dots, X_d)^t$ has a multivariate normal distribution with mean vector $\mu = (\mu_1, \dots, \mu_d)$ and positive-definite dispersion matrix Σ if its distribution function is given by:

$$\nu_{\mu, \Sigma}(\mathbf{x}) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_d} \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp \left(-\frac{1}{2} (\epsilon - \mu)^t \Sigma^{-1} (\epsilon - \mu) \right) d\epsilon_1 \dots d\epsilon_d,$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_d)^t$ and $\mathbf{x} = (x_1, \dots, x_d)^t$. Note that for the Normal distribution, $\Sigma = \text{Cov}[X]$, where $\text{Cov}[X]$ denotes the variance-covariance matrix of X . It follows from Sklar's theorem that this multivariate distribution gives rise to a unique copula. The copula of a random vector is invariant under strictly increasing transformations of the random vector (see Nelsen (1999)). Therefore, the copula of a $\nu_{\mu, \Sigma}$ -distribution is identical to that of a $\nu_{0, P}$ -distribution, where P is the correlation matrix implied

by the dispersion matrix Σ . In what follows, we will implicitly assume that Σ refers to the correlation matrix and we will work with the standardized version of the multivariate normal distribution, denoted with ν_Σ . The d -dimensional normal copula with correlation matrix Σ is then defined and denoted by:

$$C_\Sigma(u_1, \dots, u_d) = \nu_\Sigma(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)), \text{ for all } (u_1, \dots, u_d) \in [0, 1]^d,$$

where Φ denotes the distribution function of the univariate standard normal distribution. A simulation algorithm for the normal copula can be found in the discussion paper by Wang related to the paper by Frees and Valdez (1998) or in Embrechts et al. (2003).

- **The Student copula:**

The d -dimensional random vector $X = (X_1, \dots, X_d)^t$ has a (non-singular) multivariate Student distribution with m degrees of freedom ($m > 0$), mean vector μ and positive-definite dispersion matrix Σ if its distribution function is given by:

$$t_{m,\mu,\Sigma}(\mathbf{x}) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} \frac{\Gamma(\frac{m+d}{2})|\Sigma|^{-1/2}}{\Gamma(\frac{m}{2})(m\pi)^{d/2}} \left[1 + \frac{1}{m}(\epsilon - \mu)^t \Sigma^{-1}(\epsilon - \mu) \right]^{-\frac{m+d}{2}} d\epsilon_1 \dots d\epsilon_d,$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_d)^t$ and $\mathbf{x} = (x_1, \dots, x_d)^t$. The multivariate Student distribution with $m = 1$ is also called the multivariate Cauchy distribution. Note that for the Student distribution with m degrees of freedom ($m > 2$), we have $\Sigma = \frac{m}{m-2} \text{Cov}[X]$ (see Demarta and McNeil (2005) and references therein). The covariance matrix is only defined for $m > 2$. As for the normal copula, in what follows, we will implicitly assume that Σ refers to the correlation matrix and denote the standardized version of the multivariate Student distribution with m degrees of freedom with $t_{m,\Sigma}$. The Student copula with m degrees of freedom and correlation matrix Σ is then defined and denoted by:

$$C_{m,\Sigma}(u_1, \dots, u_d) = t_{m,\Sigma}(t_m^{-1}(u_1), \dots, t_m^{-1}(u_d)), \text{ for all } (u_1, \dots, u_d) \in [0, 1]^d.$$

A simulation algorithm for the Student copula can be found in Embrechts et al. (2003).

The normal copula is certainly one of the most popular copulas in practice. The dependence structure between different risks can be taken into account by means of a correlation matrix. It is symmetric (in the sense that it is equal to its survival copula) and has no tail dependence (see section 4.3), making it not sufficiently flexible for contexts where extreme outcomes of the margins may be more correlated. The Student copula exhibits tail dependence (again see section 4.3), which makes it more flexible than the normal copula. However, it remains symmetric.

3.3 Archimedean Copulas

A popular class of copulas are the so-called Archimedean copulas, which were described in Genest and MacKay (1986a) and Genest and MacKay (1986b). Let $\varphi : [0, 1] \rightarrow [0, +\infty[$ be some continuous, strictly decreasing and convex function for which $\varphi(1) = 0$. Every such function φ generates a bivariate copula C_φ :

$$C_\varphi(u_1, u_2) = \begin{cases} \varphi^{-1}[\varphi(u_1) + \varphi(u_2)] & \text{if } \varphi(u_1) + \varphi(u_2) \geq \varphi(0), \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

φ is called the generator of the Archimedean copula C_φ . The independence copula is Archimedean with generator $\varphi(t) = -c \ln(t)$, where c is an arbitrary constant in $]0, +\infty[$.

It is possible to create multivariate Archimedean copulas from their bivariate version. Therefore, assume a bivariate Archimedean copula with generator φ . Now define the function $C_\varphi^{[d]}$ by the following iteration for $d \geq 3$:

$$C_\varphi^{[d]}(u_1, \dots, u_d) = C_\varphi(C_\varphi^{[d-1]}(u_1, \dots, u_{d-1}), u_d), \text{ for all } (u_1, \dots, u_d) \in [0, 1]^d. \quad (9)$$

$C_\varphi^{[d]}$ is a copula for all $d \geq 2$ if and only if φ^{-1} is completely monotonic in \mathbb{R}^+ (see Kimberling (1974)). A function g is said to be completely monotonic on the interval J if it is continuous and has derivatives of all orders which alternate in sign, i.e.:

$$(-1)^k \frac{d^k}{dt^k} g(t) \geq 0 \text{ for all } t \in J \text{ and } k \in \{0, 1, 2, \dots\}. \quad (10)$$

If the generator $\varphi(t)$ is the inverse of the Laplace transform of a distribution function G on \mathbb{R}^+ satisfying $G(0) = 0$, the following procedure can be used to simulate the Archimedean copula defined by:

$$C_\varphi(u_1, \dots, u_d) = \varphi^{-1}(\varphi(u_1) + \dots + \varphi(u_d)), \quad (u_1, \dots, u_d) \in [0, 1]^d$$

(see Marshall and Olkin (1988)):

- Generate S with distribution function G such that the Laplace transform of G is φ^{-1} .
- Generate a d -dimensional uniform random vector (U_1, \dots, U_d) on $[0, 1]^d$ such that all U_i , for $i \in \{1, \dots, d\}$, are independent.
- Then $V_i = \varphi^{-1}\left(-\frac{\ln(U_i)}{S}\right)$, $i \in \{1, \dots, d\}$, is a uniform random vector on $[0, 1]^d$ with copula C_φ .

