Optimal Insurance Coverage of a Durable Consumption Good with a Premium Loading in a Continuous Time Economy

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Outline

Providing insurance is a costly activity

- Private insurance markets and social insurance programs have been instrumental in improving welfare over time.
- However, providing insurance needs positive premium loading.
- Therefore, insurance premium $>$ expected loss.
- When insurance is costly, the choice of insurance coverage is not simple.

Static Models

- Mossin (1968) showed that full insurance coverage is not optimal.
- Arrow (1971) showed that it is optimal to purchase a deductible insurance.
- These classic literature takes a *static* economy *without* consumption .
- Therefore the agent cannot hedge against the shock of loss by reducing his consumption over time.
- To study the interaction between the risk of loss and consumption, we should use a continuous time model.

Continuous Time Models for Investment and Consumption

- Since Merton (1969), a considerable number of studies have been conducted on *intertemporal consumption and investment* strategy in a continuous time economy.

1. Merton (1969, 1971): The optimal portfolio is equal to the tangency portfolio which is given under the CAPM (*static model*).
2. Merton (1973): When future investment opportunities are time-varying, the optimal portfolio *differs* from the tangency portfolio.

Intertemporal Consumption and Investment with Insurance

Intertemporal consumption and investment problems *with insurance* can be classified into two groups.

1. Extension of Merton (1969,1971)
   - Briys (1986), Gollier (1994)
   - Investment strategy and consumption decision are separated from insurance choice.
   - The results for insurance are consistent with the classic models (Arrow, Mossin).
   - Optimal portfolio is given by CAPM.
2. Extension of Merton (1973)
   - Moore and Young (2006), Gollier (2002, 2003)
   - Investment strategy and consumption decision cannot be separated from insurance choice.
   - The results are inconsistent with the classic models and CAPM.

Introduce a durable good

Our model:

- We consider *a durable consumption good* which can be insured as a case when results are inconsistent with the static models and CAPM.

The reason for inconsistency with the static models:

- When durable goods provides utility,
  - consumption by durable goods at risk will be substituted by perishable consumption.
- When durable goods can be traded and give time-varying investment opportunities,
  - the investment strategy is affected through the risk of durable goods held.

Optimal Consumption and Investment with a Durable Good

More about durable goods:

- Insurable assets are often durable consumption goods such as housing and motorcars.
  - It is necessary to investigate the insurance demand for durable consumption goods.
  - Little attention has been given to the research of optimal insurance coverage of durable consumption goods.

On the other hand:

- Hindy and Huang (1993), Detemple and Giannikos (1996), Cuoco and Liu (2000), Cocco (2004), Cauley, Pavlov and Schwarts (2005) and Grossman and Laroque (1990)
- We follow Damgaard, Fuglsbjerg and Munk (2003) that extend Grossman and Laroque (1990).

Features of durable goods

In our model, we take into account the following features of a durable good:

- A durable good can be stored and provides utility to its owner over a period of time.
- A durable good can be resold and acts as a physical asset.
- The stock of a durable good depreciates at a certain rate over time.
- The unit price of durable goods follows a geometric Brownian motion and is partly correlated with the price of a financial risky asset.
- Beside these basic features, we assume a durable good can be insured against damage or loss.

Objective and Approach

- Objective:
  - (A) To determine optimal insurance coverage for a durable consumption good.
  - (B) To investigate how optimal policies are affected from the risk of damage and the premium loading of its insurance.
- Approach:
  - Extend Damgaard et al. (2003)
- New Feature:
  - Durable goods can be insured against damage or loss

Settings

- Financial assets:
  $dS_0(t)/S_0(t) = rdt$, $dS(t)/S(t) = \mu dt + \sigma_S dw_1(t)$
- The unit price of a durable good:

$$\frac{dP(t)}{P(t)} = \mu_P dt + \sigma_{P1} dw_1(t) + \sigma_{P2} dw_2(t) \qquad (2)$$

- The amount of durable goods

$$\frac{dK(t)}{K(t)} = -\delta dt + \lambda \ell dt - \ell dN(t), \qquad (3)$$

- Insurance payment: $p(t) \geq 0$
- Insurance premium: $q(t) = \lambda \phi p(t)$, $\phi > 1$
- ϕ represents loading factor.

