

Transform Approach for Operational Risk Modeling: Value-at-Risk and Tail Conditional Expectation

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Operational Risk

- A capital charge for operational risk is required to the financial institutions.
- The Basel Committee for Banking Supervision (2006) defines operational risk as follows: “The risk of losses resulting from inadequate or failed internal processes, people and systems or from external events”

Table: A list of loss event types (level 1) of operational risks

Event-Type Category (Level 1)	Definition
Internal fraud	Losses due to acts of a type intended to defraud, misappropriate property or circumvent regulations, the law or company policy, excluding diversity/discrimination events, which involves at least one internal party
External fraud	Losses due to acts of a type intended to defraud, misappropriate property or circumvent law, by a third party
Employment Practices and Workplace Safety	Losses arising from acts inconsistent with employment, health or safety laws or agreement, from payment of personal injury claims, or from diversity/discrimination events
Clients, Products & Business Practices	Losses arising from an unintentional or negligent failure to meet a professional obligation to specific clients (including fiduciary and suitability requirements), or from the nature or design of a product
Damage to Physical Assets	Losses arising from loss or damage to physical assets from natural disaster or other events
Business disruption and system failures	Losses arising from disruption or system failures
Execution, Delivery & Process Management	Losses from failed transaction processing or process management, from relations with trade counterparties and vendors

An example of mismanagement of operational risk

- The collapse of Britain's Barings Bank in February 1995 is perhaps the quintessential tale of operational risk management gone wrong. Over a course of days, Barings, Britain's oldest merchant bank, went from apparent strength to bankruptcy. The estimated loss was **£700million**.
- A similar even more severe failure came to light in January 2008 at the French bank Societe Generale. First estimates of the Societe Generale loss are around **\$US 7bn**.
- Both failures were completely unexpected and were caused by the actions of a single trader.

Structure of the New Accord (Basel II)

- Due to Basel II: International Convergence of Capital Measurement and Capital Standards: A Revised Framework - Comprehensive Version, the financial institutions need to have a risk management tool to measure operational losses (<http://www.bis.org/publ/bcbs128.htm>):
 - Pillar 1: Minimum capital requirements (risk management);
 - Pillar 2: Supervisory review of capital adequacy;
 - Pillar 3: Public disclosure.

Advanced Measurement Approach (AMA)

- It allows banks to use their internally generated risk estimates.
- The Committee is **not** specifying the approach or distributional assumptions used to generate the operational risk measures for regulatory capital purposes.
- Incorporation of risk diversification benefits allowed.
- An example of AMA approach: **Loss Distribution Approach (LDA)**.

Mathematical notations for LDA

- Considering one line of business, let $X_i^{(k)}$, $i = 1, 2, \dots$, be the loss amounts from type k operational risk, which are assumed to be independent and identically distributed with distribution function $H(x)$ ($x > 0$), then the total loss arising from type k operational risk up to time t is defined by $L_t^{(k)} = \sum_{i=1}^{N_t} X_i^{(k)}$, where $k = 1, 2, \dots, d$ and N_t is the total number of losses up to time t . The grand total loss is hence given by

$$L_t = \sum_{k=1}^d L_t^{(k)}.$$

Risk measure and total capital charge

- According to the Basel II Advanced Measurement Approach (AMA) guidelines leading to the Loss Distribution Approach (LDA), the financial institutions may use the Value at Risk (VaR or the q -quantile) as a risk measure to decide the capital amount required for next t years' operational risk, i.e.

$$\text{VaR}_{99.9\%}(L_t)$$

- However to obtain the $\text{VaR}_{99.9\%}(L_t)$, it requires to derive the joint distribution of the total loss random vector $\left(L_t^{(1)}, L_t^{(2)}, \dots, L_t^{(d)}\right)$, which is

a challenging task. Accordingly, the Basel II AMA guidelines propose to use

$$\sum_{k=1}^d \text{VaR}_{99.9\%}(L_t^{(k)})$$

for a capital charge and consider a diversification effect of the latter under appropriate correlation assumptions to reduce it, i.e.

$$\begin{aligned} & \text{VaR}_{99.9\%}(L_t) \\ = & \text{VaR}_{99.9\%}\left(L_t^{(1)} + L_t^{(2)} + \dots + L_t^{(d)}\right) \leq \sum_{k=1}^d \text{VaR}_{99.9\%}(L_t^{(k)}). \end{aligned}$$

Issues on operational losses

- Loss amounts show **extremes**: Distribution for $L_t^{(k)} = \sum_{i=1}^{N_t} X_i^{(k)}$ or $X_i^{(k)}$, where $k = 1, 2, \dots, d$ needs to be **heavy-tailed** distribution.
- Loss occurrence times are irregularly spaced in time: A Poisson process may be used for N_t initially and advanced point processes such as the Cox process can be used.
- Dependence between operational risk sources.

