Applying Option Pricing Theory to Flood Insurance and Other Catastrophe Risks

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Key Features

- 'Complete' markets in which a pricing kernel (or Radon Nikodym derivative) exists, or full set of Arrow-Debreu state prices are available.
- Markets are, most likely, dynamically incomplete such that pricing by no-arbitrage and continuous trading is not possible because:
  - Some, but not all, prices are observable
  - Underlying (or derivatives) not market traded or lacks liquidity
- Obtained preference free closed form solutions by making strong assumption about the distribution of the underlying, and that at least one fair price is available
- An extensive range of distributions for the underlying is possible by using monotonic transformation.

Advantage of this framework

- Dynamically incomplete markets where perfect/good hedge is not a requirement.
- Useful if the underlying (or derivative) is not market traded or lacks liquidity, has no 'spot' price, cannot be stored (electricity) or where there is no continuous time process (e.g. flood).
- Functional form of the pricing kernel can be inferred from market prices of similar contracts.

Key Foundation Papers

- Rubinstein (1976, BJEMS)
- Brennan (1979, JF)
- Heston (1993, RFS)
- Camara (2003, JF)
- Schroder (2004, JF)

Parallel Development in Insurance

- Hans and Shiu (1994)
    - Esscher transform
- Wang (2000, 2002, 2003)
    - Distortion operator
    - Equilibrium pricing transform
    - Buhlman's 1980 economic model

Premium Principles

Premium principle	Letter	A	B	C	D	E	F	G	H	I	J	K
Property	Name	Net	Exp'd value	Var	Std dev	Exp	Esscher	PH	Equiv utility	Wang	Swiss	Dutch
Number	Name											
1	Independent	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
2	Risk load	Y	Y	Y	Y	Y	N (Y if $Z = X$)	Y	Y	Y	Y	Y
3	Not unjustified	Y	N	Y	Y	Y	Y	Y	Y	Y	Y	Y
4	Max loss	Y	N	N	N	Y	Y	Y	Y	Y	Y	Y
5	Translation	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N
6	Scale	Y	Y	N	Y	N	N	Y	N	Y	N	Y
7	Additivity	Y	Y	N	N	N	N	N	N	N	N	N
8	Subadditivity	Y	Y	N	N	N	N	Y	N	Y	N	Y
9	Superadditivity	Y	Y	N	N	N	N	N	N	N	N	N
10	Add indep.	Y	Y	Y	N	Y	N	N	N	N	N	N
11	Add comono.	Y	Y	N	N	N	N	Y	N	Y	N	N
12	Monotone	Y	Y	N	N	Y	N	Y	Y	Y	Y	Y
13	FSD	Y	Y	N	N	Y	N	Y	Y	Y	Y	Y
14	SL	Y	Y	N	N	Y	N	Y	Y	Y	Y	Y
15	Continuity	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y

Basic Framework

p_i is the probability of state i.

q_i is the state contingent claim on state i, paying \$1 if state i occurs in return for $\$q_i$ as premium;

$$\begin{aligned} \sum_i q_i &= 1, \\ 0 &< q_i \leq 1. \end{aligned}$$

$x_{T,i}$ is a risky cash flow at time T and state i.

$F_{t,T}$ is the time t risk neutral forward price of $x_{T,i}$;

$$F_{t,T} = \sum_i q_i x_{T,i}.$$

ϕ_i is is the forward pricing kernal,

$$\phi_i = \frac{q_i}{p_i},$$

or the probability deflated state price (cf. rn derivative);

$$E\left(\phi_i\right) = \sum_i p_i \phi_i = \sum_i q_i = 1$$

Forward price can also be written as

$$F_{t,T} = \sum_i p_i \phi_i x_{T,i}$$

Objective Function

Wealth maximisation as objective:

$$\max_{W_{T,i}} E\left[U\left(W_T\right)\right] = \sum_i p_i U\left(W_{T,i}\right)$$

Subject to budget constraint:

$$\sum_i W_{T,i} q_i B_{t,T} = W_t .$$

Investor implicitly chooses a set of contigent claims q_i depending on the cash flow payoff patterns.

