Modelling Dependence of Interest, Inflation and Stock Market returns

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Background

- *Started as part of 'CAS Research Working Party on Correlations and Dependencies among all Risk Sources'.*
- *Earlier version published in CAS 2006 Winter Forum.*
- *Original aims:*
  - *Explore the use of copulas for practical problems. Practitioners often still prefer simpler methods due to difficulties in implementation.*
  - *Study the impact of 'dependence over time' on the present value of a liability;*
- *ASTIN 2008 version:*
  - *Also explored three dimensional copula, not just two dimensions as before.*

Background

- Why?

‘*Negative correlations among asset classes, so evident during an expansion, can collapse as all asset prices fall together, undermining the strategy of improving risk/reward trade-offs through diversification.’*

Alan Greenspan, FT 16 March 2008 ‘We will never have a perfect model for risk’.

=> ***There is a real need for more realistic modelling of dependence!***

Theory

- What is a Copula?

In mathematical terms: a multivariate distribution function of a number of Uniform[0,1] distributed variables: $C[0,1]^n \rightarrow [0,1]$.

In more practical terms: a full and unambiguous definition of the dependence of multiple stochastic variables.

=> Whenever you simulate two or more stochastic variables in combination, you are using a Copula!!

For example: *Independence* is a copula;
Comonotonicity is a copula;

Linear correlation is not a copula!

Example

Suppose X~ N(0,1), Y ~N(0,1)

Q1: What is the distribution of X+Y?

Answer: Don't know. Dependence of X and Y not specified.

Suppose X~ N(0,1), Y ~N(0,1), $\rho(X,Y) = 0.5$

Q 2. What is the variance of X+Y?

Answer: 1+2*0.5 +1 =3

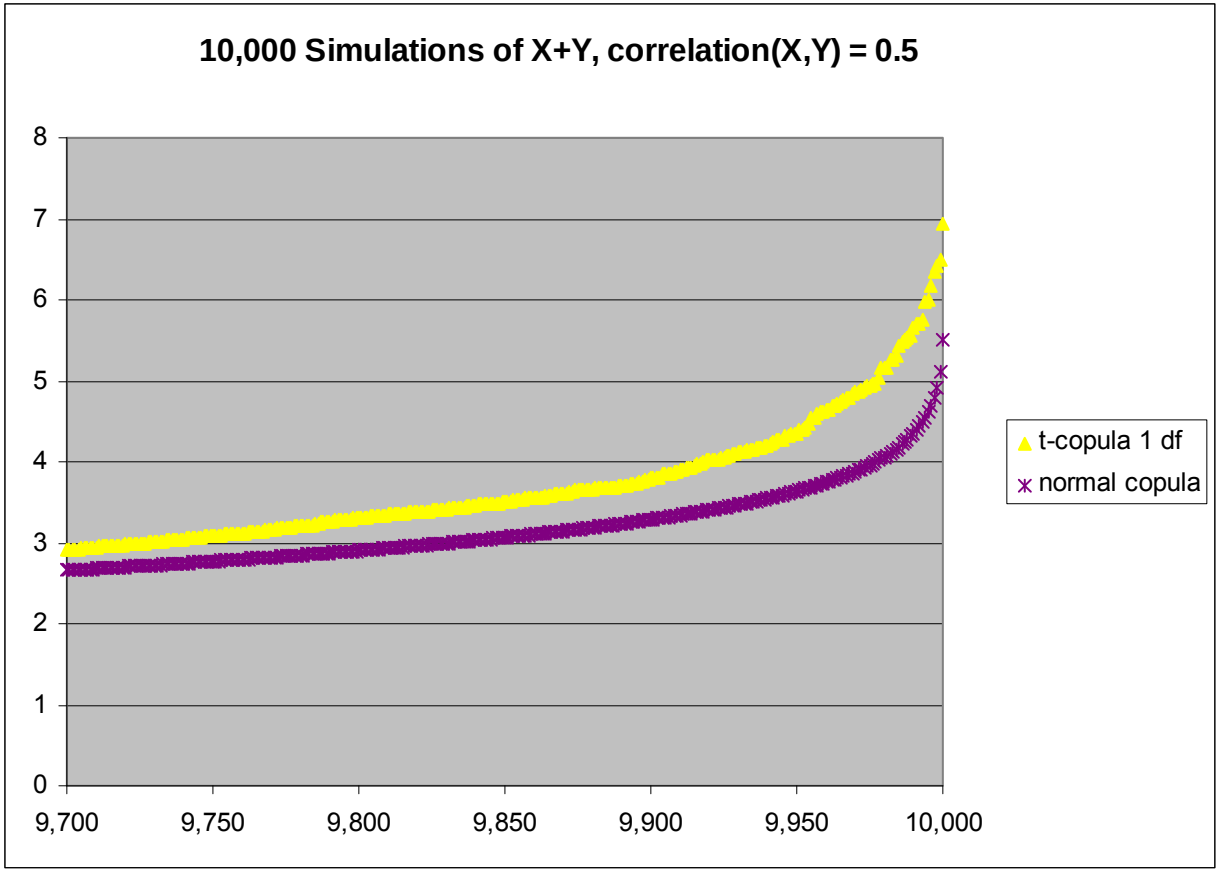
Q 3. What is the distribution of X+Y?