Assumption

- Financial securities and durable goods can be bought in unlimited quantities and are infinitely divisible.
- Financial securities can be sold short but durable goods can not be sold short.
- There are no transaction costs.

The wealth process

- The wealth of the agent: $X(t) = \theta_0(t) + \theta(t) + K(t)P(t)$
- $\theta_0(t), \theta(t)$:The amount invested in risk-free and risky assets.
- The wealth price process of the agent

$$
\begin{aligned}
dX(t) \quad = \quad & \Big(r(X(t) - K(t)P(t) - \theta(t)) + \mu\theta(t) - C(t) - p(t) \\
& +(\mu_P - \delta + \lambda\ell)K(t)P(t)\Big)dt \\
& + \Big(\theta(t)\sigma + K(t)P(t)\sigma_{P1}\Big)dw_1(t) \\
& + K(t)P(t)\sigma_{P2}dw_2(t) \\
& + \Big(q(t-) - \ell P(t)K(t)\Big)dN(t) \qquad (4)
\end{aligned}
$$

Utility function

- Infinite time horizon: $T = \infty$
- An utility function:

$$U(c,k) = \frac{1}{1-\gamma}\left(c^{\beta}k^{1-\beta}\right)^{1-\gamma}$$

- A solvency condition:

$$X(\eta) = X(\eta-) - \ell P(\eta-)K(\eta-) + q(\eta) > 0 \qquad (5)$$

- Admissible strategies:

$$\mathcal{A}(x,k,p) = \{(\theta,k,c,q):\ k>0,\ c>0,\ x-\ell pk+q>0, q\geq 0\}$$

Value function

- We will maximize an expected utility:

$$J^S(x,p) = E\left[\int_0^\infty \mathrm{e}^{-\rho t} U(C(t),K(t))dt\right], \tag{6}$$

w.r.t. the policy of an agent:
$S = \{S_t : t > 0\}, S_t = (\theta(t), K(t), C(t), q(t)) \in \mathcal{A}, \forall t > 0.$

- Then maximized value of expected utility (a value function)

$$V(x,p) = \sup_{S_t \in \mathcal{A},\ t>0} J^S(x,p) \tag{7}$$

satisfies HJB equation (as presented in the next slide).

HJB equation

$$
\begin{aligned}
\rho V(x,p) = \\
\sup_{S\in\mathcal{A}} \Bigg\{ & \frac{1}{1-\gamma}\left(c^{\beta}k^{1-\beta}\right)^{1-\gamma} \\
& + \Big(r(x-pk) + \theta(\mu - r) + (\mu_P - \delta)kp - c - \lambda\phi q\Big)\frac{\partial V}{\partial x}(x,p) \\
& + \frac{1}{2}\Big(\theta^2\sigma^2 + k^2p^2\sigma_P^2 + 2\theta\sigma\sigma_{P1}kp\Big)\frac{\partial^2 V}{\partial x^2}(x,p) + \mu_P p\frac{\partial V}{\partial p}(x,p) \\
& + \frac{1}{2}\sigma_P^2 p^2\frac{\partial^2 V}{\partial p^2}(x,p) + \Big(\theta\sigma\sigma_{P1} + \sigma_P^2 kp\Big)p\frac{\partial^2 V}{\partial x\partial p}(x,p) \\
& + \lambda\Big(V(x-\ell kp + q, p) - V(x,p) - \ell kp\frac{\partial V}{\partial x}(x,p)\Big)\Bigg\} \qquad (8)
\end{aligned}
$$

The way to solve the equation is straightforward.