For an extensive list of one-parameter bivariate families of Archimedean copulas, we refer to Nelsen (1999). 3 popular Archimedean copulas are the Clayton, the Frank and the Gumbel-Hougaard copulas. These copulas have been given different names by different authors. We again refer to Nelsen (1999) for an overview.

- **Clayton's copula:**

For any $\alpha > 0$, the d -dimensional Clayton copula is defined and denoted by:

$$C_{C,\alpha}(u_1, \dots, u_d) = (u_1^{-\alpha} + \dots + u_d^{-\alpha} - 1)^{-1/\alpha}, \text{ for all } (u_1, \dots, u_d) \in [0, 1]^d.$$

The generator of Clayton's copula is:

$$\varphi_{C,\alpha}(t) = \frac{1}{\alpha}(t^{-\alpha} - 1), \text{ where } t \in [0, 1] \text{ and } \alpha > 0.$$

Its inverse is the Laplace transform of a Gamma random variable $S \sim \text{Gamma}(1/\alpha, 1)$.

- **Frank's copula:**

For any $\alpha > 0$, the Frank copula can be defined in general dimensions $d \geq 2$ (see Nelsen (1999)). It is defined and denoted by:

$$C_{F,\alpha}(u_1, \dots, u_d) = -\frac{1}{\alpha} \ln \left(1 + \frac{\prod_{i=1}^d (\exp(-\alpha u_i) - 1)}{\exp(-\alpha) - 1} \right), \text{ for all } (u_1, \dots, u_d) \in [0, 1]^d. \quad (11)$$

The generator of Frank's copula is:

$$\varphi_{F,\alpha}(t) = -\ln \left[\frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1} \right] \text{ where } t \in [0, 1] \text{ and } \alpha > 0.$$

Its inverse is equal to the Laplace transformation of a discrete distribution S with a probability density function given by:

$$\mathbb{P}[S = s] = \frac{(1 - e^{-\alpha})^s}{s\alpha}, \text{ for } s \in \{1, 2, \dots\}. \quad (12)$$

In the bivariate case, definition (11) also gives rise to a copula for $\alpha < 0$. In Frank (1979), it is shown that this copula is the only Archimedean copula satisfying $C(u_1, u_2) = \overline{C}(u_1, u_2)$ for all $(u_1, u_2) \in [0, 1]^2$.

• **Gumbel-Hougaard copula:**

For any $\alpha \geq 1$, the Gumbel-Hougaard copula can be defined in general dimensions $d \geq 2$ (see Nelsen (1999)). It is defined and denoted by:

$$C_{G,\alpha}(u_1, \dots, u_d) = \exp(-((-\ln(u_1))^\alpha + \dots + (-\ln(u_d))^\alpha)^{1/\alpha}), \text{ for all } (u_1, \dots, u_d) \in [0, 1]^d.$$

The generator of the Gumbel-Hougaard copula is:

$$\varphi_{G,\alpha}(t) = (-\ln(t))^\alpha \text{ where } t \in [0, 1] \text{ and } \alpha \geq 1.$$

Its inverse is equal to the Laplace transform of a positive stable random variable $S \sim St(1/\alpha, 1, \gamma, 0)$ with

$$\gamma = \left(\cos \left(\frac{\pi}{2\alpha} \right) \right)^\alpha$$

and $\alpha > 1$. In order to generate a Stable random variable $S \sim St(\alpha, \beta, \gamma, \delta)$ with $\alpha \in (0, 2] \setminus \{1\}$ and $\beta \in [-1, 1]$, one can use the following procedure (see Weron (1996)):

- Generate $V \sim \mathcal{U}[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $W \sim Expo(1)$.
- Set $B_{\alpha,\beta} = \frac{\arctan(\beta \tan(\frac{\pi\alpha}{2}))}{\alpha}$ and $S_{\alpha,\beta} = [1 + \beta^2 \tan^2(\frac{\pi\alpha}{2})]^{1/(2\alpha)}$.
- Compute $S \sim St(\alpha, \beta, \gamma, \delta)$ as:

$$\begin{aligned} X &= S_{\alpha,\beta} \frac{\sin(\alpha(V + B_{\alpha,\beta}))}{(\cos(V))^{1/\alpha}} \left[\frac{\cos(V - \alpha(V + B_{\alpha,\beta}))}{W} \right]^{(1-\alpha)/\alpha} \\ S &= \gamma X + \delta \end{aligned}$$

Note that this procedure is not valid for $\alpha = 1$, which is not relevant within the context of simulating a Gumbel copula.

We refer to Nelsen (1999) for some limiting and special cases for the Archimedean copulas defined above. For each of the three examples, the parameter α can be interpreted as a measure for the strength of the dependence.

Archimedean copulas are more flexible than the normal and Student copula in the sense that they are not necessarily symmetric and can exhibit either upper or lower tail dependence (see section 4.3). For multidimensional problems, they may be less flexible since the dependence structure is based on solely one parameter, implying the same dependence structure between all margins.

4 Measures of Dependence

In this section, we focus on three well-known dependence measures. Kendall's tau and the measures of tail dependence of a random couple can be defined as a function of their copula.

