Optimal Insurance Coverage of a Durable Consumption Good with a Premium Loading in a Continuous Time Economy

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Outline

1. Introduction
2. A Model
3. Comparative Statics
4. Numerical Examples
5. Conclusions
Earlier studies

Providing insurance is a costly activity

- Private insurance markets and social insurance programs have been instrumental in improving welfare over time.
- However, providing insurance needs positive premium loading.
- Therefore, insurance premium $> \text{expected loss}$.
- When insurance is costly, the choice of insurance coverage is not simple.
Earlier studies

Static Models

• Mossin (1968) showed that full insurance coverage is not optimal.
• Arrow (1971) showed that it is optimal to purchase a deductible insurance.
• These classic literature takes a static economy without consumption.
• Therefore the agent cannot hedge against the shock of loss by reducing his consumption over time.
• To study the interaction between the risk of loss and consumption, we should use a continuous time model.
Since Merton (1969), a considerable number of studies have been conducted on *intertemporal consumption and investment* strategy in a continuous time economy.

1. Merton (1969, 1971): The optimal portfolio is equal to the *tangency portfolio* which is given under the CAPM (*static model*).

2. Merton (1973): When future investment opportunities are *time-varying*, the optimal portfolio *differs* from the tangency portfolio.
Intertemporal consumption and investment problems *with insurance* can be classified into two groups.

   - Briys (1986), Gollier (1994)
   - Investment strategy and consumption decision are separated from insurance choice.
   - The results for insurance are **consistent** with the classic models (Arrow, Mossin).
   - Optimal portfolio is given by **CAPM**.

2. **Extension of Merton (1973)**
   - Investment strategy and consumption decision **cannot** be separated from insurance choice.
   - The results are **inconsistent** with the classic models and **CAPM**.
Earlier studies

Introduce a durable good

Our model:

- We consider a *durable consumption good* which can be insured as a case when results are inconsistent with the static models and CAPM.

The reason for inconsistency with the static models:

- When durable goods provides utility,
  - consumption by durable goods at risk will be substituted by perishable consumption.

- When durable goods can be traded and give time-varying investment opportunities,
  - the investment strategy is affected through the risk of durable goods held.
Earlier studies

Optimal Consumption and Investment with a Durable Good

More about durable goods:

- Insurable assets are often durable consumption goods such as **housing** and **motorcars**.
  - It is necessary to investigate the insurance demand for durable consumption goods.
  - Little attention has been given to the research of optimal insurance coverage of durable consumption goods.

On the other hand:

In our model, we take into account the following features of a durable good:

- A durable good can be stored and provides utility to its owner over a period of time.
- A durable good can be resold and acts as a physical asset.
- The stock of a durable good depreciates at a certain rate over time.
- The unit price of durable goods follows a geometric Brownian motion and is partly correlated with the price of a financial risky asset.
- Beside these basic features, we assume a durable good can be insured against damage or loss.
Objective and Approach

Objective:
(A) To determine optimal insurance coverage for a durable consumption good.
(B) To investigate how optimal policies are affected from the risk of damage and the premium loading of its insurance.

Approach:
- Extend Damgaard et al. (2003)

New Feature:
- Durable goods can be insured against damage or loss
Financial assets:
\[ dS_0(t)/S_0(t) = r dt, \quad dS(t)/S(t) = \mu dt + \sigma_S dw_1(t) \]

The unit price of a durable good:
\[ \frac{dP(t)}{P(t)} = \mu_P dt + \sigma_{P1} dw_1(t) + \sigma_{P2} dw_2(t) \]  
(2)

The amount of durable goods
\[ \frac{dK(t)}{K(t)} = -\delta dt + \lambda d\ell dt - \ell dN(t), \]  
(3)

Insurance payment:  \[ p(t) \geq 0 \]

Insurance premium:  \[ q(t) = \lambda \phi p(t), \quad \phi > 1 \]

\( \phi \) represents loading factor.
Assumption

- Financial securities and durable goods can be bought in unlimited quantities and are infinitely divisible.

- Financial securities can be sold short but durable goods can not be sold short.

