Reinsurance Contract Valuation When the Liabilities are of Fractional Brownian Motion type

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July 6, 2008

- The majority of the actuarial literature assumes that liabilities are driven by a process with independent increments.

- We drop this assumption using fractional Brownian motion as the driving stochastic process.

- We study the valuation of reinsurance contracts for liabilities exhibiting long range dependence modelled by fractional Brownian motion.

- Using fractional Itô calculus and ideas from option pricing theory we derive a partial differential equation the solution of which provides the value of the reinsurance policy.

- An analytical solution is found for this equation and the results obtained by this approach are compared with the results obtained by Monte-Carlo simulation.

Brownian Motion

A process $\{B_t, t \geq 0\}$, with the following properties is called standard Brownian motion

- $B_0 = 0$,
- B_{t_1}, $B_{t_2} - B_{t_1}, ..., B_{t_m} - B_{t_{m-1}}$ are independent for every $m \geq 1$ and $0 = t_0 < t_1 < ... < t_m < \infty$ for all $t, s \geq 0$.
- $B_t - B_s \tilde{} N(0, t - s)$ for every $0 < s < t < \infty$
- the sample path $t \to B_t(\omega)$ is continuous, for every $\omega \in \Omega$.

Brownian motion is one of the most significant processes since it stands as the prototype

a) process with stationary and independent increments

b) Markov process

c) Martingale with continuous sample paths

d) Gaussian process

Fractional Brownian motion

- A continuous centered Gaussian process $\{B_t^H, t \geq 0\}$, $B_0^H = 0$, is a fractional Brownian motion with Hurst parameter $H \in (0,1)$ if its covariance function is given by

$$E[B_t^H B_s^H] = \frac{1}{2} Var(B_1^H)(t^{2H} + s^{2H} - |t - s|^{2H}),$$

for all $t, s \geq 0$.

- We will consider the normalised version of this process by taking

$$Var(B_1^H) = 1.$$

- For $H = 1/2$, from fractional Brownian motion we take the standard Brownian motion.

- Fractional Brownian motion is not a semimartingale.
- Fractional Brownian motion is not a Martingale.
- Fractional Brownian motion is not a Markov process.

Self - Similarity Property of Fractional Brownian Motion

Fractional Brownian motion has the following self-similarity property: For any constant $\alpha > 0$, the processes $\{B^H_{at}\}$ and $\{\alpha^H B^H_t\}$ have the same distribution. Since all processes are centered Gaussian, in order to show that $\{B^H_{at}\} \stackrel{d}{=} \{a^H B^H_t\}$ it is enough to show the equality in covariance.

$$E[B^H_{at} B^H_{as}] = E[(a^H B^H_t)(a^H B^H_s)].$$

Long Memory Property of Fractional Brownian Motion

Fractional Brownian motion is characterized by the Hurst exponent H which determines the sign of the covariance of the past and the future increments. When $H > \frac{1}{2}$ the covariance is positive and when $H < \frac{1}{2}$ the covariance is negative. With positive covariance we mean that after some positive (negative) incerements it is more probable to get more positive (negative) increments.

Fractional Brownian Motion as a Modelling Tool

Fractional Brownian motion is used to model a wide variety of stochastic data arising in

- enigineering and physics, i.e. network traffic data, levels of a river, turbulence in an incompressible fluid flow, see e.g. Shiryaev (1999)

- financial mathematics, log returns of the stock prices, the electricity price in a liberated electricity market, foreign exhange rates, weather derivatives.

- FBM has been used recently to model the claims an insurance business may face, see eg.Michna (1998, 1999). Michna investigated ruin probabilities and first passage times for self-similar processes. He proposed self-similar processes as a risk model with claims appearing in good and bad periods. Then, he obtained FBM with drift as a limit risk process.

Stochastic Integration with respect to Fractional Brownian Motion

- In the paper of Duncan, Hu and Pasik-Duncan (2000) a stochastic integral over fractional Brownian motion of Hurst exponent $1/2 < H < 1$ has been defined.

- The stochastic integral $\int_0^t f_s dW_s^H$ over deterministic functions f is defined to provide a zero mean, Gaussian random variable with variance

$$\int_0^\infty \int_0^\infty f_s f_t \phi(s,t) ds dt$$

where

$$\phi(s,t) = H(2H-1) \mid s-t \mid^{2H-2} .$$

- The stochastic integral $\int_0^t F_s dW_s^H$ can be defined over stochastic processes F as the limit

$$\int_0^t F_s dW_s^H = \lim_{\Delta \to 0} \sum_{i=0}^{n-1} F_{t_i} \diamond (W_{t_{i+1}}^H - W_{t_i}^H)$$

By $\diamond$ we denote the Wick product which is defined by

$$\varepsilon(f) \diamond \varepsilon(g) = \varepsilon(f+g)$$

where

$$\varepsilon(f) := exp\left\{\int_0^\infty f_t dW_t^H - \frac{1}{2}\int_0^\infty \int_0^\infty f_s f_t \phi(s,t) ds dt\right\}$$

is the stochastic exponential of the deterministic function f which is such that

$$|\int_0^\infty \int_0^\infty f_s f_t \phi(s,t) ds dt| < \infty.$$

Duncan et al provide the following generalization of Ito's lemma in the case of fractional Brownian motion.