Answer: don't know

If (X,Y) follow a multivariate normal distribution, then X+Y is also normal.

Multivariate Normality is a copula

Example


10,000 Simulations of X+Y, correlation(X,Y) = 0.5
8
7
6
5
4
3
2
1
0
9,700
9,750
9,800
9,850
9,900
9,950
10,000
t-copula 1 df
normal copula

Article

Main Question:

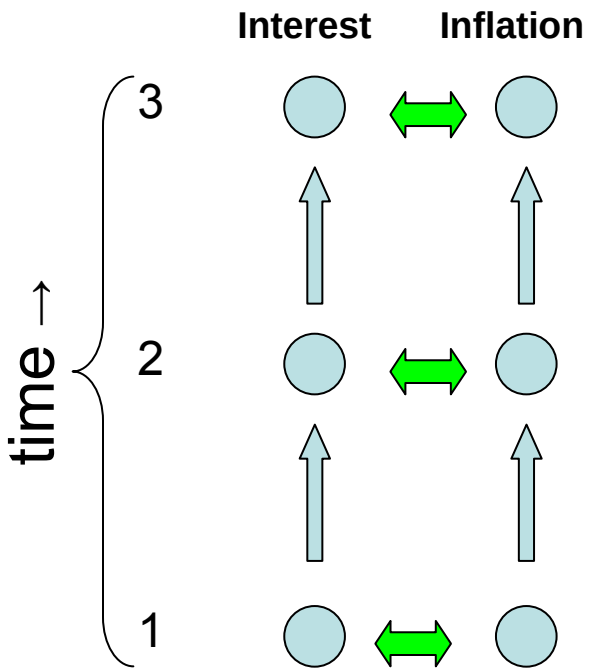
What is the distribution of the present value of a claims liability under the impact of uncertain future interest and (line specific) inflation rates?

Approach:

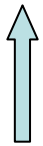
- *Model interest and inflation as time series;*
- *Model dependence between interest and inflation.*

Modelling structure

How to combine time series with dependence of interest and inflation within the same structure?



Stochastic dependence of error terms



Expectation of x(t) conditionally given x(t-1), x(t-2)

Realisations of interest/inflation rates

In formulas

Interest (CIR)

$$Int(t) = max\{0, Int(t-1) + a[b - Int(t-1)] + \sqrt{Int(t-1)}\varepsilon_{int}(t)\}$$

Inflation

$$ln[Inf(t)] = c_0 + c_1\, ln[Inf(t-1) + c_2\, ln[Inf(t-2)] + \varepsilon_{l\_inf}(t)$$

Dependence of $\varepsilon_{l\_inf}(t)$ *and* $\varepsilon_{int}(t)$ *? => Find copula*

Copulas

- The model has been fitted using over 20 years of historic experience.
- From the fitted model, we obtain pairs of residuals $(\varepsilon_{l\_inf}(t), \varepsilon_{int}(t))$.
- From the pairwise residuals we try to fit a copula between $\varepsilon_{l\_inf}(t)$ and $\varepsilon_{int}(t)$.