- There are no transaction costs.
The wealth process

- The wealth of the agent: \( X(t) = \theta_0(t) + \theta(t) + K(t)P(t) \)
- \( \theta_0(t), \theta(t) \): The amount invested in risk-free and risky assets.
- The wealth price process of the agent

\[
\begin{align*}
    dX(t) &= \left( r(X(t) - K(t)P(t) - \theta(t)) + \mu\theta(t) - C(t) - p(t) \\
            &\quad + (\mu_P - \delta + \lambda\ell)K(t)P(t) \right) dt \\
            &\quad + \left( \theta(t)\sigma + K(t)P(t)\sigma_{P1} \right) dw_1(t) \\
            &\quad + K(t)P(t)\sigma_{P2} dw_2(t) \\
            &\quad + \left( q(t-) - \ell P(t)K(t) \right) dN(t)
\end{align*}
\] (4)
Utility function

- Infinite time horizon: \( T = \infty \)
- An utility function:

\[
U(c, k) = \frac{1}{1-\gamma} \left( c^{\beta} k^{1-\beta} \right)^{1-\gamma}
\]

- A solvency condition:

\[
X(\eta) = X(\eta-) - \ell P(\eta-) K(\eta-) + q(\eta) > 0 \quad (5)
\]

- Admissible strategies:

\[
\mathcal{A}(x, k, p) = \{ (\theta, k, c, q) : k > 0, c > 0, x - \ell p k + q > 0, q \geq 0 \}
\]
We will maximize an expected utility:

$$J^S(x, p) = E \left[ \int_0^\infty e^{-\rho t} U(C(t), K(t)) dt \right],$$

w.r.t. the policy of an agent:

$$S = \{S_t: t > 0\}, \quad S_t = (\theta(t), K(t), C(t), q(t)) \in \mathcal{A}, \forall t > 0.$$

Then maximized value of expected utility (a value function)

$$V(x, p) = \sup_{S_t \in \mathcal{A}, \ t > 0} J^S(x, p)$$

satisfies HJB equation (as presented in the next slide).
HJB equation

\[ \rho V(x, p) = \sup_{S \in A} \left\{ \frac{1}{1 - \gamma} (c^\beta k^{1-\beta})^{1-\gamma} \right. \]

\[ + \left. \left( r(x - pk) + \theta(\mu - r) + (\mu_P - \delta)kp - c - \lambda \phi q \right) \frac{\partial V}{\partial x}(x, p) \right. \]

\[ + \frac{1}{2} \left( \theta^2 \sigma^2 + k^2 p^2 \sigma_P^2 + 2\theta \sigma \sigma_P k p \right) \frac{\partial^2 V}{\partial x^2}(x, p) + \mu_P \frac{\partial V}{\partial p}(x, p) \]

\[ + \frac{1}{2} \sigma_P^2 p^2 \frac{\partial^2 V}{\partial p^2}(x, p) + \left( \theta \sigma \sigma_P + \sigma_P^2 k p \right) p \frac{\partial^2 V}{\partial x \partial p}(x, p) \]

\[ + \lambda \left( V(x - \ell kp + q, p) - V(x, p) - \ell kp \frac{\partial V}{\partial x}(x, p) \right) \right\} \]

(8)

The way to solve the equation is straightforward.
Set up

How to Solve

HJB equation (r.h.s is rewritten by a function G):

$$\rho V(x, p) = \sup_{\theta \in \mathbb{R}, c > 0, k > 0, q \geq 0} G \left( V, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2}, \frac{\partial V}{\partial p}, \frac{\partial^2 V}{\partial p^2}, \frac{\partial^2 V}{\partial x \partial p}, \theta, c, k, q \right)$$

subject to $x - \ell pk + q > 0$

1. Reducing the dimensionality: $v(y) = p^{\beta(1-\gamma)} V(x, p)$, $y = x/p$.
2. Give first order condition for $(\theta, c, k)$ ignoring $c, k \geq 0$ and $x - \ell pk + q > 0$.
3. Consider first order condition for $q$ (KKT Condition):

$$q^* = 0 \quad \text{and} \quad v'(0) \leq 0 \quad \text{or} \quad q^* > 0 \quad \text{and} \quad v'(q^*) = 0$$

4. After tedious manipulation, a candidate control policies are given by the feed back forms as in Merton (1973).
The optimal strategies are to keep a constant fraction of wealth:

\[ \bar{\theta}(t) = \alpha_{\theta} \bar{X}(t), \quad \bar{\theta}_0(t) = \alpha_0 \bar{X}(t), \quad \bar{K}(t)P(t) = \alpha_k \bar{X}(t), \]
\[ \bar{C}(t) = \alpha_c \bar{X}(t), \quad \bar{q}(t) = \alpha_q \bar{X}(t) \]

- Constants \( \alpha_{\theta}, \alpha_0, \alpha_c \) and \( \alpha_q \) will be given by an explicit function of \( \alpha_k \).
- While \( \alpha_k \) will be given by a root of an implicit function \( F(\alpha_k) = 0 \).
An explicit solution