4.1 Pearson's correlation

Consider two random variables X_1 and X_2 with margins with finite variance. Pearson's correlation coefficient is then defined and denoted by:

$$\rho_P(X_1, X_2) = \frac{\text{Cov}[X_1, X_2]}{\sqrt{\text{Var}[X_1]\text{Var}[X_2]}}. \quad (13)$$

$\rho_P(X_1, X_2)$ is a measure for the degree of linear relationship between X_1 and X_2 . A substantial drawback of Pearson's correlation is that it is not invariant under strictly increasing transformations t_1 and t_2 . I.e., in general, $\rho_P(t_1(X_1), t_2(X_2))$ is not equal to $\rho_P(X_1, X_2)$. It follows from the Cauchy-Schwartz inequality that $\rho_P(X_1, X_2)$ is always in $[-1, 1]$. If X_1 and X_2 are independent, then $\rho_P(X_1, X_2) = 0$. As explained in Embrechts et al. (2002), it is possible to construct a random couple with almost zero correlation for which the components are co- or countermonotonic. This contradicts the intuition that small correlation implies weak dependence.

4.2 Kendall's rank correlation

Consider two random variables X_1 and X_2 and let Y_1 and Y_2 denote two other random variables with the same joint distribution but independent of X_1 and X_2 . Kendall's tau is defined as:

$$\begin{aligned} \rho_\tau(X_1, X_2) &= \mathbb{E}[\text{sign}[(X_1 - Y_1)(X_2 - Y_2)]] \\ &= \mathbb{P}[(X_1 - Y_1)(X_2 - Y_2) > 0] - \mathbb{P}[(X_1 - Y_1)(X_2 - Y_2) < 0]. \end{aligned}$$

Hence, if $\rho_\tau(X_1, X_2)$ is positive, there is a higher probability of having an upward slope in the relation between X_1 and X_2 and conversely if $\rho_\tau(X_1, X_2)$ is negative. If $\rho_\tau(X_1, X_2) = 0$, we intuitively expect upward slopes with the same probability as downward slopes. Kendall's tau is invariant under strictly monotone transformations. This implies that $\rho_\tau(X_1, X_2)$ only depends on the copula of (X_1, X_2) . It can easily be verified that Kendall's tau for a flipped copula is the same as for the original copula.

4.3 Tail Dependence

The coefficient of upper and lower tail dependence of a random couple (X_1, X_2) with marginal distribution functions F_1 and F_2 are respectively defined and denoted by:

$$\begin{aligned} \lambda_U &= \lim_{v \rightarrow 0} \mathbb{P}[X_1 > \overline{F}_1^{-1}(v) | X_2 > \overline{F}_2^{-1}(v)] \\ \lambda_L &= \lim_{v \rightarrow 0} \mathbb{P}[X_1 \leq F_1^{-1}(v) | X_2 \leq F_2^{-1}(v)], \end{aligned}$$

where $\bar{F}_i(x) = 1 - F_i(x)$ for $i \in \{1, 2\}$. As explained in Denuit et al. (2005), if F_1 and F_2 are continuous, the tail dependence coefficients can be written as:

$$\lambda_U^C = \lim_{v \rightarrow 1} \frac{1 - 2v + C(v, v)}{1 - v}$$

$$\lambda_L^C = \lim_{v \rightarrow 0} \frac{C(v, v)}{v},$$

where C denotes the copula of the random couple (X_1, X_2) .

4.4 Examples

In table 3, we summarize the Kendall's tau and the coefficients of tail dependence for the copulas defined in section 3.

Copula	ρ_τ	λ_L	λ_U
C_I	0	0	0
C_U	1	1	1
C_L	-1	0	0
C_α	$2 \arcsin(\alpha)/\pi$	0 if $\alpha < 1$ and 1 if $\alpha = 1$	
$C_{m,\alpha}$	$2 \arcsin(\alpha)/\pi$	$2t_{m+1} \left(-\sqrt{m+1} \sqrt{\frac{1-\alpha}{1+\alpha}} \right)$	
$C_{C,\alpha}$	$1 - \frac{4}{\alpha} + \frac{\alpha}{\alpha+2} \int_0^\alpha \frac{t}{e^t-1} dt$	0	$2^{-1/\alpha}$
$C_{F,\alpha}$	$1 - \frac{1}{\alpha}$	0	0
$C_{G,\alpha}$	$1 - \frac{1}{\alpha}$	$2 - 2^{1/\alpha}$	0

Table 3: Dependence measures for different copulas.

For Kendall's tau, see Lindskog et al. (2003) for proofs for the normal and Student copula and Nelsen (1999) for the other examples. For the tail dependence, see Embrechts et al. (2002) or Demarta and McNeil (2005) for proofs regarding the Student and the normal copula. The Clayton copula exhibits lower but no upper tail dependence. For applications with asymptotic dependence in the upper tail, it may be interesting to work with the flipped Clayton copula. This copula has also been called the heavy right tail copula (see Venter (2001)).

Below, we will plot simulations for some of the copulas defined in section 3. In order to make the copulas comparable in some way, we ensure that they all have a Kendall tau of 0.5. This implies that the copulas have parameters and tail dependence as specified in table 4. For the Student copula, we choose 4 degrees of freedom.

Copula	ρ_τ	α	λ_L	λ_U
C_α	0.5	0.707	0	0
$C_{4,\alpha}$	0.5	0.707	0.397	0.397
$C_{C,\alpha}$	0.5	2	0.707	0
$C_{F,\alpha}$	0.5	5.736	0	0
$C_{G,\alpha}$	0.5	2	0	0.586

Table 4: Parameters and tail dependence for a Kendall tau of 0.5.

In figure 1 and 2, we show 10.000 simulations for the Normal and the Student copula. For both copulas, we make a zoom at (1,1).