Fit of Copulas

- Frees and Valdez (1998) describe a procedure to fit ‘Archimedean’ Copulas.
- Archimedean copulas are symmetric in all variables =>
- The dependence between each pair of variables is the same.

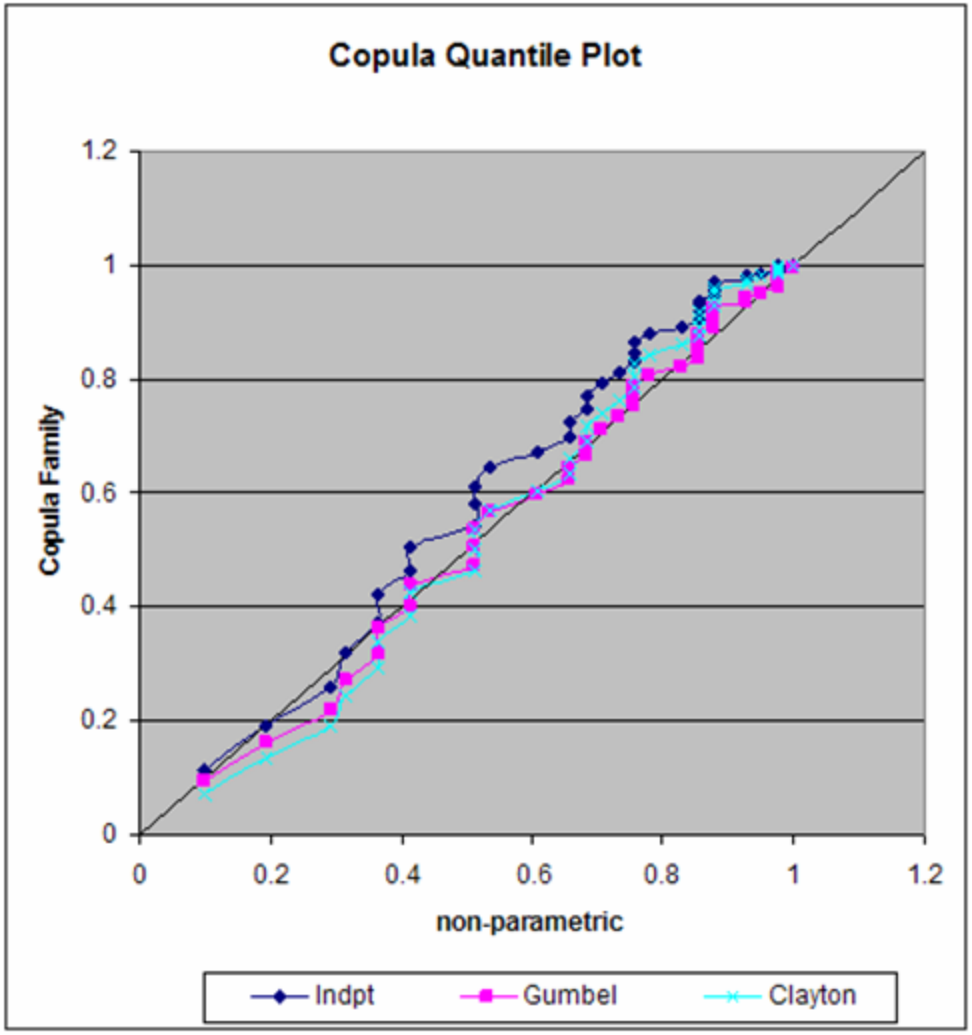
Fit of Copulas

We use the fitting procedure of Frees & Valdez to test the fit of three Archimedan copulas:

- Independent Copula;
- Gumbel Copula (right tailed);
- Clayton Copula (left tailed);

The procedure uses a quantile plot similar to the one used to fit a univariate distribution. Realisations are plotted versus expected values under the Copula assumption.