An implicit function

Definition of $F(\alpha_k)$

Define a function $F(\alpha_k)$ as

$$F(\alpha_k) = \begin{cases} 
\Lambda_0 + \Lambda_1 \alpha_k + \Lambda_2 \alpha_k^2 \\
+ \frac{\lambda}{\gamma} \left\{ (1 - \ell \alpha_k)^{-\gamma} \left( 1 + \frac{\beta \gamma \ell}{1 - \beta} \alpha_k \right) - \left( 1 + \frac{\gamma \ell}{1 - \beta} \alpha_k \right) \right\}, & \alpha_k < \hat{\alpha}_k \\
\Lambda'_0 + \left( \Lambda_1 + \frac{\lambda (\phi - 1) \ell}{1 - \beta} \right) \alpha_k + \Lambda_2 \alpha_k^2, & \alpha_k \geq \hat{\alpha}_k, \end{cases}$$

where

$$\hat{\alpha}_k = \frac{1 - \phi - \frac{1}{\gamma}}{\ell}, \quad \Lambda'_0 = \Lambda_0 + \frac{\lambda (\phi - 1)}{\gamma} + \lambda \phi \left( \phi - \frac{1}{\gamma} - 1 \right)$$

and where $\Lambda_0, \Lambda_1, \Lambda_2$ are constants which does not depend on $\phi, \lambda$ and $\ell$.

- We note that the domain of function is divided into two parts at threshold $\hat{\alpha}$ as illustrated in the next slide.
An Explicit Solution

A nature of the implicit function

\[ F(x) \]

No insurance is optimal

Partial insurance is optimal

- Note that \( \hat{\alpha}_k \) will give a **deductible** level of insurance coverage.
- We will next show you an existence of a unique root.
A root of the implicit function

Assumption

If $F(\hat{\alpha}_k) > 0$ then

$$\Lambda_0 < -\frac{1}{2}(1 - \gamma)\sigma^2_{P_2}\alpha^2_k + \frac{\lambda}{\gamma} \left\{ 1 + (1 - \ell \alpha_k)^{-\gamma}(-1 + \ell \gamma \alpha_k) \right\}.$$  

If $F(\hat{\alpha}_k) \leq 0$ then $\Lambda'_0 < -\frac{1}{2}(1 - \gamma)\sigma^2_{P_2}\alpha^2_k$.

Lemma

Under Assumption 1, the implicit equation $F(\alpha_k) = 0$ has a unique positive root.

The assumption will become the transversality condition.
An Explicit Solution

Theorem 1

Under Assumption 1, the value function for the problem is given by

$$
\bar{V}(x, p) = \frac{1}{1 - \gamma} \alpha_v p^{-(1 - \beta)(1 - \gamma)} x^{1 - \gamma}
$$

and the controls are given in feedback form as

$$
\bar{\theta}(t) = \alpha_\theta \bar{X}(t), \quad \bar{\theta}_0(t) = \alpha_0 \bar{X}(t), \quad \bar{K}(t) = \alpha_k \bar{X}(t)/P(t), \quad \bar{C}(t) = \alpha_c \bar{X}(t), \quad \bar{q}(t) = \alpha_q \bar{X}(t)
$$

where $\bar{X}(t)$ is the wealth process generated by these controls and where constants $\alpha_v, \alpha_\theta$ are written by

$$
\alpha_v = \alpha_c \beta^{(1 - \gamma) - 1} \alpha_k \beta^{-1} (\gamma - 1) \beta
$$

$$
\alpha_\theta = \frac{\mu - r}{\gamma^2 \sigma S^2} + \left(\beta - (\alpha_k + \beta - 1) \gamma - 1\right) \frac{\sigma P_1}{\gamma \sigma S}
$$

$$
\alpha_0 = 1 - \alpha_\theta - \alpha_k
$$

and where ...
(and where) $\alpha_k$ is a root of the equation $F(\alpha_k) = 0$ and where constants $\alpha_q, \alpha_c$ are given by as follows:

$$\alpha_q = \begin{cases} \ell \alpha_k - \left(1 - \phi \frac{1}{\gamma}\right), & \hat{\alpha}_k \leq \alpha_k < \infty \\ 0, & 0 < \alpha_k < \hat{\alpha}_k, \end{cases}$$