How to Solve

HJB equation (r.h.s is rewritten by a function G):

$$\rho V(x,p) = \sup_{\theta\in R, c>0, k>0, q\geq 0} G\left(V, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2}, \frac{\partial V}{\partial p}, \frac{\partial^2 V}{\partial p^2}, \frac{\partial^2 V}{\partial x \partial p}, \theta, c, k, q\right)$$

subject to $x - \ell pk + q > 0$

1. Reducing the dimensionality: $v(y) = p^{\beta(1-\gamma)}V(x,p), y = x/p$.
2. Give first order condition for (θ, c, k) ignoring $c, k \geq 0$ and $x - \ell pk + q > 0$.
3. Consider first order condition for q (KKT Condition):

$$(q^* = 0 \quad \textit{and} \quad v'(0) \leq 0) \quad \textit{or} \quad (q^* > 0 \quad \textit{and} \quad v'(q^*) = 0)$$

4. After tedious manipulation, a candidate control policies are given by the feed back forms as in Merton (1973).

A feedback form

- The optimal strategies are to keep a constant fraction of wealth:

$$\bar{\theta}(t) = \alpha_\theta \bar{X}(t), \quad \bar{\theta}_0(t) = \alpha_0 \bar{X}(t), \quad \bar{K}(t)P(t) = \alpha_k \bar{X}(t),$$
$$\bar{C}(t) = \alpha_c \bar{X}(t), \quad \bar{q}(t) = \alpha_q \bar{X}(t)$$

- Constants $\alpha_\theta, \alpha_0, \alpha_c$ and α_q will be given by an explicit function of α_k.
- While α_k will be given by a root of an implicit function $F(\alpha_k) = 0$.

An implicit function

Definition of $F(\alpha_k)$

Define a function $F(\alpha_k)$ as

$$F(\alpha_k) = \begin{cases} \Lambda_0 + \Lambda_1\alpha_k + \Lambda_2\alpha_k^2 \\ +\frac{\lambda}{\gamma}\left\{(1-\ell\alpha_k)^{-\gamma}\left(1+\frac{\beta\gamma\ell}{1-\beta}\alpha_k\right) - \left(1+\frac{\gamma\ell}{1-\beta}\alpha_k\right)\right\}, & \alpha_k < \hat{\alpha}_k \\ \Lambda_0' + \left(\Lambda_1 + \frac{\lambda(\phi-1)\ell}{1-\beta}\right)\alpha_k + \Lambda_2\alpha_k^2, & \alpha_k \geq \hat{\alpha}_k, \end{cases}$$

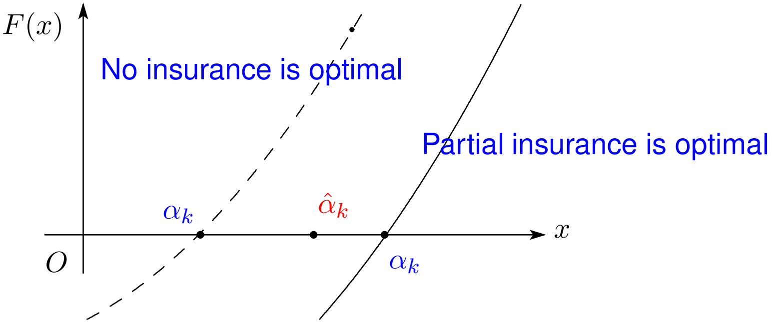
where

$$\hat{\alpha}_k = \frac{1-\phi^{-\frac{1}{\gamma}}}{\ell}, \quad \Lambda_0' = \Lambda_0 + \frac{\lambda(\phi-1)}{\gamma} + \lambda\phi(\phi^{-\frac{1}{\gamma}} - 1)$$

and where $\Lambda_0, \Lambda_1, \Lambda_2$ are constants which does not depend on ϕ, λ and ℓ.

- We note that the domain of function is divided into two parts at threshold $\hat{\alpha}$ as illustrated in the next slide.