Lagrangian Multiplier

$$\begin{aligned} L &= \sum_i p_i U\left(W_{T,i}\right) + \lambda\left[W_t B_{t,T}^{-1} - \sum_i q_i W_{T,i}\right] \\ \frac{\partial L}{\partial W_{T,i}} &= p_i U'\left(W_{T,i}\right) - q_i \lambda = 0 \qquad \text{(FOC)} \end{aligned}$$

Sum over state i gives

$$\begin{aligned} \lambda \sum_i q_i &= \sum_i p_i U'\left(W_{T,i}\right) \\ \lambda &= E\left[U'\left(W_{T,i}\right)\right] \end{aligned}$$

Substitute back into FOC:

$$\frac{q_i}{p_i} = \frac{U'\left(W_{T,i}\right)}{\lambda} = \frac{U'\left(W_{T,i}\right)}{E\left[U'\left(W_{T,i}\right)\right]} = \phi_i.$$

Asset Specific Pricing Kernel

The key to this pricing framework is to infer the asset specific pricing kernel:

$$F = \underset{x,\phi}{E}\left[\phi x\right] = \underset{x}{E}\left[\underset{\phi}{E}\left[\phi | x\right] x\right] = \underset{x}{E}\left[\psi x\right]$$

$$\psi = \underset{\phi}{E}\left[\phi | x\right]$$

- Only the asset specific pricing kernel is needed for pricing contracts contingent on x.
- Brennan (1979) ψ is the projection of ϕ onto the space of x.

Preference Free Option Pricing

Underlying	$F_{t,T}(x_T) = \underset{x}{E}(\psi x)$
Derivatives	$F_{t,T}[g(x_T)] = \underset{x}{E}[\psi g(x_T)]$

where $g(\cdot)$ is the payoff function.

Black-Schole Example

$$
\begin{aligned}
F\left[g(x)\right] &= \int g(x)\,\psi(x)\,f(x)\,dx \\
&= \int h(\ln x)\,\psi(\ln x)\,f(\ln x)\,d\ln x \\
&= \int g(x)\,\widehat{f}(x)\,dx \\
&= \int g(x)\,f^{*}(x)\,dx
\end{aligned}
$$

$$
\begin{aligned}
f(\ln x) &= \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2\sigma^2}\left(\ln x - \mu_x\right)^2} \\
\psi(x) &= \alpha x^{\beta} \\
\widehat{f}(x) &= \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2\sigma^2}\left[\ln x - \left(\mu_x + \beta\sigma_x^2\right)\right]^2} \\
f^{*}(x) &= \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2\sigma^2}\left[\ln x - \left(F - \frac{1}{2}\sigma_x^2\right)\right]^2}
\end{aligned}
$$

Sufficient conditions for RNVR

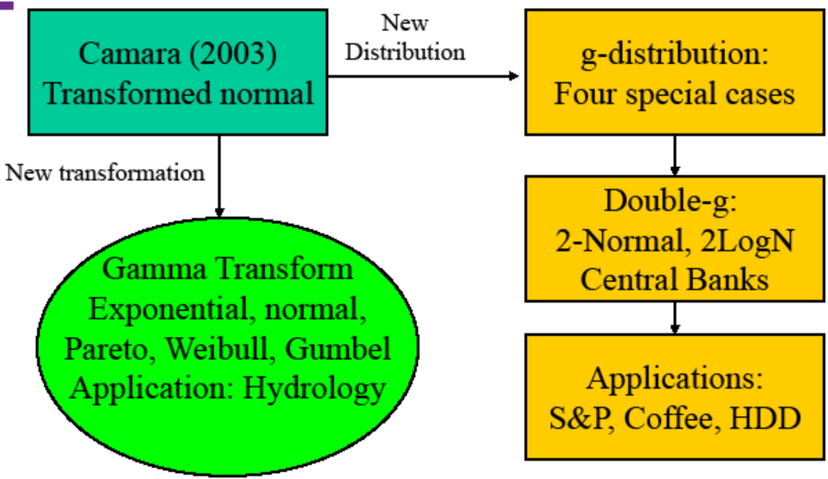
Heston (1993)

"It is all down to the functional forms of x and ψ."

Rubinstein	$\ln W_T$	$U' = W^{\gamma}$
Brennan	W_T	$U' = \exp(\gamma W)$
Camara	$h_W(W_T)$	$U' = \exp[\gamma h_W(W)]$

It turns out that, for transform normal and any normal class, the functional form of $h_W(\cdot)$ is not relevant. The solution depends on $\rho_{W,x}$ and σ_W only.