Tail conditional expectation (TCE) (also known as TailVaR)

- We also examine the tail conditional expectation (TCE) (also known as TailVaR) defined by

$$\mathbb{E} \left\{ L_t^{(k)} \mid L_t^{(k)} \geq \text{VaR}_q(L_t^{(k)}) \right\}$$

to obtain the capital amount required from type k operational risk for next t years.

- The TCE tells us how great the losses are as it takes an average over the worst cases and therefore takes into account the tail distribution of the losses. However, the VaR only looks at a quantile and it does not tell us how great losses are.

Overview

- In order to quantify the aggregate losses from all lines of operational risk, we employ an actuarial risk model, i.e. we consider the classical compound Poisson/Cox model.
- We assume the loss size X follows Loggamma, Fréchet and truncated Gumbel distribution to deal with extreme losses in practice. We also use an exponential distribution for the case of non-extreme losses.
- For simplicity, in this paper, we ignore the correlation assumptions, i.e. we assume that $L_t^{(k)}$, $k = 1, 2, \dots, d$ are independent each other but not identical.

- The Laplace transform of the distribution of the aggregate losses.
- The capital amount required for next t years' operational risk: the VaR and TCE.
- Transform analysis and numerical illustrations of VaR and TCE.

The distribution of the total loss $\mathbb{P}\left(L_t^{(k)} \leq l\right)$ via the Laplace transform of

$$\text{the distribution of } L_t^{(k)} = \sum_{i=1}^{N_t} X_i^{(k)}$$

- In order to evaluate 'VaR_{99.9%}($L_t^{(k)}$)', it is necessary for us to calculate the distribution of the aggregate loss $\mathbb{P}\left(L_t^{(k)} \leq l\right)$. However it is difficult to derive it explicitly.
- We consider using the Laplace transform as it can be inverted to calculate 'VaR_{99.9%}($L_t^{(k)}$)' numerically.

Choice of a counting process N_t in $L_t^{(k)} = \sum_{i=1}^{N_t} X_i^{(k)}$

- As we can see in Table, fraud, business disruption, execution error and system failure etc. are primary events. In order to measure the occurrence of operational losses out of these primary events, we need a counting process to deal with **deterministic** or **stochastic** nature of their arrival rates in practice. Therefore it is natural to use point processes to consider series of operational losses. The simplest one is using a homogeneous Poisson process that has deterministic frequency.

N_t follows a homogeneous Poisson process with loss frequency λ

- Assuming that the loss arrival process N_t follows a homogeneous Poisson process with loss frequency λ and that $L_0^{(k)} = 0$, the Laplace transform of the the distribution of total loss $L_t^{(k)}$ is given by

$$\mathbb{E} \left\{ e^{-\nu L_t^{(k)}} \right\} = \exp \left[-\lambda \left\{ 1 - \hat{m}(\nu) \right\} t \right]$$

where $\nu \geq 0$ and $\hat{m}(\nu) = \int_0^{\infty} e^{-\nu x} dH(x) < \infty$.

The L.T. of the distribution of $L_t^{(k)}$ using an Exponential

- Using an exponential loss size distribution, i.e.

$$h(x) = a \exp(-ax), \quad x \geq 0, \quad a > 0,$$

the Laplace transform of the the distribution of aggregate loss $L_t^{(k)}$ is given by

$$\mathbb{E} \left\{ e^{-\nu L_t^{(k)}} \right\} = \exp \left\{ -\lambda \left(\frac{\nu}{a + \nu} \right) t \right\}.$$

The L.T. of the distribution of $L_t^{(k)}$ using a Loggamma

- Using a Loggamma loss size distribution, i.e.

$$h(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} (\ln x)^{\alpha-1} x^{-\beta-1}, \quad x > 1, \beta > 0 \text{ and } \alpha > 0,$$

the Laplace transform of the distribution of aggregate loss $L_t^{(k)}$ is given by

$$\mathbb{E} \left\{ e^{-\nu L_t^{(k)}} \right\} = \exp \left[-\lambda t + \frac{\lambda t}{\alpha \Gamma(\alpha)} \int_0^\infty \exp \left\{ -\nu \exp(\beta^{-1} z^{1/\alpha}) - z^{1/\alpha} \right\} dz \right],$$

where $z = (\beta \ln x)^\alpha$.

The L.T. of the distribution of $L_t^{(k)}$ using a Fréchet

- Using a Fréchet loss size distribution, i.e.

$$h(x) = \frac{\varsigma}{\sigma} \left(\frac{x-\mu}{\sigma} \right)^{-\varsigma-1} \exp \left\{ - \left(\frac{x-\mu}{\sigma} \right)^{-\varsigma} \right\}, \quad x \geq \mu, \mu > 0, \sigma > 0 \text{ and } \varsigma > 0$$

the Laplace transform of the distribution of aggregate loss $L_t^{(k)}$ is given by

$$\mathbb{E} \left\{ e^{-\nu L_t^{(k)}} \right\} = \exp \left[-\lambda t + \lambda t \int_0^{\infty} \exp \left\{ -\nu(\mu + \sigma z^{-1/\varsigma}) - z \right\} dz \right],$$

where $z = \left(\frac{x-\mu}{\sigma} \right)^{-\varsigma}$.