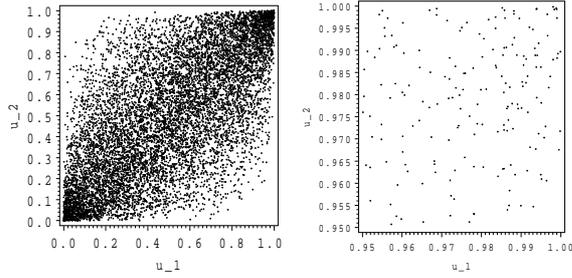


Figure 1: $C_{0.707}$ -copula

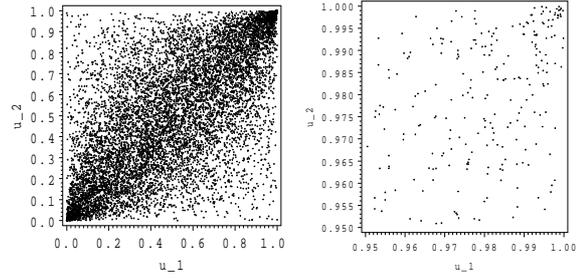


Figure 2: $C_{4, 0.707}$ -copula

We see that the Student and the Normal copula have a rather similar shape. For the Student copula, there are still some points around (1,0) and (0,1) whereas for the Normal copula, these regions are almost empty. In the zooms, we observe that for the Student copula, there is a stronger concentration of the points around (1,1) than for the Normal copula, this illustrates the tail dependence.

In figure 3, we show 10.000 simulations for a Clayton copula. We make a zoom both around (0,0) and (1,1).

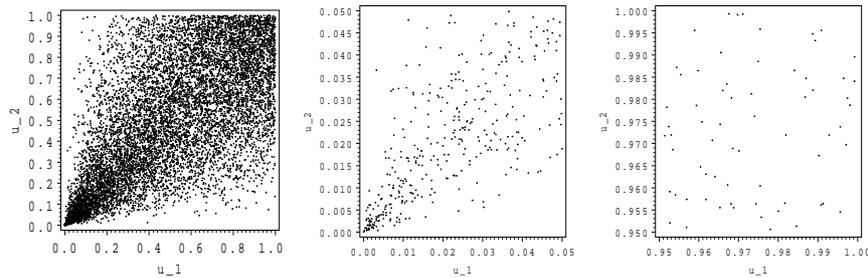


Figure 3: $C_{C, 2}$ -copula.

The shape of this copula is totally different than for the Normal and the Student copula. In the tails, we detect a totally different behavior. The zoom at (1,1) looks more or less like an independent copula whereas the zoom at (0,0) exhibits some kind of conic shape with a strong concentration around the origin.

In figure 4 and 5, we show 10.000 simulations for a Frank and a Gumbel-Hougaard copula. For both copulas, we make a zoom around (1,1).

In figure 4, we detect the symmetry for the Frank copula. The shape however is very different from that of the Normal and the Student copula. Above the first bisector, the Normal and Student copula seem to exhibit a concave shape whereas that of the Frank copula looks more convex. The zoom of the Frank copula around (1,1) again looks like the independent copula whereas the zoom for the Gumbel-Hougaard copula shows a concentration around (1,1).

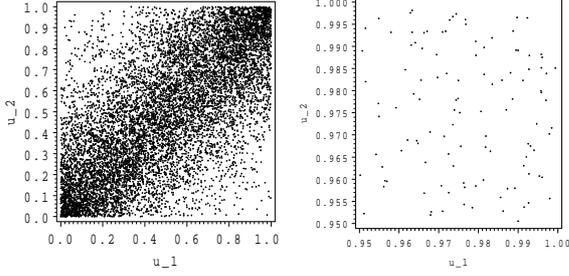


Figure 4: $C_F, 5.736$ -copula

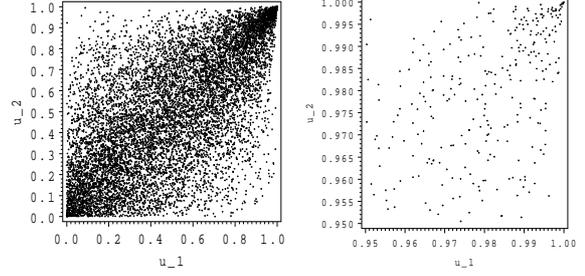


Figure 5: $C_G, 2$ -copula

5 Analysis of Residual Risk using TVaR

For every analysis we make in this section, we assume that all margins are identical. This allows to make comparisons between 2- and 5-dimensional results and creates some symmetry which makes interpretation more straightforward. A similar analysis can of course be made if margins are different. We look at exponential and lognormal margins. The results for exponential margins will allow to verify the accuracy of the simulations with the results obtained in section 2. X has a lognormal distribution with parameters μ and σ^2 ($\sigma > 0$) if it has the probability density function as specified by (14):

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}} \text{ for all } x > 0. \quad (14)$$

It is well known that if $X \sim LN(\mu, \sigma^2)$,

$$\mathbb{E}[X] = e^{\mu+\sigma^2/2} \text{ and} \quad (15)$$

$$\text{Var}[X] = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}. \quad (16)$$

In order to work with margins which are comparable, we will fix their mean and variance. It follows easily from (15) and (16) that in order to have a lognormal distribution with mean and variance $\frac{1}{\lambda}$, we need the following parameters:

$$\mu = -\frac{1}{2} \ln(2\lambda^2) \text{ and } \sigma = \sqrt{\ln(2\lambda^2) - 2\ln(\lambda)}. \quad (17)$$

The same analysis can of course be made for all kinds of margins which can be simulated.

5.1 Merger of 2 Identical Companies

In table 5, we give summary statistics for the residual risk for the merger and for the sum of the separate subsidiaries for exponential margins with parameter $\lambda = \frac{1}{50}$. The results are based on 1.000.000 simulations. In order to make the results comparable, the parameters of the copulas are chosen to obtain a Kendall tau of 0.5, except for the comonotonic, countermonotonic and independent copula of course. The subscripts .95 and .99 in table 5 indicate the level of the TVaR. The columns $\mathbb{P}_{.95}$ and $\mathbb{P}_{.99}$ give the estimated probabilities that the residual risk is 0 when a TVaR at level 0.95 or 0.99 is used as a solvency level. The numerical accuracy of the results in the independent case can be compared with table 1. Except for the comonotonic case, the simulations for the margins are not necessarily the same. The comparison of the results for the mean and the sum of the TVaR's for the separate subsidiaries gives an idea about their numerical variability.