A nature of the implicit function



- Note that $\hat{\alpha}_k$ will give a deductible level of insurance coverage.
- We will next show you an existence of a unique root.

A root of the implicit function

Assumption

If $F(\hat{\alpha}_k) > 0$ *then*

$$\Lambda_0 < -\frac{1}{2}(1-\gamma)\sigma_{P2}^2\alpha_k^2 + \frac{\lambda}{\gamma}\left\{1 + (1-\ell\alpha_k)^{-\gamma}(-1+\ell\gamma\alpha_k)\right\}.$$

If $F(\hat{\alpha}_k) \leq 0$ *then* $\Lambda_0' < -\frac{1}{2}(1-\gamma)\sigma_{P2}^2\alpha_k^2$.

Lemma

Under Assumption 1, the implicit equation $F(\alpha_k) = 0$ *has a unique positive root.*

The assumption will become the transversality condition.

Theorem 1

Under Assumption 1, the value function for the problem is given by

$$\bar{V}(x,p) = \frac{1}{1-\gamma}\alpha_v p^{-(1-\beta)(1-\gamma)} x^{1-\gamma} \tag{9}$$

and the controls are given in feedback form as

$$\bar{\theta}(t) = \alpha_\theta \bar{X}(t), \bar{\theta}_0(t) = \alpha_0 \bar{X}(t), \bar{K}(t) = \alpha_k \bar{X}(t)/P(t), \bar{C}(t) = \alpha_c \bar{X}(t), \bar{q}(t) = \alpha_q \bar{X}(t) \tag{10}$$

where $\bar{X}(t)$ is the wealth process generated by these controls and where constants α_v, α_θ are written by

$$\alpha_v = {\alpha_c}^{\beta(1-\gamma)-1} {\alpha_k}^{(\beta-1)(\gamma-1)} \beta \tag{11}$$

$$\alpha_\theta = \frac{\mu - r}{\gamma\sigma_S^2} + \Big(\beta - (\alpha_k + \beta - 1)\gamma - 1\Big)\frac{\sigma_{P1}}{\gamma\sigma_S} \tag{12}$$

$$\alpha_0 = 1 - \alpha_\theta - \alpha_k \tag{13}$$

and where ...

Theorem 1 (cont.)

(and where) α_k *is a root of the equation* $F(\alpha_k) = 0$ *and where constants* α_q, α_c *are given by as follows:*

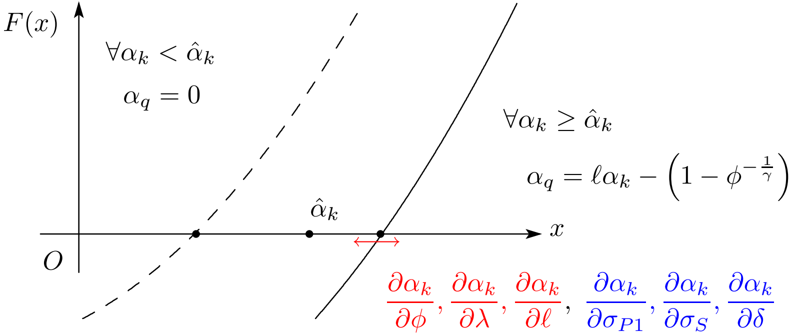
$$\alpha_q = \begin{cases} \ell\alpha_k - \left(1 - \phi^{-\frac{1}{\gamma}}\right), & \hat{\alpha}_k \leq \alpha_k < \infty \\ 0, & 0 < \alpha_k < \hat{\alpha}_k, \end{cases} \quad (14)$$

$$\alpha_c = \begin{cases} -\beta\Lambda_0 - \frac{1}{2}\beta(1-\gamma)\sigma_{P2}^2\alpha_k^2 \\ \quad -\beta\left[\frac{\lambda(\phi-1)}{\gamma} + \lambda\phi(\phi^{-\frac{1}{\gamma}} - 1)\right], & \hat{\alpha}_k \leq \alpha_k < \infty \\ -\beta\Lambda_0 - \frac{1}{2}\beta(1-\gamma)\sigma_{P2}^2\alpha_k^2 \\ \quad + \frac{\lambda\beta}{\gamma}\left\{1 + (1-\ell\alpha_k)^{-\gamma}(-1+\ell\gamma\alpha_k)\right\}, & 0 < \alpha_k < \hat{\alpha}_k. \end{cases} \quad (15)$$