The L.T. of the distribution of $L_t^{(k)}$ using a truncated Gumbel

- And using a truncated Gumbel loss size distribution,

$$h(x) = \frac{\exp\{\exp(\zeta/\eta)\}}{\exp\{\exp(\zeta/\eta)\}-1} \frac{1}{\eta} \exp\left\{-\frac{x-\zeta}{\eta} - \exp\left(-\frac{x-\zeta}{\eta}\right)\right\}, \quad x \geq 0, \zeta > 0 \text{ and } \eta > 0,$$

the Laplace transform of the distribution of aggregate loss $L_t^{(k)}$ is given by

$$\mathbb{E}\left\{e^{-\nu L_t^{(k)}}\right\} = \exp(-\lambda t) \times \exp\left[\begin{array}{l} \lambda t \left\{ \frac{\exp\{\exp(\zeta/\eta)\}}{\exp\{\exp(\zeta/\eta)\}-1} \right\} \times \\ \exp(-\nu\zeta) \Gamma(\nu\eta + 1; e^{\zeta/\eta}) \end{array}\right], \text{ where } z =$$

$$\exp\left(-\frac{x-\zeta}{\eta}\right) \text{ and } \Gamma(\phi; \varphi) \equiv \int_0^\phi z^{\phi-1} e^{-z} dz.$$

A Cox process for N_t in $L_t^{(k)} = \sum_{i=1}^{N_t} X_i^{(k)}$

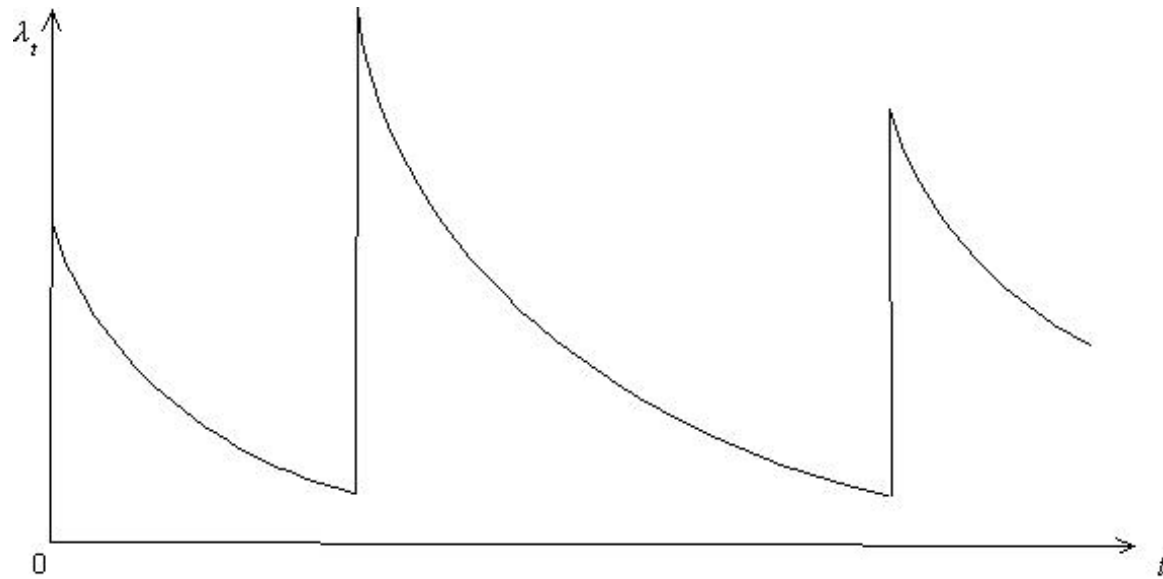
- To deal with stochastic nature of operational loss arrival in practice, we consider a Cox process as an alternative point process. The Cox process provides flexibility by letting the intensity not only depend on time but also allowing it to be a stochastic process. Therefore the Cox process can be viewed as a two step randomisation procedure. A process λ_t is used to generate another process N_t by acting its intensity. That is, N_t is a Poisson process conditional on λ_t which itself is a stochastic process.