Copula	Variable	$\mathbb{E}_{.95}$	$\sigma_{.95}$	$\mathbb{E}_{.99}$	$\sigma_{.99}$	$\mathbb{P}_{.95}$	$\mathbb{P}_{.99}$	TVaR _{.95}	TVaR _{.99}
C_U	RR_X	1.826	18.9	0.368	8.60	0.982	0.996	399	562
	$RR_{X_{1;2}}$	1.826	18.9	0.368	8.60	0.982	0.996	399	562
	Div Ben	0%	0%	0%	0%	0.000	0.000	0%	0%
C_L	RR_X	0.909	9.5	0.189	4.43	0.982	0.996	235	315
	$RR_{X_{1;2}}$	1.838	13.4	0.373	6.20	0.963	0.993	400	560
	Div Ben	51%	29%	49%	29%	0.018	0.004	41%	44%
C_I	RR_X	1.060	10.8	0.208	4.80	0.981	0.996	296	389
	$RR_{X_{1;2}}$	1.829	13.4	0.367	6.07	0.963	0.993	400	560
	Div Ben	42%	20%	43%	21%	0.018	0.004	26%	31%
$C_{0.707}$	RR_X	1.606	16.7	0.321	7.55	0.982	0.996	368	510
	$RR_{X_{1;2}}$	1.843	15.8	0.369	6.93	0.969	0.993	399	562
	Div Ben	13%	-6%	13%	-9%	0.013	0.003	8%	9%
$C_{4, 0.707}$	RR_X	1.727	18.1	0.356	8.31	0.982	0.996	373	526
	$RR_{X_{1;2}}$	1.829	16.9	0.369	7.55	0.971	0.994	399	561
	Div Ben	6%	-7%	3%	-10%	0.010	0.002	7%	6%
$C_{C, 2}$	RR_X	1.133	11.4	0.213	4.88	0.981	0.996	330	430
	$RR_{X_{1;2}}$	1.830	13.5	0.368	6.08	0.964	0.993	399	562
	Div Ben	38%	15%	42%	20%	0.017	0.004	17%	23%
$\bar{C}_{C, 2}$	RR_X	1.832	19.0	0.366	8.44	0.982	0.996	390	553
	$RR_{X_{1;2}}$	1.831	18.4	0.367	8.18	0.976	0.995	398	560
	Div Ben	0%	-3%	0%	-3%	0.005	0.001	2%	1%
$C_{F, 5.736}$	RR_X	1.201	12.0	0.220	5.12	0.981	0.996	347	451
	$RR_{X_{1;2}}$	1.841	13.7	0.367	6.16	0.965	0.993	399	560
	Div Ben	35%	13%	40%	17%	0.016	0.004	13%	19%
$C_{G, 2}$	RR_X	1.811	19.0	0.373	8.78	0.982	0.996	385	544
	$RR_{X_{1;2}}$	1.834	18.0	0.375	8.30	0.974	0.995	400	560
	Div Ben	1%	-5%	1%	-6%	0.008	0.002	4%	3%
$\bar{C}_{G, 2}$	RR_X	1.449	14.9	0.281	6.49	0.981	0.996	354	479
	$RR_{X_{1;2}}$	1.852	14.8	0.368	6.30	0.967	0.993	400	561
	Div Ben	22%	-1%	24%	-3%	0.015	0.003	11%	15%

Table 5: Risk measures of residual risk for 2 exponential risks ($\rho_\tau = 0.5$).

In table 5, we observe that among all dependence structures, there is a maximum diversification benefit on the TVaR for the merger of about 41% on the 0.95-level and 44% on the 0.99-level in the countermonotonic case. When we compare the normal and the Student copula, we observe that the Student copula requires more capital due to the tail dependence but it should be noted that the difference, certainly at a 0.95-level is small. The difference is larger if the solvency level increases. For a higher probability level, we detect a smaller difference between the average residual risk for the merger and the stand-alones for the Student copula whereas for the normal copula, it is nearly constant. We also see a clear influence of the strong tail dependence for the flipped Clayton and the Gumbel-Hougaard copula. There is almost no diversification effect in such an environment. Increasing the solvency level makes the effect on the capital even smaller. This is opposed to the situation for the Frank, the Clayton and to a lower extent also for the flipped Gumbel-Hougaard copula. These copulas exhibit a clear diversification benefit both on the capital requirement and on the mean of the residual risk, which increases with the solvency level. We observe that when looking at capital requirements, it is very important to correctly take into account the stochastic dependence in the tails. Our simulations illustrate that if strong tail dependence is present but it is not taken into account, the required capital can be substantially underestimated. With respect to the standard deviation of the residual risk, we note for most of the cases with a Kendall tau of 0.5 an increase for the merger. Only for the Clayton and the Frank copula, this statistic decreases. Note that the level of dependence we chose may be fairly high for practical considerations. Still, the increase when compared to the stand-alone scenario is never larger than 10%. The probability of default can be substantially decreased by merging the risks, mainly in situations where tail dependence is low.

In table 7, we give the same results as in table 5 but now for 2 identical lognormal margins with mean and standard deviation equal to 50 and for a Kendall tau of 0.25. It follows from (17) that the margins have parameters $\mu = 3.565$ and $\sigma = 0.8326$. The choice for Kendall's tau implies that the copulas have parameters and tail dependence as specified in table 6:

Copula	ρ_τ	α	λ_L	λ_U
C_α	0.25	0.383	0	0
$C_{4,\alpha}$	0.25	0.383	0.195	0.195
$C_{C,\alpha}$	0.25	0.667	0.354	0
$C_{F,\alpha}$	0.25	2.372	0	0
$C_{G,\alpha}$	0.25	1.333	0	0.318

Table 6: Parameters and tail dependence for a Kendall tau of 0.25.

In table 6, we see that the lower tail dependence for the Clayton copula is, just as for a Kendall tau of 0.5, larger than the upper tail dependence for the Gumbel-Hougaard and the tail dependence for Student copula.