Series of Work by Vitiello and Poon



Gamma Distribution (McKay)

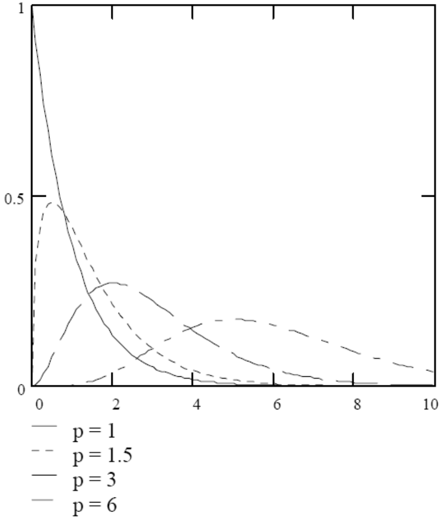
$$f(x) = \frac{a^p}{\Gamma(p)} x^{p-1} e^{-ax}$$

$$\Gamma(p) = \int_0^\infty z^{p-1} e^{-z} dz$$

$$p, a > 0,$$

$$0 \leq x < \infty.$$

Gamma Distribution



Gamma: Special Cases

$p = 1$	exponential
$p \in Z$	Erlang
$p = v/2,\ a = 1/2$	chi-squared
$p \rightarrow \infty$	normal

Gamma Studies

- Heston (1993)
- Gerber and Shiu (1994)
- Lane and Morchan (1999) – catastrophe
- Savickas (2002) – Weibull
- Schroder (2004)

Gamma Distribution Applications

- Stern and Coe (1984) – Rainfall
- Loukas,Vasiliades, Dalezios and Domenikiotis (2001) – Rainfall
- Yue, Ourada and Robee (2001) – Hydrology
- Sharda and Das (2005) - Rainfall

Tranformed Gamma

$$\begin{aligned} h(S_T) &= \mu + \sigma x \\ h_W(W_T) &= \mu_W + \sigma_W y \end{aligned}$$

where x and y are joint Gamma distribution with

$$\begin{aligned} f(x, y; p, q, a) &= \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} x^{p-1} (y - x)^{q-1} e^{-ay}, \\ a^2 &= \frac{p}{\sigma_{x,y}}, \quad \rho_{x,y} = \sqrt{\frac{p}{p+q}} \end{aligned}$$

for $y > x > 0$, and $a, p, q > 0$. As before

$$U' = \exp[\gamma h_W(W)]$$

but h_W is now Gamma transform and includes Normal as a limiting case.

Tranformed Gamma: Special Cases

For $p = 1$;

$$
\begin{array}{ll}
h(z) = \exp(z) & \text{Gumbel} \\
h(z) = z^b, & \text{Weibull} \\
h(z) = z^2/2 & \text{Rayleigh} \\
h(z) = \ b \ln(z) & \text{Pareto}
\end{array}
$$

Closed form solutions

(i) Log Gamma (Heston, 1993)

$$h\left(S_T\right) = \ln\left(S_T\right)$$

(ii) Log chi-squared $\left(p = \frac{1}{2}\nu\right)$

$$h\left(S_T\right) = \frac{1}{2}\ln\left(S_T\right)$$

(iii) Weibull $(p = 1)$

$$h\left(S_T\right) = \sigma\left[\left(S_T - \mu\right)/\,\sigma\right]b + \mu$$

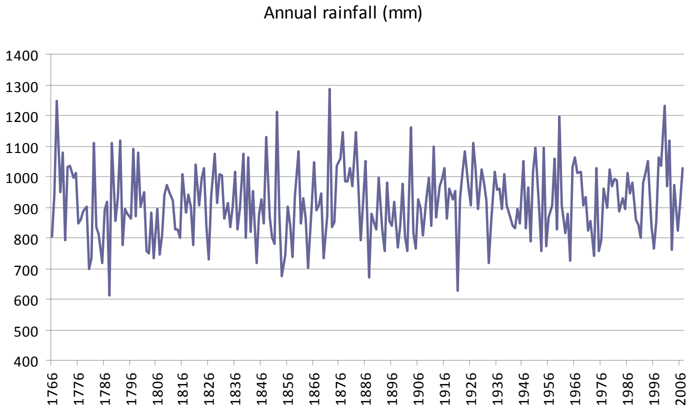
(iv) Log Gumbel $(p = 1)$

$$h\left(S_T\right) = \sigma\exp\left[\left(S_T - \mu\right)/\,\sigma\right] + \mu$$