(14)

$$\alpha_c = \begin{cases} -\beta \Lambda_0 - \frac{1}{2} \beta(1 - \gamma) \sigma_P^2 \alpha_k^2 \\ -\beta \left[ \frac{\lambda(\phi - 1)}{\gamma} + \lambda \phi \left(\phi \frac{1}{\gamma} - 1\right) \right], & \hat{\alpha}_k \leq \alpha_k < \infty \\ -\beta \Lambda_0 - \frac{1}{2} \beta(1 - \gamma) \sigma_P^2 \alpha_k^2 \\ + \frac{\lambda \beta}{\gamma} \left\{ 1 + (1 - \ell \alpha_k)^{-\gamma}(-1 + \ell \gamma \alpha_k) \right\}, & 0 < \alpha_k < \hat{\alpha}_k. \end{cases}$$

(15)
An Explicit Solution

Why an actual coverage depends on stock price parameters

\[ F(\alpha_k) = \begin{cases} 
\Lambda_0 + \Lambda_1 \alpha_k + \Lambda_2 \alpha_k^2 + g(\lambda, \ell), & \alpha_k < \hat{\alpha}_k \\
\Lambda_1 + \frac{\lambda \ell (\phi - 1)}{1 - \beta} \alpha_k + \Lambda_2 \alpha_k^2, & \alpha_k \geq \hat{\alpha}_k 
\end{cases} \]

\[ \alpha_k \mapsto (\alpha_c, \alpha_\theta, \alpha_q) \]

\[ \forall \alpha_k < \hat{\alpha}_k \]
\[ \alpha_q = 0 \]

\[ \forall \alpha_k \geq \hat{\alpha}_k \]
\[ \alpha_q = \ell \alpha_k - \left(1 - \phi^{\frac{1}{\gamma}}\right) \]
Effects of premium loading on investment policies

Effects on holding amount of durable goods

Lemma

Assuming $\phi > 1$, the optimal holding policy $\alpha_k$ satisfies:

$$\frac{\partial \alpha_k}{\partial \phi} < 0, \quad \hat{\alpha}_k \leq \alpha_k < \infty;$$

$$\frac{\partial \alpha_k}{\partial \phi} = 0, \quad 0 < \alpha_k < \hat{\alpha}_k;$$

$$\frac{\partial \alpha_k}{\partial \lambda} < 0, \quad \frac{\partial \alpha_k}{\partial \ell} < 0, \quad 0 < \alpha_k < \infty.$$

- When partial coverage is optimal, an increase in $\phi$ decreases demand for durable goods.
- Increases in $\lambda$ and $\ell$ decrease demand for durable consumption goods where partial insurance or no insurance is optimal.
Effects of premium loading on investment policies

Effects on financial strategies

**Proposition**

Assume \( \phi > 1 \) and \( 0 < \beta < 1 \) then premium loadings \( \phi \) can affect the optimal financial investment strategies as follows:

(i) \( \hat{\alpha}_k \leq \alpha_k < \infty \)

\[
\frac{\partial \alpha_\theta}{\partial \phi} = -\frac{\sigma_{P1}}{\sigma_S} \frac{\partial \alpha_k}{\partial \phi} = \begin{cases} 
(+) & \sigma_{P1} > 0, \\
(-) & \sigma_{P1} \leq 0,
\end{cases}
\]

\[
\frac{\partial \alpha_0}{\partial \phi} = \frac{\sigma_{P1} - \sigma_S}{\sigma_S} \frac{\partial \alpha_k}{\partial \phi} = \begin{cases} 
(-) & \sigma_{P1} - \sigma_S > 0, \\
(+) & \sigma_{P1} - \sigma_S \leq 0.
\end{cases}
\]

(ii) \( 0 < \forall \alpha_k < \hat{\alpha}_k \)

\[
\frac{\partial \alpha_\theta}{\partial \phi} = 0, \quad \frac{\partial \alpha_0}{\partial \phi} = 0.
\]

An increase in \( \phi \) affects investment decision through the change in demand for durable goods.
Effects of premium loading on insurance and consumption policies

### Effects of premium loading

Assume $\phi > 1$, then

(i) $\hat{\alpha}_k \leq \forall \alpha_k < \infty$

\[
\frac{\partial \alpha_q}{\partial \phi} = \ell \frac{\partial \alpha_k}{\partial \phi} - \frac{1}{\gamma} \phi^{-\frac{1}{\gamma}-1} = (-),
\]

\[
\frac{\partial \alpha_C}{\partial \phi} = -\beta(1 - \gamma)\sigma^2_{P2} \alpha_k \frac{\partial \alpha_k}{\partial \phi} - \beta \lambda \left( \frac{1}{\gamma} - 1 \right) \left( 1 - \phi^{-\frac{1}{\gamma}} \right)
\]

\[
= -( - ) - ( + ) = ( \pm ).
\]