Copula	Variable	$\mathbb{E}_{.95}$	$\sigma_{.95}$	$\mathbb{E}_{.99}$	$\sigma_{.99}$	$\mathbb{P}_{.95}$	$\mathbb{P}_{.99}$	TVaR.95	TVaR.99
C_U	RR_X	2.817	34.23	0.737	19.68	0.983	0.997	417	674
	$RR_{X_{1;2}}$	2.817	34.23	0.737	19.68	0.983	0.997	417	674
	Div Ben	0%	0%	0%	0%	0.000	0.000	0%	0%
C_L	RR_X	1.583	19.49	0.405	10.88	0.984	0.997	265	408
	$RR_{X_{1;2}}$	2.831	24.57	0.733	13.90	0.967	0.993	417	674
	Div Ben	44%	21%	45%	22%	0.017	0.003	36%	40%
C_I	RR_X	1.662	19.91	0.416	10.87	0.983	0.997	314	466
	$RR_{X_{1;2}}$	2.804	24.33	0.738	13.88	0.967	0.993	417	674
	Div Ben	41%	18%	44%	22%	0.016	0.003	25%	31%
$C_{0.383}$	RR_X	1.983	23.23	0.495	13.09	0.983	0.997	350	531
	$RR_{X_{1;2}}$	2.807	25.15	0.748	14.54	0.968	0.993	417	676
	Div Ben	29%	8%	34%	10%	0.014	0.003	16%	21%
$C_{4, 0.383}$	RR_X	2.294	28.65	0.642	17.68	0.983	0.997	359	573
	$RR_{X_{1;2}}$	2.833	28.41	0.763	16.95	0.971	0.994	418	680
	Div Ben	19%	-1%	16%	-4%	0.013	0.003	14%	16%
$C_C, 0.667$	RR_X	1.721	20.09	0.430	11.30	0.983	0.997	329	484
	$RR_{X_{1;2}}$	2.829	24.41	0.752	14.31	0.967	0.993	417	676
	Div Ben	39%	18%	43%	21%	0.016	0.003	21%	28%
$\bar{C}_C, 0.667$	RR_X	2.546	31.21	0.682	18.69	0.984	0.997	379	614
	$RR_{X_{1;2}}$	2.807	29.93	0.740	17.64	0.973	0.994	417	675
	Div Ben	9%	-4%	8%	-6%	0.011	0.002	9%	9%
$C_F, 2.372$	RR_X	1.757	20.44	0.438	11.67	0.983	0.997	339	498
	$RR_{X_{1;2}}$	2.831	24.47	0.746	14.38	0.967	0.993	418	677
	Div Ben	38%	16%	41%	19%	0.015	0.003	19%	26%
$C_G, 1.333$	RR_X	2.487	31.65	0.667	18.10	0.984	0.997	371	599
	$RR_{X_{1;2}}$	2.839	30.52	0.746	17.14	0.972	0.994	418	677
	Div Ben	12%	-4%	11%	-6%	0.011	0.002	11%	11%
$\bar{C}_G, 1.333$	RR_X	1.852	21.80	0.465	12.53	0.983	0.997	338	505
	$RR_{X_{1;2}}$	2.818	24.86	0.750	14.72	0.968	0.993	416	674
	Div Ben	34%	12%	38%	15%	0.015	0.003	19%	25%

Table 7: Risk measures of residual risk for 2 lognormal risks ($\rho_\tau = 0.25$).

In table 7, we observe that among all dependence structures, there is a maximum diversification benefit on the TVaR for the merger of about 36% on the 0.95-level and 40% on the 0.99-level. If we exclude the comonotonic case, we again have a minimum diversification benefit for the flipped Clayton copula, which is about 9% both at a 0.95 and 0.99 probability level. For the Gumbel-Hougaard copula, the diversification benefit on the TVaR now also remains nearly constant in function of the probability level, whereas for

the Student copula, it now even increases with the probability level. However, for each of the latter 3 copulas, the average residual risk still shows a lower relative diversification benefit when increasing the probability level. We see a clear influence of the fatter tails for the lognormal marginals both on the TVaR and on the risk measures of the residual risk. The subadditivity of the TVaR is never such that it increases the average residual risk of the merger. However, in cases with (strong) positive tail dependence, the diversification benefit on the residual risk remains small if we compare with the independent situation and with the situations without tail dependence. The probabilities of default are slightly larger than for the exponential margins, which have less fat tails.

5.2 Merger of 5 Identical Companies

In table 8, we give the results of the same simulations as in table 5 but in 5 dimensions. Note that for the normal and the Student copula, all elements in the correlation matrix, apart from those on the diagonal, are equal to the value specified in table 4. This may in practice of course be not very realistic but we make this choice in order to use copulas which are comparable with respect to their Kendall tau. For the normal and Student copulas, it is possible to use different dependence structures between the margins (if we look at them 2 by 2). For multivariate Archimedean copulas, the dependence structure between the margins (2 by 2) is always the same and based on solely one parameter.