Fit of Copulas



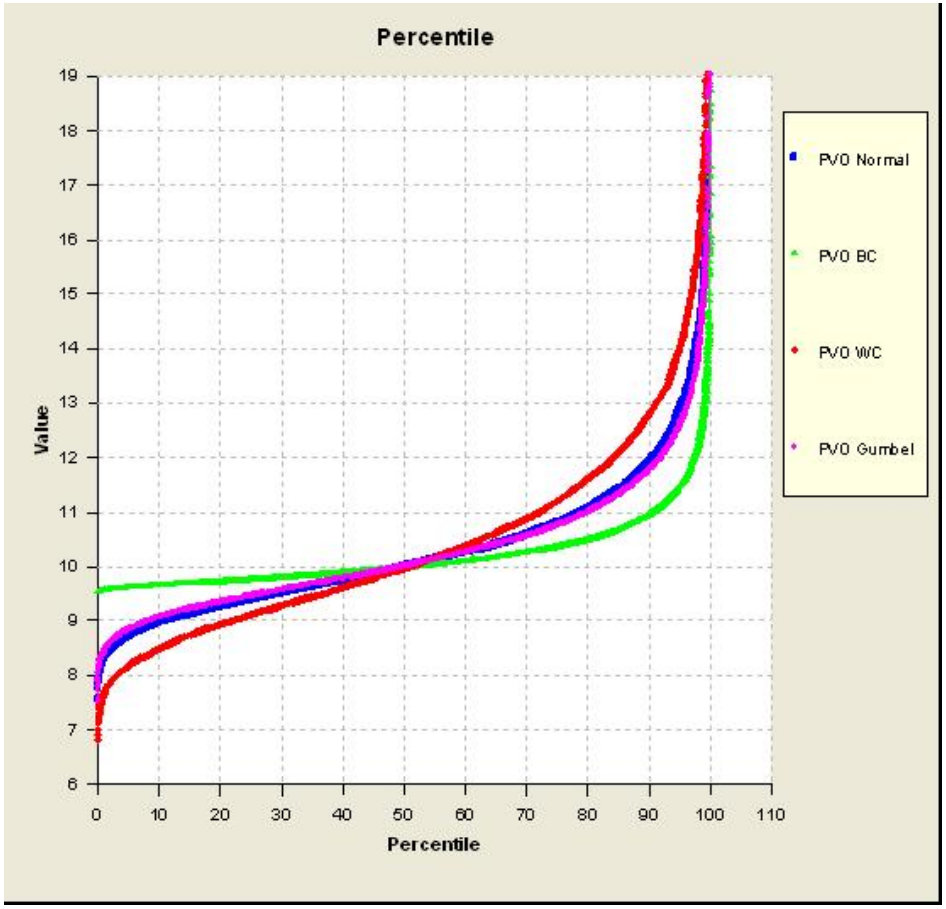
Result of Copula Fit

Copula	Parameter α	Sum of squared distances
Independent	n/a	0.169048
Gumbel	1.27	0.025189
Clayton	0.38	0.080636

N.B. the linear correlation between $\varepsilon_{l\_inf}(t)$ and $\varepsilon_{int}(t)$ equals 0.19 with a standard error of 0.15.

Although the linear correlation is not significant, the Copula plot clearly suggests the two variables are not independent.

Distribution of PV



Results

- The Gumbel copula provides the best fit.
- In this case study the multivariate Normal Copula gives almost the same result.
- If we extend with a third variable, then the Gumbel is not so suitable anymore, as all dependence relations have to be the same.
- The Normal Copula on the other hand allows for different dependence between different variables.

Recap

- Why use copulas?
- What is (not) a copula?
- How do we choose the ‘right’ copula?
- How to define model for interest + inflation rates in multiple future years taking into account all dependencies.
- In article: also consider three dimensional case: interest, inflation and stock market returns.

Take-aways

- There is a need for modelling of dependencies beyond linear correlations. Copulas and time series are useful in practice for this purpose.
- The class of Archimedean copulas allows for inspection of the goodness of fit of the copula in a graphical plot. However applicability is mostly limited to the two-dimensional case.
- In any simulation with multiple stochastic variables, a copula is used. The normal (and also t-copula) allow for the specification of multiple dependence relations.

Make everything as simple as possible, but no simpler.

- Albert Einstein