**Insurance coverage** decreases through both the decrease in demand for durable goods and increase in deductible level.  
**Consumption** is affected by: (1) the decrease of protection for durable consumption goods, (2) increase of insurance premium.
Effects of intensity

Corollary

Assume \( \phi > 1 \), then (i) \( \hat{\alpha}_k \leq \forall \alpha_k < \infty \)

\[
\frac{\partial \alpha_q}{\partial \lambda} = \ell \frac{\partial \alpha_k}{\partial \lambda} = (-),
\]

\[
\frac{\partial \alpha_C}{\partial \lambda} = -\beta(1 - \gamma)\sigma_P^2 \alpha_k \frac{\partial \alpha_k}{\partial \lambda} - \beta \left\{ \frac{\phi - 1}{\gamma} + \phi \left( \phi \frac{1}{\gamma} - 1 \right) \right\}
\]

\[
= \neg (-) - (+) = (\pm).
\]

(ii) \( 0 < \forall \alpha_k < \hat{\alpha}_k \)

\[
\frac{\partial \alpha_q}{\partial \lambda} = 0
\]

\[
\frac{\partial \alpha_C}{\partial \lambda} = -\beta(1 - \gamma)\sigma_P^2 \alpha_k \frac{\partial \alpha_k}{\partial \lambda} \frac{\beta}{\gamma} \left\{ 1 + (1 - \ell \alpha_k)^{-\gamma}(-1 + \ell \gamma \alpha_k) \right\}
\]

\[
= \neg (-) + (\pm) = (\pm).
\]
Assume $\phi > 1$, then

(i) $\hat{\alpha}_k \leq \forall \alpha_k < \infty$

\[
\frac{\partial \alpha_q}{\partial \ell} = (+), \quad \frac{\partial \alpha_C}{\partial \ell} = -\beta(1 - \gamma)\sigma^2_P \alpha_k \frac{\partial \alpha_k}{\partial \ell} = (+).
\]

(ii) $0 < \forall \alpha_k < \hat{\alpha}_k$

\[
\frac{\partial \alpha_q}{\partial \ell} = 0, \quad \frac{\partial \alpha_C}{\partial \ell} = (+).
\]

An increase in $\ell$ increases insurance demand although demand for insured asset decreases.
Parameter settings

\[ r = 0.02, \quad \mu = 0.04, \quad \sigma_S = 0.20, \quad \mu_P = 0.03, \quad \delta = 0.02, \]
\[ \sigma_{P_1} = 0.07, \quad \sigma_{P_2} = 0.07, \quad \gamma = 0.5, \quad \beta = 0.5, \quad \rho = 0.03, \]
\[ \lambda = 1/50, \quad \ell = 0.8, \quad \phi = 1.2. \]
Impact of risk aversion

Impact of changing $\gamma$ when $\phi = 1$

- $\alpha_k$ changes in $\gamma$ just as $\alpha_C$.
- The effects from risk aversion measure $\gamma$ perishable consumptions is not uniform as implied in Merton (1969).
- $\gamma$ also dose not uniformly affect holding policies for durable goods.
Impact of risk aversion

Impact of changing $\phi$ and variation in $\gamma$

- $\alpha_q = \ell \alpha_k - \left(1 - \phi^{-\frac{1}{\gamma}}\right)$
  - A deductible level is decreasing in risk aversion measure $\gamma$.
- However an increase in risk aversion $\gamma$ can decrease demand for durable goods.
- Sometimes, an increase in $\gamma$ can decrease the actual demand for insurance.
Impact of financial risk on insurance policies

Impact of volatilities by changing $\phi$

- The optimal deductible level cannot be affected by financial risk although $\alpha_q$ is influenced by financial risk through $\alpha_k$.
- $\sigma_{P1}$, $\sigma_S$ have a large impact on $\alpha_q$. 
Impact of financial risk on insurance policies

Impact of volatilities by changing $\lambda$

- Deductible level (threshold): $\hat{\alpha}_k = 0.38$.
- Changes in volatilities can diminish demand for insurance.
• We show the optimal insurance policy for durable consumption goods with positive premium loading in a methods of Merton (1973).

• Analytical Results:
  ▶ How premium loading affects investment policies
  ▶ How financial risks affect the optimal insurance policy.

• Numerical Results:
  ▶ The change in $\sigma_S, \sigma_{P1}$ and $\delta$ has a large impact on $\alpha_q$.

• Future works
  ▶ We intend to explore the problem of transaction costs on durable goods trading.