For a Cox process's intensity λ_t : Shot noise process

- Losses arising from the mismanagement of operational risks depend on the intensity of primary events. One of the processes that can be used to measure the impact of primary events is the shot noise process. The shot noise process is particularly useful in loss arrival process as it measures the frequency, magnitude and time period needed to determine the effect of primary events. As time passes, the shot noise process decreases as more and more losses are figured out. This decrease continues until another event occurs which will result in a positive jump in the shot noise process. Therefore the shot noise process can be used as the parameter of a Cox process to measure the number of operational losses, i.e. we will use it as an intensity function to generate a Cox process. We will adopt the shot noise process used by Cox & Isham (1980):

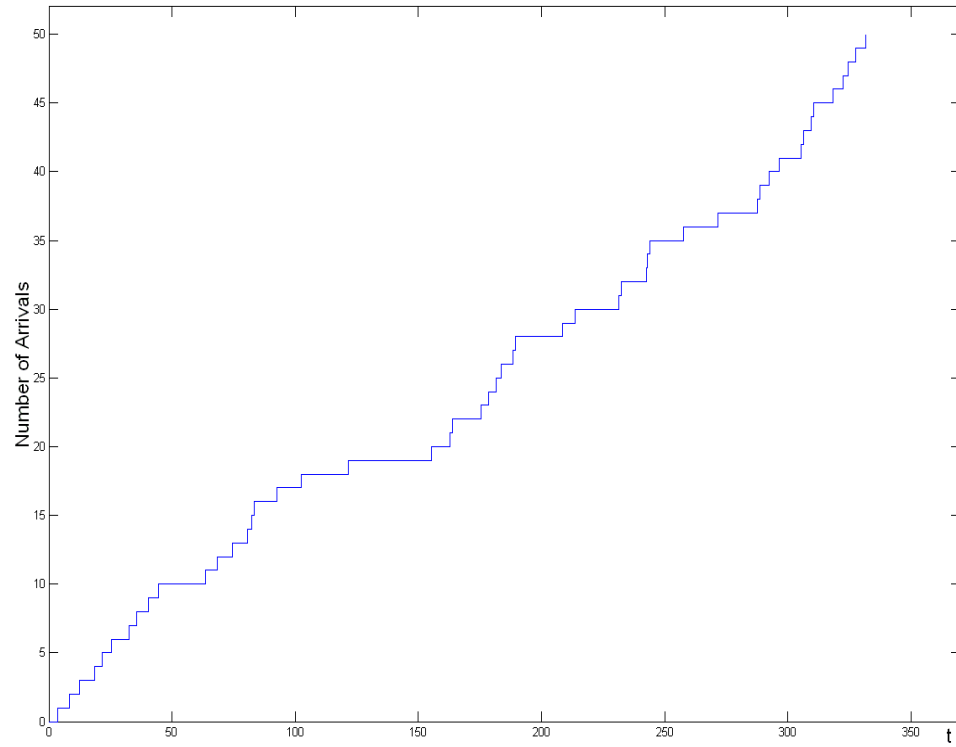
$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i=1}^{M_t} Y_i e^{-\delta(t-S_i)}, \text{ where}$$

- λ_0 is the initial value of λ_t that is carried on from primary events incurred previously;
- $\{Y_i\}_{i=1,2,\dots}$ is a sequence of independent and identically distributed random variables with distribution function $G(y)$ ($y > 0$) and $E(Y) < \infty$ (i.e. magnitude of contribution of primary event i to intensity);
- $\{S_i\}_{i=1,2,\dots}$ is the sequence representing the event times of a Poisson process with constant intensity ρ ;
- δ is the rate of exponential decay.



Graph illustrating shot noise process

- Now let us assume that the loss arrival process N_t follows a Cox process with its intensity λ_t . Below figure illustrates a Cox process with shot noise intensity.



Graph illustrating the Cox process with shot noise intensity

The Laplace transform (L.T.) of the distribution of $L_t^{(k)} = \sum_{i=1}^{N_t} X_i^{(k)}$

- Assuming that the loss arrival process N_t follows a Cox process with its intensity λ_t and that $L_0^{(k)} = 0$, the Laplace transform of the the distribution of total loss $L_t^{(k)}$ is given by

$$\mathbb{E} \left\{ e^{-\nu L_t^{(k)}} \mid \lambda_0 \right\} = \mathbb{E} \left[\exp \left\{ - \left\{ 1 - \hat{m}(\nu) \right\} \Lambda_t \right\} \mid \lambda_0 \right],$$

where $\Lambda_t = \int_0^t \lambda_s ds$ and where λ_0 is assumed to be known.

The Laplace transform (L.T.) of the distribution of $L_t^{(k)} = \sum_{i=1}^{N_t} X_i^{(k)}$

- Assuming that jump size of primary event follows an exponential distribution, i.e. $g(y) = b \exp(-by)$, $y > 0$, $b > 0$ and λ_t is stationary, the explicit expression for the L. T. is given by

$$\mathbb{E} \left\{ e^{-\nu L_t^{(k)}} \right\} = \left(\frac{\delta b + \left\{ 1 - \hat{m}(\nu) \right\} (1 - e^{-\delta t})}{\delta b e^{-\delta t}} \right)^{\frac{b\rho}{\delta b + \left\{ 1 - \hat{m}(\nu) \right\}}} e^{-\frac{\rho}{\delta}}.$$

- The Laplace transform of the distribution of total loss $L_t^{(k)}$ using an Exponential, a Loggamma, a Fréchet and a truncated Gumbel can be easily obtained.