Copula	Variable	E _{.95}	$\sigma_{.95}$	E _{.99}	$\sigma_{.99}$	P _{.95}	P _{.99}	TVaR _{.95}	TVaR _{.99}
C_U	RR_X	4.614	47.5	0.897	21.1	0.981	0.996	999	1400
	$RR_{X_{1;5}}$	4.614	47.5	0.897	21.1	0.981	0.996	999	1400
	Div Ben	0%	0%	0%	0%	0.000	0.000	0%	0%
C_I	RR_X	1.340	13.4	0.258	6.0	0.981	0.996	533	649
	$RR_{X_{1;5}}$	4.575	21.2	0.927	9.7	0.911	0.982	999	1405
	Div Ben	71%	37%	72%	38%	0.070	0.015	47%	54%
$C_{0.707}$	RR_X	3.608	37.3	0.736	17.1	0.982	0.996	870	1198
	$RR_{X_{1;5}}$	4.578	32.7	0.938	14.0	0.944	0.987	997	1408
	Div Ben	21%	-14%	22%	-22%	0.038	0.009	13%	15%
$C_{4, 0.707}$	RR_X	4.117	43.4	0.871	20.0	0.982	0.996	888	1263
	$RR_{X_{1;5}}$	4.585	38.7	0.931	17.2	0.954	0.990	996	1402
	Div Ben	10%	-12%	6%	-17%	0.028	0.006	11%	10%
$C_C, 2$	RR_X	1.800	17.6	0.300	6.7	0.981	0.996	707	857
	$RR_{X_{1;5}}$	4.623	22.3	0.914	9.6	0.917	0.982	1000	1402
	Div Ben	61%	21%	67%	30%	0.063	0.014	29%	39%
$\bar{C}_C, 2$	RR_X	4.604	48.0	0.918	21.7	0.982	0.996	966	1363
	$RR_{X_{1;5}}$	4.613	45.7	0.917	20.7	0.970	0.994	998	1396
	Div Ben	0%	-5%	0%	-5%	0.012	0.002	3%	2%
$C_F, 5.736$	RR_X	2.120	20.1	0.345	7.6	0.980	0.996	782	960
	$RR_{X_{1;5}}$	4.588	22.9	0.923	9.8	0.923	0.982	997	1403
	Div Ben	54%	12%	63%	22%	0.057	0.014	22%	32%
$C_G, 2$	RR_X	4.594	47.7	0.914	21.3	0.982	0.996	946	1337
	$RR_{X_{1;5}}$	4.630	43.5	0.913	19.2	0.960	0.992	1001	1397
	Div Ben	1%	-10%	0%	-11%	0.022	0.005	6%	4%
$\bar{C}_G, 2$	RR_X	2.799	28.2	0.522	12.0	0.981	0.996	801	1045
	$RR_{X_{1;5}}$	4.604	27.6	0.918	11.1	0.935	0.985	999	1403
	Div Ben	39%	-2%	43%	-9%	0.046	0.012	20%	25%

Table 8: Risk measures of residual risk for 5 exponential risks ($\rho_\tau = 0.5$).

As we could expect, the diversification benefit on the TVaR and the default probability is now always larger than in the 2-dimensional situation. For the average residual risk, we have a higher diversification benefit as well. From our list of copulas, now the independent situation of course leads to the highest diversification benefit. For the rest, the same conclusions as in dimension 2 hold. In cases where the volatility of the residual risk

of the merger increased compared to the stand-alone situation, it now increases more and vice versa if it decreased. If our main interest is the average residual risk or the default probability, in these examples, the TVaR is never too subadditive.

In table 9, we give the same results as in table 7 but for 5 identical lognormal risks.

Copula	Variable	E. ₉₅	σ ₉₅	E. ₉₉	σ ₉₉	P. ₉₅	P. ₉₉	TVaR ₉₅	TVaR ₉₉
C_U	RR_X	7.116	88.0	1.906	52.0	0.983	0.997	1049	1698
	$RR_{X_{1;5}}$	7.116	88.0	1.906	52.0	0.983	0.997	1049	1698
	Div Ben	0%	0%	0%	0%	0.000	0.000	0%	0%
C_I	RR_X	2.078	23.7	0.495	12.3	0.983	0.997	564	755
	$RR_{X_{1;5}}$	7.018	38.3	1.875	22.2	0.920	0.983	1040	1691
	Div Ben	70%	38%	74%	45%	0.063	0.013	46%	55%
$C_{0.383}$	RR_X	3.624	40.8	0.851	21.9	0.983	0.996	749	1072
	$RR_{X_{1;5}}$	7.066	43.9	1.886	24.4	0.931	0.985	1045	1695
	Div Ben	49%	7%	55%	10%	0.051	0.012	28%	37%
$C_{4, 0.383}$	RR_X	4.897	61.1	1.338	38.4	0.984	0.997	797	1239
	$RR_{X_{1;5}}$	7.105	59.3	1.883	35.3	0.945	0.988	1045	1692
	Div Ben	31%	-3%	29%	-9%	0.039	0.009	24%	27%
$C_C, 0.667$	RR_X	2.244	24.8	0.542	14.1	0.982	0.997	636	847
	$RR_{X_{1;5}}$	6.980	38.4	1.937	24.1	0.922	0.984	1039	1696
	Div Ben	68%	35%	72%	41%	0.060	0.013	39%	50%
$\bar{C}_C, 0.667$	RR_X	5.990	74.6	1.661	46.3	0.984	0.997	885	1433
	$RR_{X_{1;5}}$	7.054	69.3	1.894	42.2	0.954	0.991	1042	1688
	Div Ben	15%	-8%	12%	-10%	0.029	0.006	15%	15%
$C_F, 2.372$	RR_X	2.582	27.9	0.532	13.3	0.982	0.996	693	918
	$RR_{X_{1;5}}$	7.044	39.7	1.851	22.2	0.924	0.983	1042	1690
	Div Ben	63%	30%	71%	40%	0.058	0.013	34%	46%
$C_G, 1.333$	RR_X	5.901	72.6	1.615	43.9	0.984	0.997	855	1411
	$RR_{X_{1;5}}$	7.012	64.1	1.861	37.8	0.946	0.989	1041	1695
	Div Ben	16%	-13%	13%	-16%	0.038	0.008	18%	17%
$\bar{C}_G, 1.333$	RR_X	2.752	30.6	0.649	17.2	0.982	0.997	683	934
	$RR_{X_{1;5}}$	6.998	40.3	1.883	24.1	0.926	0.984	1040	1689
	Div Ben	61%	24%	66%	29%	0.056	0.012	34%	45%

Table 9: Risk measures of residual risk for 5 lognormal risks ($\rho_\tau = 0.25$).

In the case the risks are independent, we again observe a very substantial benefit on the required capital. For all copulas, the relative decrease of the TVaR and of the average residual risk is again more important than when we compare with the 2-dimensional case. Just as we observed by comparing table 5 and 8, if in 2 dimensions, the standard deviation of the residual risk of the merger increased compared to that of the sum of the stand-alone companies, it now increases more and vice versa if it decreased.

In table 10, we give the same results as in table 9 but for 5 identical lognormal risks with mean 50 and a coefficient of variation (COV) of 25%. This means that we use the parameters $\mu = 3.882$ and $\sigma = 0.2462$ for the margins. This may correspond better to the shape of aggregate distributions for different lines of business.