Why an actual coverage depends on stock price parameters

$$F(\alpha_k) = \begin{cases} \Lambda_0 + \Lambda_1 \alpha_k + \Lambda_2 \alpha_k^2 + g(\lambda, \ell), & \alpha_k < \hat{\alpha}_k \\ h(\phi, \lambda, \Lambda_0) + \left(\Lambda_1 + \dfrac{\lambda \ell (\phi - 1)}{1 - \beta} \right) \alpha_k + \Lambda_2 \alpha_k^2, & \alpha_k \geq \hat{\alpha}_k, \end{cases}$$

$$\alpha_k \mapsto (\alpha_c, \alpha_\theta, \alpha_q)$$



Effects on holding amount of durable goods

Lemma

Assuming $\phi > 1$, the optimal holding policy α_k satisfies:

$$
\begin{aligned}
&\frac{\partial \alpha_k}{\partial \phi} < 0, && && \hat{\alpha}_k \le \alpha_k < \infty; \\
&\frac{\partial \alpha_k}{\partial \phi} = 0, && && 0 < \alpha_k < \hat{\alpha}_k; \\
&\frac{\partial \alpha_k}{\partial \lambda} < 0, && \frac{\partial \alpha_k}{\partial \ell} < 0, && 0 < \alpha_k < \infty.
\end{aligned}
$$

- When partial coverage is optimal, an increase in ϕ decreases demand for durable goods.
- Increases in λ and ℓ decrease demand for durable consumption goods where partial insurance or no insurance is optimal.

Effects on financial strategies

Proposition

Assume $\phi > 1$ and $0 < \beta < 1$ then premium loadings ϕ can affect the optimal financial investment strategies as follows:

(i) $\hat{\alpha}_k \leq \alpha_k < \infty$

$$\frac{\partial \alpha_\theta}{\partial \phi} = -\frac{\sigma_{P1}}{\sigma_S}\frac{\partial \alpha_k}{\partial \phi} = \begin{cases} (+), & \sigma_{P1} > 0, \\ (-), & \sigma_{P1} \leq 0, \end{cases},$$

$$\frac{\partial \alpha_0}{\partial \phi} = \frac{\sigma_{P1} - \sigma_S}{\sigma_S}\frac{\partial \alpha_k}{\partial \phi} = \begin{cases} (-), & \sigma_{P1} - \sigma_S > 0, \\ (+), & \sigma_{P1} - \sigma_S \leq 0. \end{cases}$$

(ii) $0 < \forall \alpha_k < \hat{\alpha}_k$. $\dfrac{\partial \alpha_\theta}{\partial \phi} = 0, \quad \dfrac{\partial \alpha_0}{\partial \phi} = 0.$

An increase in ϕ affects investment decision through the change in demand for durable goods.

Effects of premium loading

Assume $\phi > 1$, then
(i) $\hat{\alpha}_k \leq \forall \alpha_k < \infty$

$$\begin{aligned} \frac{\partial \alpha_q}{\partial \phi} &= \ell \frac{\partial \alpha_k}{\partial \phi} - \frac{1}{\gamma} \phi^{-\frac{1}{\gamma}-1} = (-), \\ \frac{\partial \alpha_C}{\partial \phi} &= -\beta(1-\gamma)\sigma_{P2}^2 \alpha_k \frac{\partial \alpha_k}{\partial \phi} - \beta\lambda \left(\frac{1}{\gamma} - 1\right)\left(1 - \phi^{-\frac{1}{\gamma}}\right) \\ &= -(-) - (+) = (\pm). \end{aligned}$$

Insurance coverage decreases through both the decrease in demand for durable goods and increase in deductible level.
Consumption is affected by: (1) the decrease of protection for durable consumption goods, (2) increase of insurance premium.