Inverting the Fast Fourier transform using the L.T. obtained

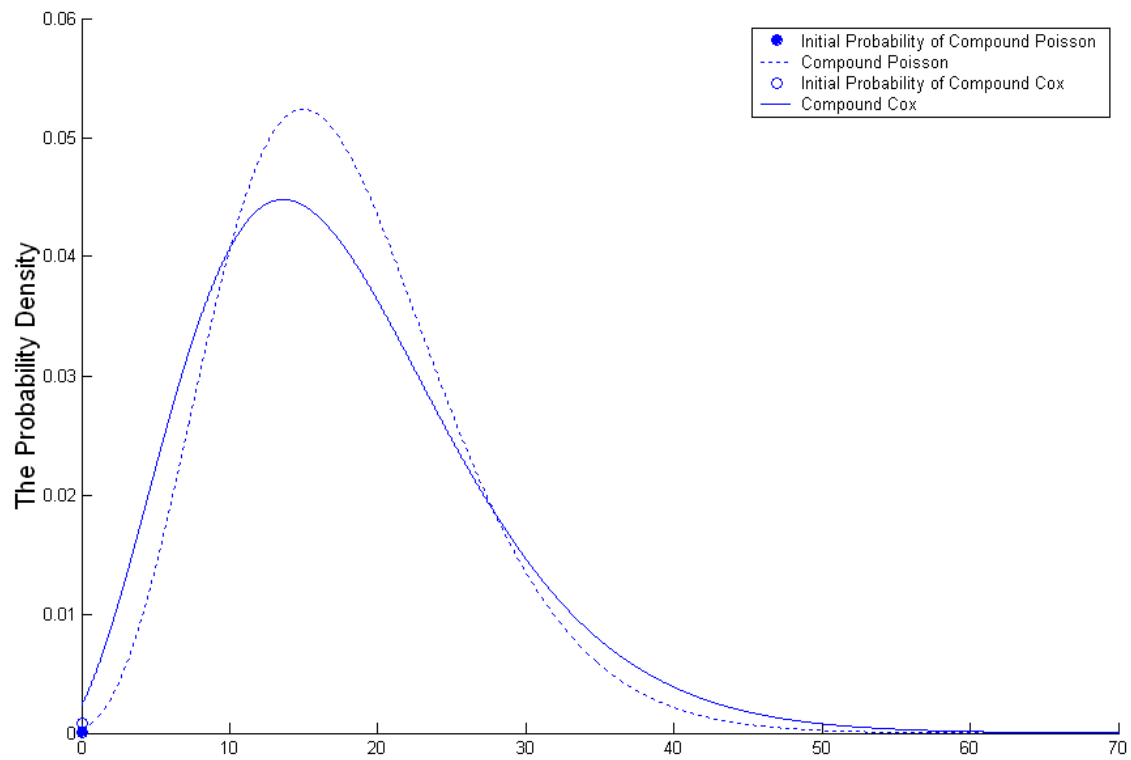
- The VaR can be expressed as

$$\text{VaR}_{99.9\%}(L_t^{(k)}) = \inf \left\{ l \in \mathbb{R} : P(L_t^{(k)} > l) \leq 0.001 \right\}.$$