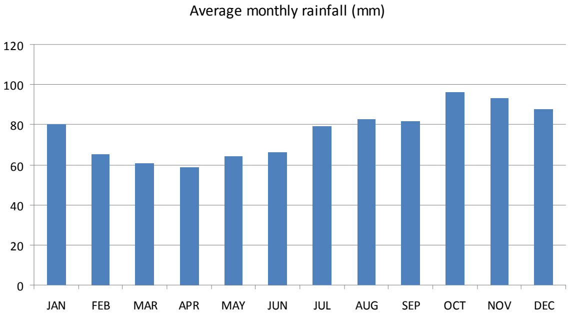
An English Flood Problem

- Gulf stream moving south due to climate change, and joined up as a static ring, the outcome of which is prolong rain in the same location
- England is highly densely populated, and drainage system not designed for such a rainfall pattern.
- As flash flood becomes more common, the use of disaggregate data (i.e. month-by-month or week-by-week) becomes more important.

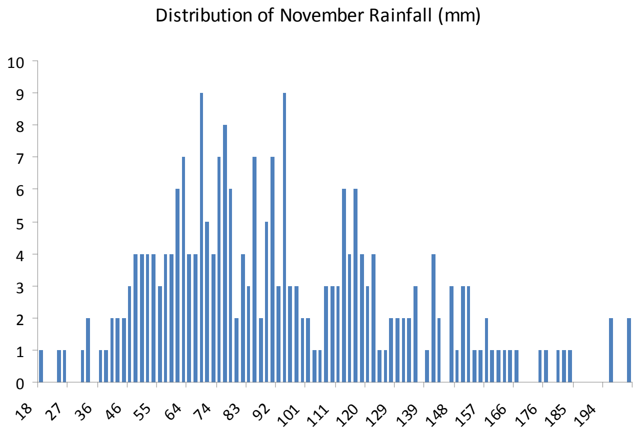
UK Met Office, Hadley Centre rainfall precipitation data for England and Wales over the period from 1766 to 2007



English Seasonally Rainfall Average (1766 to 2007)



November is the Killer Month



Summary statistics for monthly rainfall (1766 to 2007)

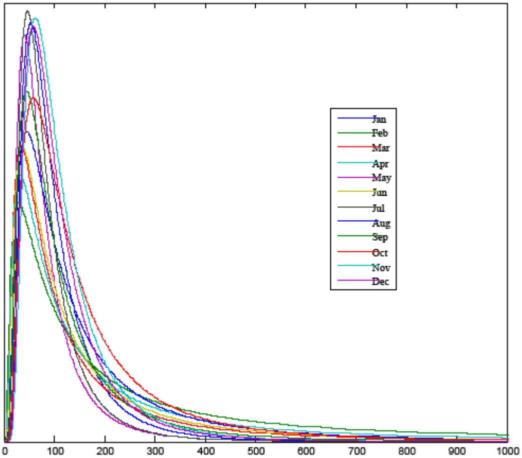
	Mean	Median	Std Dev	Kurtosis	Skewness	Min	Max
JAN	80.38	77.75	34.34	2.458	0.220	4.4	176.8
FEB	65.03	60.35	32.05	2.736	0.489	3.6	158.6
MAR	60.92	57.40	29.45	3.448	0.573	5.6	177.5
APR	58.98	58.35	26.54	3.173	0.368	7.1	142.6
MAY	64.22	62.30	27.23	3.169	0.527	7.9	151.8
JUN	66.24	62.40	30.02	3.089	0.575	4.3	157.1
JUL	79.19	74.70	34.42	2.728	0.411	8.2	182.6
AUG	82.56	80.50	32.83	3.435	0.484	9.1	192.9
SEP	81.48	77.75	37.24	2.702	0.397	8.0	189.5
OCT	95.91	96.65	38.43	2.683	0.114	8.8	218.1
NOV	93.08	86.10	37.59	2.972	0.646	17.0	202.5
DEC	87.87	89.00	38.35	2.948	0.382	8.9	193.9
ANN	915.87	904.75	118.04	3.054	0.256	612.0	1284.9
Ann_M	76.32	75.40				51.00	107.08