In this setting, the tail is substantially smaller than for a COV of 100%. Therefore, the TVaR is reduced importantly. The relative diversification benefit on the TVaR for the merger varies between 52% of the value in table 9 for the independent copula and 38% for the Gumbel copula (both at a 99%-level). At a 95%-level, the relative diversification benefit on the average residual risk for the mergers is reduced most importantly for the Gumbel and the flipped Clayton copula (resp. 4% and 26% of the diversification benefit in table 9). For the other copulas, it varies between 63% of the level in table 9 for the Student copula and 93% of this level for the independent copula. A comparable conclusion can be drawn

at a 99%-level. For all simulations, the diversification “benefit” on the standard deviation of the residual risk is lower than in table 9. The probabilities that the residual risk is zero are slightly lower than in table 9. The differences between the default probabilities for the merger and the stand-alones increase with about 10% in these scenarios.

Copula	Variable	$\mathbb{E}_{.95}$	$\sigma_{.95}$	$\mathbb{E}_{.99}$	$\sigma_{.99}$	$\mathbb{P}_{.95}$	$\mathbb{P}_{.99}$	TVaR _{.95}	TVaR _{.99}
C_U	R_X	0.737	7.52	0.140	3.20	0.981	0.996	405	468
	$R_{X_{1:5}}$	0.737	7.52	0.140	3.20	0.981	0.996	405	468
	Div Ben	0%	0%	0%	0%	0.000	0.000	0%	0%
C_I	R_X	0.256	2.54	0.045	1.02	0.981	0.996	313	335
	$R_{X_{1:5}}$	0.737	3.36	0.141	1.47	0.910	0.982	405	469
	Div Ben	65%	24%	68%	30%	0.071	0.015	23%	29%
$C_{0.383}$	R_X	0.458	4.58	0.085	1.95	0.981	0.996	355	393
	$R_{X_{1:5}}$	0.743	3.91	0.139	1.56	0.923	0.983	405	468
	Div Ben	38%	-17%	39%	-25%	0.058	0.013	12%	16%
$C_{4, 0.383}$	R_X	0.600	6.41	0.125	2.96	0.982	0.996	360	412
	$R_{X_{1:5}}$	0.744	5.18	0.143	2.25	0.937	0.987	405	469
	Div Ben	19%	-24%	12%	-31%	0.045	0.009	11%	12%
$C_C, 0.667$	R_X	0.287	2.77	0.049	1.10	0.980	0.996	334	359
	$R_{X_{1:5}}$	0.738	3.42	0.141	1.47	0.912	0.982	405	469
	Div Ben	61%	19%	65%	26%	0.068	0.015	17%	23%
$\bar{C}_C, 0.667$	R_X	0.716	7.33	0.139	3.28	0.981	0.996	377	438
	$R_{X_{1:5}}$	0.746	5.93	0.142	2.60	0.949	0.990	405	468
	Div Ben	4%	-24%	2%	-26%	0.033	0.007	7%	7%
$C_F, 2.372$	R_X	0.331	3.13	0.053	1.17	0.980	0.996	347	375
	$R_{X_{1:5}}$	0.744	3.47	0.141	1.46	0.914	0.982	405	469
	Div Ben	56%	10%	62%	20%	0.066	0.014	14%	20%
$C_G, 1.333$	R_X	0.735	7.53	0.141	3.32	0.981	0.996	375	439
	$R_{X_{1:5}}$	0.740	5.68	0.142	2.51	0.939	0.988	405	469
	Div Ben	1%	-33%	1%	-32%	0.043	0.009	7%	6%
$\bar{C}_G, 1.333$	R_X	0.341	3.37	0.062	1.38	0.981	0.996	340	370
	$R_{X_{1:5}}$	0.740	3.61	0.142	1.49	0.918	0.982	405	469
	Div Ben	54%	7%	57%	8%	0.063	0.014	16%	21%

Table 10: Risk measures of residual risk for 5 lognormal risks with 25% COV ($\rho_\tau = 0.25$).

In all 5-dimensional examples, the TVaR seems to give an acceptable benefit to the diversification for the merger. The subadditivity of the TVaR is never such that the average residual risk and the probability of default of the merger increase compared to the situation of the stand-alones.

6 Conclusion

Our examples have clearly demonstrated the possible diversification benefit when risks are being merged. This benefit plays not only on the required capital but also on the residual risk after capital allocation and on the default probability. If the TVaR is used as a risk measure, merging risks can be in the interest both of the shareholders and of the regulator defending the interests of the policyholders. When using the average residual risk as a benchmark, our examples demonstrate that the TVaR is not too subadditive under a wide range of dependence structures. The same holds when the default probability is considered. The standard deviation of the residual risk for the merger can be both larger and smaller than the sum of the residual risk for the stand alone companies, depending on the strength of the (tail) dependence. These results make us believe that the TVaR is a very valid candidate for a risk measure providing, in itself, a basis for compromise between the interests of the shareholders and the regulator.

Based on a simulation model taking various copulas for describing the dependence structure between the margins, we have seen that results for copulas which all have the same Kendall tau may be very different. This clearly illustrates the importance of the tails in general and the effect of tail dependence in particular for capital calculation purposes. If possible tail dependence in the data is being neglected, this may lead to a capital relief for a merger which is too large.

In case of positive upper tail dependence and a high value for Kendall's tau, the diversification benefit on the capital relatively decreases if we increase the solvency level. If Kendall's tau and the strength of the upper tail dependence decrease, this is not necessarily true. When there is no upper tail dependence, a higher solvency level leads to a higher relative diversification benefit, both for higher and lower values for Kendall's tau.

In a multi-dimensional setting, Archimedean copulas based on one parameter may lack flexibility to effectively capture the dependence structure. The normal and Student copulas are more flexible in this sense since they allow to work with more parameters based on a correlation matrix. The normal copula has the drawback that it does not allow to model tail dependence. We have seen that the tail dependence for the Student copula drives the results for the normal and Student copula further apart for higher probability levels. Further analysis based on real-life data should be made to see whether the Student copula is flexible enough to capture the tail dependence in an appropriate manner.

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