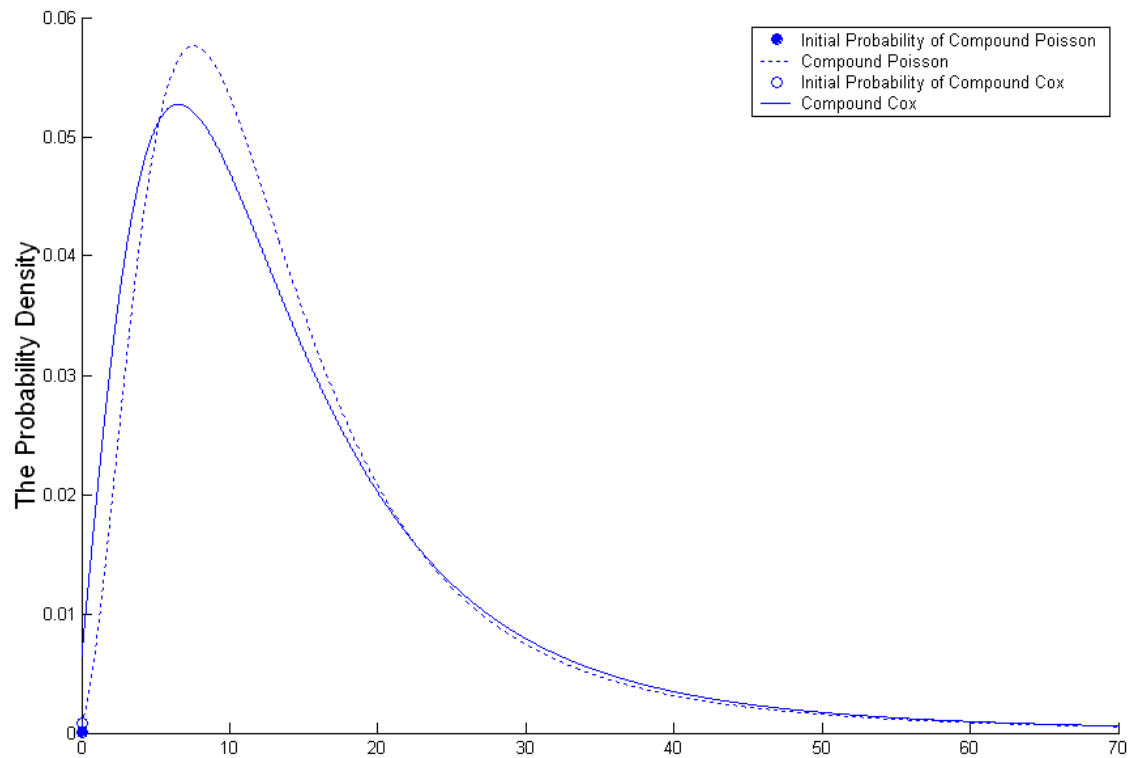
- The TCE can be expressed by

$$\begin{aligned} \text{TCE}_{99.9\%}(L_t^{(k)}) &= \mathbb{E} \left\{ L_t^{(k)} \mid L_t^{(k)} \geq \text{VaR}_{99.9\%}(L_t^{(k)}) \right\} = \\ &= \frac{\mathbb{E} \left[L_t^{(k)} I \left\{ L_t^{(k)} \geq \text{VaR}_{99.9\%}(L_t^{(k)}) \right\} \right]}{0.001}. \end{aligned}$$

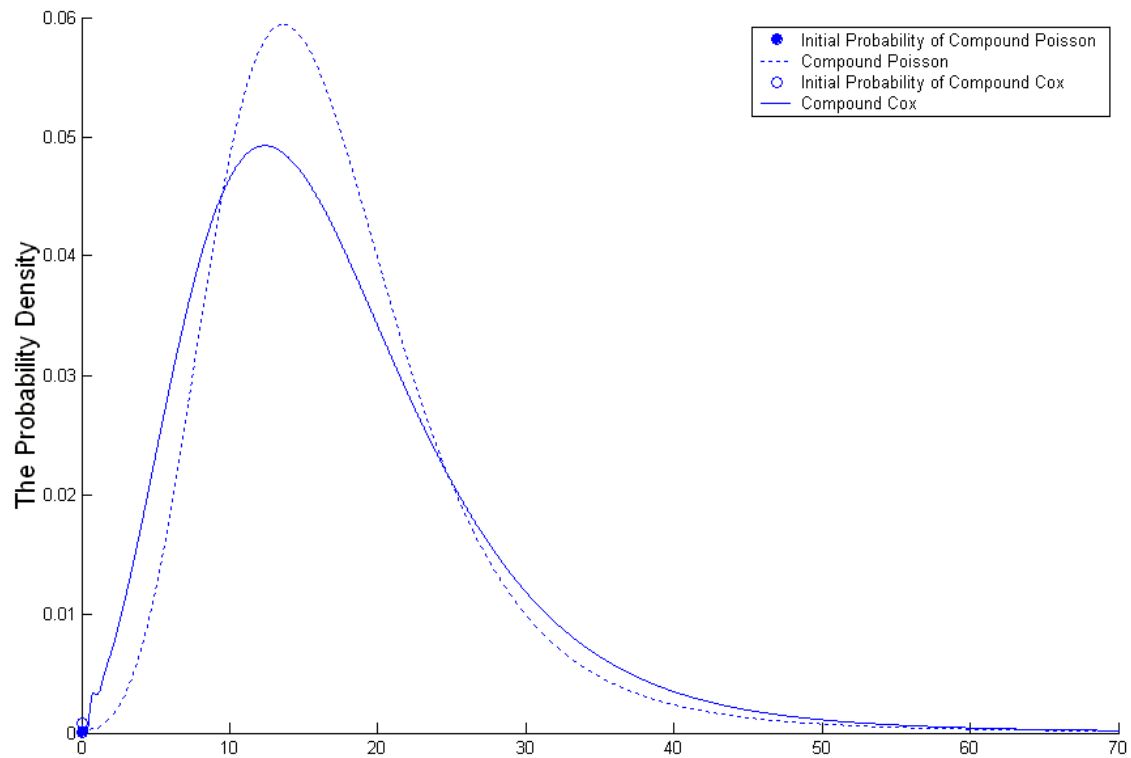
- As it is not possible for us to obtain the distribution/density of $L_t^{(k)}$ explicitly, we invert the Fast Fourier transform using the L.T. of aggregate loss to approximate the VaR (Castleman 1996; Gonzalez and Woods 2002 and Gonzalez et al. 2004).
- The below figures are the distributions of aggregate loss of $L_t^{(k)}$.