Effects of intensity

Corollary

Assume $\phi > 1$, *then* (i) $\hat{\alpha}_k \leq \forall \alpha_k < \infty$

$$\begin{aligned} \frac{\partial \alpha_q}{\partial \lambda} &= \ell \frac{\partial \alpha_k}{\partial \lambda} = (-), \\ \frac{\partial \alpha_C}{\partial \lambda} &= -\beta(1-\gamma)\sigma_{P2}^2 \alpha_k \frac{\partial \alpha_k}{\partial \lambda} - \beta \left\{ \frac{\phi - 1}{\gamma} + \phi \left(\phi^{-\frac{1}{\gamma}} - 1 \right) \right\} \\ &= -(-) - (+) = (\pm). \end{aligned}$$

(ii) $0 < \forall \alpha_k < \hat{\alpha}_k$

$$\begin{aligned} \frac{\partial \alpha_q}{\partial \lambda} &= 0 \\ \frac{\partial \alpha_C}{\partial \lambda} &= -\beta(1-\gamma)\sigma_{P2}^2 \alpha_k \frac{\partial \alpha_k}{\partial \lambda} \frac{\beta}{\gamma} \left\{ 1 + (1 - \ell \alpha_k)^{-\gamma} (-1 + \ell \gamma \alpha_k) \right\} \\ &= -(-) + (\pm) = (\pm). \end{aligned}$$

Effects of loss proportion

Corollary

Assume $\phi > 1$, *then*

(i) $\hat{\alpha}_k \leq \forall\alpha_k < \infty$

$$\frac{\partial \alpha_q}{\partial \ell} = (+), \quad \frac{\partial \alpha_C}{\partial \ell} = -\beta(1-\gamma)\sigma_{P2}^2 \alpha_k \frac{\partial \alpha_k}{\partial \ell} = (+).$$

(ii) $0 < \forall\alpha_k < \hat{\alpha}_k$

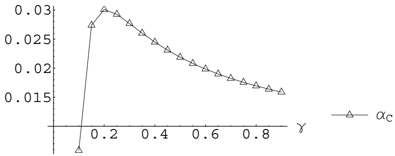
$$\frac{\partial \alpha_q}{\partial \ell} = 0, \quad \frac{\partial \alpha_C}{\partial \ell} = (\pm).$$

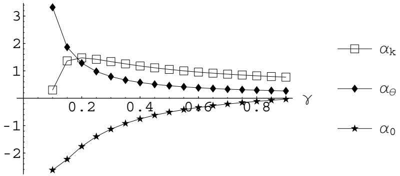
An increase in ℓ increases insurance demand although demand for insured asset decreases.

Parameter settings

$$
\begin{array}{lllll}
r = 0.02, & \mu = 0.04, & \sigma_S = 0.20, & \mu_P = 0.03, & \delta = 0.02, \\
\sigma_{P1} = 0.07, & \sigma_{P2} = 0.07, & \gamma = 0.5, & \beta = 0.5, & \rho = 0.03, \\
\lambda = 1/50, & \ell = 0.8, & \phi = 1.2. & &
\end{array} \tag{24}
$$

Impact of changing γ when $\phi = 1$

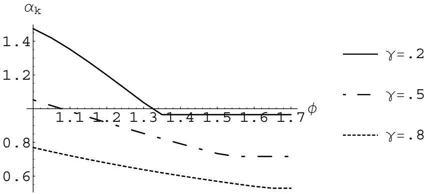


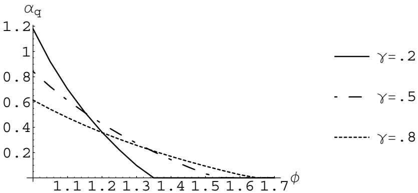


α_k changes in γ just as α_C.