Gamma estimation on month and annual rainfall

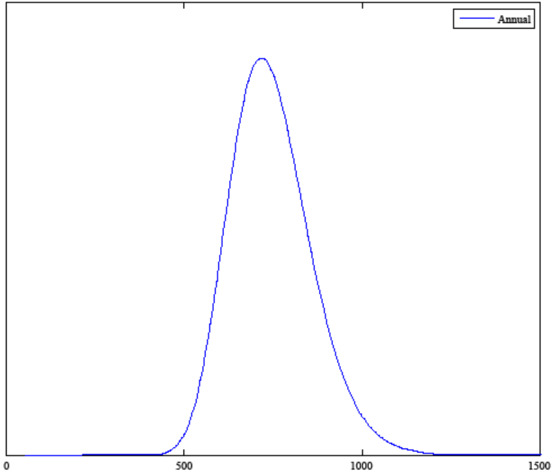
log(x)	p	1/a	stdev(p)	stdev(1/a)	nLogL	Mean	StdDev
Jan	45.250	0.090	5.487	0.011	122.206	4.054	0.6027
Feb	11.400	0.322	1.368	0.039	198.681	3.667	1.0862
Mar	18.397	0.201	2.219	0.025	169.363	3.706	0.8641
Apr	14.699	0.252	1.769	0.031	183.659	3.701	0.9652
May	37.395	0.102	4.531	0.012	126.256	3.804	0.6220
Jun	18.878	0.200	2.278	0.024	169.909	3.768	0.8672
Jul	44.796	0.088	5.432	0.011	118.560	3.927	0.5867
Aug	23.335	0.170	2.820	0.021	163.050	3.968	0.8214
Sep	29.441	0.133	3.563	0.016	146.352	3.927	0.7237
Oct	32.212	0.130	3.901	0.016	148.949	4.183	0.7370
Nov	50.046	0.084	6.071	0.010	120.607	4.211	0.5952
Dec	46.462	0.089	5.635	0.011	123.291	4.141	0.6075
Annual	1884.519	0.003	229.356	0.000	-63.142	6.581	0.1516

Note: stdev is the standard deviation and nLogL is the negative of Log-likelihood value.

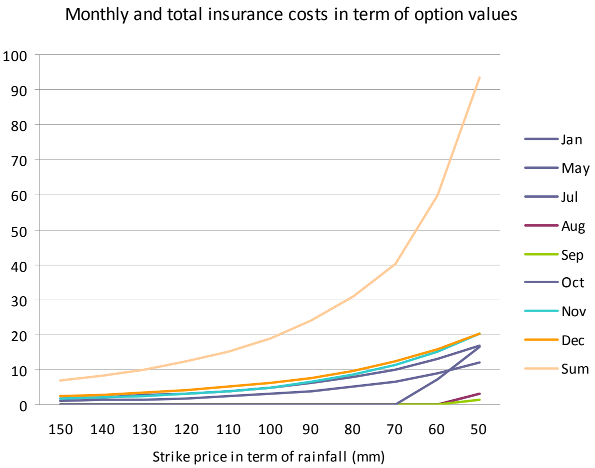
Monthly rainfall distribution (1766-2007)



Annual rainfall distribution (1766-2007)



Saving up for rainy day



Contract consideration

- Option framework allow flexible pricing design, e.g. cap, tranches.
- Suitable for market securitisation as in CAT bond and weather derivatives.
- More appropriate if the forward (or futures) market for rainfall index is establshed also. Otherwise, one would need a fair (forward) price for calibration, i.e. comparable insurance premium for similar contract can be used to back out the preference parameter.
- Credit risk is smaller if exchange traded; need to consider carefully the implication of “margin call” and “marked to market” on current cash flows.
- Need trustworthy weather stations as in the HDD/CDD contracts.

Contract consideration (continue)

- Useful for re-insurance, but need to consider basis risk; mapping into flood economic loss of specific area (e.g. residential vs. commercial) and possibly non-linear relationships between flood claims and rainfall index.
- More efficient for administration; payout can be settled immediately after a big flood, and there is less concern about claimant's credibility or cheating.
- Similar development has already taken place, e.g. CCRIF (Caribbean Catastrophe Risk Ibsurance Facility) where a mutual fund was set up by the Caribbean countries for managing the insurance based on a Hurricane index and an equake index. The facility is provided by Munich Re and Llyod's.

Conclusion

- Non-Gaussian; let distribution reflects the real world.
- Utility is concave, monotonic, twice differentiable. User decides function or back it out if sufficient price information is available.
- Need some prices and distributional parameters; calibration is fast due to analytical solutions.
- Could produce hedge ratio quickly.
- Current work restricts to European options only.
- It is a different way of thinking about derivative prices; a cross between no-arbitrage and insurance.

Thank You!

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