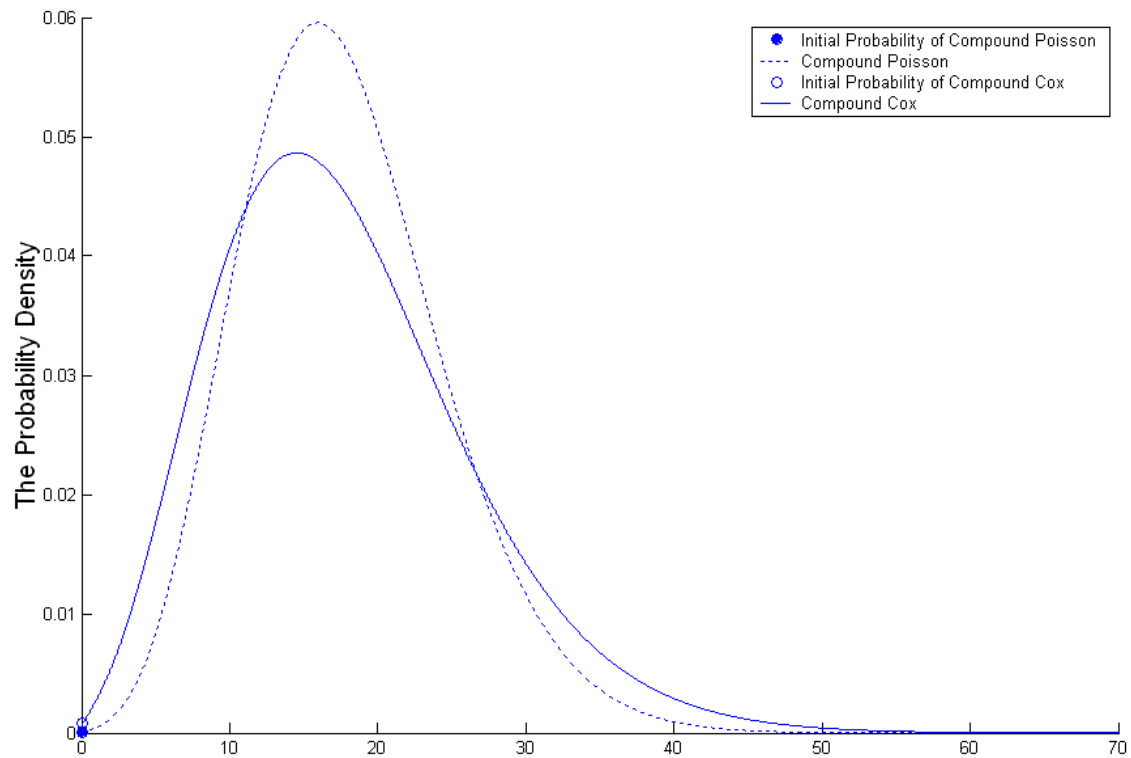
The distribution of total loss with respect to Poisson/Cox process with Exponential loss size distribution



The distribution of total loss with respect to Poisson/Cox process with Loggamma loss size distribution



The distribution of total loss with respect to Poisson/Cox process with Fréchet loss size distribution



The distribution of total loss with respect to Poisson/Cox process with truncated Gumbel loss size distribution

Calculating risk measures

- The parameter values used to simulate N_t and calculate the above risk measures are $\lambda = 10$, $\rho = 4$, $b = 1$, $\delta = 0.4$ and $t = 1$.
- We use the above parameter values that provide us with the same means of total loss regardless of the specification of the loss arrival process N_t to see the differences of the VaRs and TCEs due to the tails of the loss size distributions, i.e.

$$\mathbb{E}^{\text{Poisson}} \left\{ L_t^{(k)} \right\} = \mathbb{E}^{\text{Cox}} \left\{ L_t^{(k)} \right\}.$$

- In order to make the computing easier, we also choose

$$\begin{aligned} & \mathbb{E}^{\text{Exponential}}(X) \\ = & \mathbb{E}^{\text{Loggamma}}(X) \\ = & \mathbb{E}^{\text{Fréchet}}(X) \\ = & \mathbb{E}^{\text{truncated Gumbel}}(X) \\ = & \sqrt{\pi}. \end{aligned}$$