- The effects from risk aversion measure γ perishable consumptions is not uniform as implied in Merton (1969).
- γ also dose not uniformly affect holding policies for durable goods.

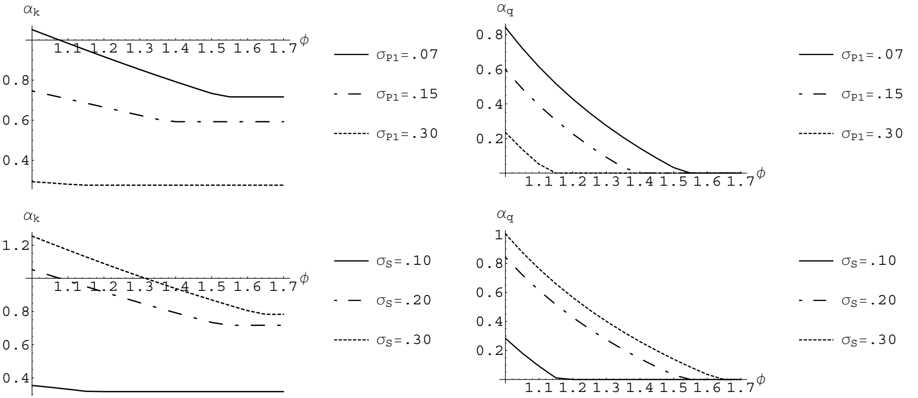
Impact of changing ϕ and variation in γ





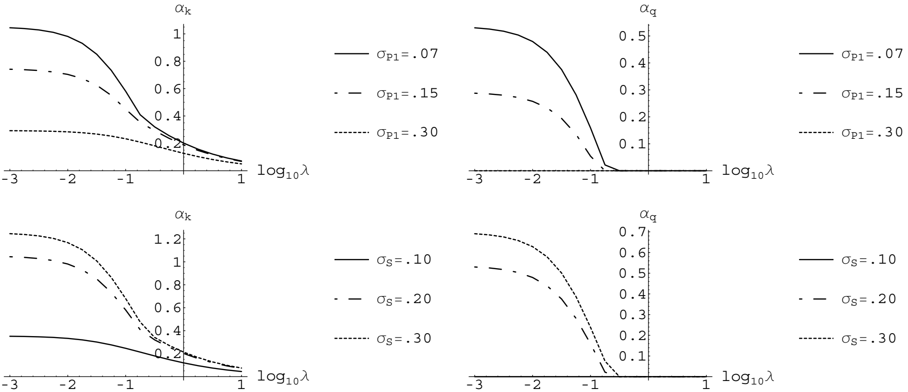
- $\alpha_q = \ell \alpha_k - \left(1 - \phi^{-\frac{1}{\gamma}}\right)$
  A deductible level is decreasing in risk aversion measure γ.
- However an increase in risk aversion γ can decrease demand for durable goods.
- Sometimes, an increase in γ can decrease the actual demand for insurance.

Impact of volatilities by changing ϕ



- The optimal deductible level cannot be affected by financial risk although α_q is influenced by financial risk through α_k
- σ_{P1}, σ_S have a large impact on α_q.

Impact of volatilities by changing λ



- Deductible level (threshold): $\hat{\alpha}_k = 0.38$.
- Changes in volatilities can diminish demand for insurance.

Summury

- We show the optimal insurance policy for durable consumption goods with positive premium loading in a methods of Merton (1973).
- Analytical Results:
  - How premium loading affects investment policies
  - How financial risks affect the optimal insurance policy.
- Numerical Results:
  - The change in σ_S,σ_{P1} and δ has a large impact on α_q.
- Future works
  - We intend to explore the problem of transaction costs on durable goods trading.