The VaRs and TCEs when loss size follows an Exponential

Table 1: Poisson process

q	$\text{VaR}_q(L_t^{(k)})$	$\text{TCE}_q(L_t^{(k)})$
0.999	49.5368	53.3579
0.99	39.8691	44.1164
0.95	32.1210	36.8966
0.9	28.3286	33.4675
0.5	16.8305	23.9688
where	$E \left\{ L_t^{(k)} \right\} = 10\sqrt{\pi},$ $\text{Var} \left\{ L_t^{(k)} \right\} = 20\pi$	

Table 2: Cox process

q	$\text{VaR}_q(L_t^{(k)})$	$\text{TCE}_q(L_t^{(k)})$
0.999	57.9834	63.0779
0.99	45.2875	50.8550
0.95	35.3129	41.4676
0.9	30.5114	37.0697
0.5	16.4034	25.1592
where	$E \left\{ L_t^{(k)} \right\} = 10\sqrt{\pi}$ $\text{Var} \left\{ L_t^{(k)} \right\} = 90.45$	

The VaRs and TCEs when loss size follows a Loggamma

Table 3: Poisson process

q	$\text{VaR}_q(L_t^{(k)})$	$\text{TCE}_q(L_t^{(k)})$
0.999	337.2020	1017.4245
0.99	99.0571	246.1027
0.95	43.7821	97.4792
0.9	31.3680	67.0061
0.5	11.9483	28.2826
where	$E \left\{ L_t^{(k)} \right\} = 10\sqrt{\pi},$ $\text{Var} \left\{ L_t^{(k)} \right\} = \infty$	

Table 4.4: Cox process

q	$\text{VaR}_q(L_t^{(k)})$	$\text{TCE}_q(L_t^{(k)})$
0.999	378.9434	1018.9412
0.99	100.9566	247.8994
0.95	45.5266	99.3334
0.9	32.7244	68.7072
0.5	11.7146	28.9562
where	$E \left\{ L_t^{(k)} \right\} = 10\sqrt{\pi},$ $\text{Var} \left\{ L_t^{(k)} \right\} = \infty$	

The VaRs and TCEs when loss size follows a Fréchet

Table 5: Poisson process

q	$\text{VaR}_q(L_t^{(k)})$	$\text{TCE}_q(L_t^{(k)})$
0.999	118.9029	218.5851
0.99	51.6413	82.8136
0.95	33.4748	48.0499
0.9	28.0027	39.2087
0.5	15.8570	24.2760
where	$E \left\{ L_t^{(k)} \right\} = 10\sqrt{\pi},$ $\text{Var} \left\{ L_t^{(k)} \right\} = \infty$	

Table 6: Cox process

q	$\text{VaR}_q(L_t^{(k)})$	$\text{TCE}_q(L_t^{(k)})$
0.999	121.1252	220.5757
0.99	55.1842	85.8063
0.95	36.5327	51.3762
0.9	30.2885	42.1960
0.5	15.5337	25.4611
where	$E \left\{ L_t^{(k)} \right\} = 10\sqrt{\pi},$ $\text{Var} \left\{ L_t^{(k)} \right\} = \infty$	

The VaRs and TCEs when loss size follows a truncated Gumbel

Table 7: Poisson process

q	$\text{VaR}_q(L_t^{(k)})$	$\text{TCE}_q(L_t^{(k)})$
0.999	43.5250	46.4355
0.99	36.0213	39.3241
0.95	29.8658	33.6550
0.9	26.7942	30.9191
0.5	17.1443	23.1380
where	$E \left\{ L_t^{(k)} \right\} = 10\sqrt{\pi},$ $\text{Var} \left\{ L_t^{(k)} \right\} = 46.58$	

Table 8: Cox process

q	$\text{VaR}_q(L_t^{(k)})$	$\text{TCE}_q(L_t^{(k)})$
0.999	53.0242	57.3668
0.99	42.1127	46.9002
0.95	33.4607	38.7967
0.9	29.2596	34.9745
0.5	16.6829	24.4921
where	$E \left\{ L_t^{(k)} \right\} = 10\sqrt{\pi},$ $\text{Var} \left\{ L_t^{(k)} \right\} = 74.19$	

Further Research

- Using general heavy-tailed distributions to deal with extreme losses in practice, e.g.

$$P(L_t^{(k)} > l) = l^{-\alpha_k} h_k(l), \quad k = 1, 2, \dots, d,$$

where α_k is the tail-index parameter and h_k is a slowly varying function.

- Multivariate distribution to measure dynamic dependence between loss processes.

- Justification of standard economic thinking, i.e. a reduction in capital charge due to diversification using VaR in the presence of extremely heavy-tailed data, i.e.

$$\begin{aligned} \text{VaR}_{99.9\%}(L_t) &= \text{VaR}_{99.9\%}\left(L_t^{(1)} + L_t^{(2)} + L_t^{(3)}\right) \\ &\leq \text{VaR}_{99.9\%}(L_t^{(1)}) + \text{VaR}_{99.9\%}(L_t^{(2)}) + \text{VaR}_{99.9\%}(L_t^{(3)}) ? \end{aligned}$$