- Ito Formula I

Let $f : R \to R$, and $f \in C^{1,2}$, then

$$f(W_T^H) - f(W_0^H) = \int_0^T f^{'}(W_s^H) dW_s^H + H \int_0^T s^{2H-1} f^{''}(W_s^H) ds, \ a.s.$$

It is interesting to see that the above formula implies the usual Ito formula for Brownian motion when $H = \frac{1}{2}$.

- Ito Formula II

Let

$$\eta_t = \int_0^t a_s dW_s^H$$

where a_t is some deterministic function such that

$$| \int_0^\infty \int_0^\infty a_s a_t \phi(s,t) ds dt | < \infty.$$

Then

$$\begin{aligned} f(t,\eta_t) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s,\eta_s) ds + \int_0^t \frac{\partial f}{\partial x}(s,\eta_s) a_s dW_s^H \\ &\quad + \int_0^t \frac{\partial^2 f}{\partial x^2}(s,\eta_s) a_s \int_0^s \phi(s,v) a_v dv ds, \ \ a.s. \end{aligned}$$

Reinsurance Contract Pricing

- Some of the most common types of reinsurance are proportional and aggregated excess reinsurance.
- A proportional reinsurance treaty means that the ceding company cedes to the reinsurer a fixed percentage of each risk of the covered portfolio, the reinsurer in return pays the same percentage of each claim and receives the same percentage of the underlying gross premiums.
- In an aggregated excess treaty the reinsurer pays the total of all claims exceeding the retention stated in the contract.

- Our aim is to study the pricing of reinsurance policies for liabilities of an insurance business exhibiting long range dependence. As a suitable model for the liabilities we choose a fractional Brownian motion with drift and Hurst exponent $H > 1/2$.
- we study the valuation of a reinsurance policy for both aggregated excess and proportional reinsurance, by modeling the contract as an Asian type option with underlying the total claim amount by the end of the contract.
- Using fractional Itô calculus we develop a partial differential equation, similar to the Black-Scholes equation, that provides the value of the contract.
- This equation is solved analytically using the technique of similarity solutions thus providing an analytic formula, in terms of special functions, for the valuation of the reinsurance contract in terms of the various parameters of the model.
- This analytic formula can be estimated very easily, without having to resort to time-consuming Monte-Carlo techniques. The results of the paper elucidate, among other things, the effect of long range dependence on the pricing of reinsurance contracts.

The model for the claims process

Let us assume that the claims process is of the form

$$\begin{aligned} dC_t &= b_t dt + \sigma_t dB_t^H \\ C_0 &= c \end{aligned}$$

where with C_t we denote the claims at time t and with b_t we denote the expectations of the claims which may model seasonalities. The term B_t^H is a fractional Brownian motion with Hurst exponent H which is used to model the long range dependence often present in insurance claims. Here we assume that $H \in \left(\frac{1}{2}, 1\right)$.

Aggregated excess reinsurance

If the total claim amount until time T is less than K, where with K we denote a certain percent of the underlying premium volume, the reinsurance company pays nothing whereas if the total claim amount is higher that K it pays the excess of K.

The payoff of this option is given by

$$\max(0, I_T - K) = (I_T - K)^+ = \left(C_0 + \int_0^T C_t dt - K \right)^+ .$$

where

$$I_T := C_0 + \int_0^T C_t dt$$

or in differential form

$$\begin{aligned} dI_t &= C_t = \int_0^t b_s ds + \int_0^t \sigma_s dB_s^H + C_0 \\ I_0 &= C_0 = c \end{aligned}$$

From an actuarial perspective the fair price for the reinsurance policy will be equal to the expected value of the discounted payoff, i.e:

$$v(T, I_0) = E\left[e^{-\delta T}\left(I_T - K\right)^+\right].$$

Proportional reinsurance

In this type of reinsurance the company pays a percentage equal to α of the total claims up to time T. The payoff is given by

$$\alpha I_T = \alpha \int_0^T C_t dt.$$

The proportional reinsurance contract is an Asian type option with final payoff αI.

A PDE for the reinsurance option price

Using techniques from fractional stochastic calculus we derive a deterministic pde for the value of the reinsurance policy.

I_T can be expressed in terms of a fractional Itô process as follows

$$I_T = I_0 + \int_0^T \Theta_t dt + \int_0^T \Sigma(T,s) dB_s^H + C_0 T.$$

where $\Theta_t = \int_0^t b_s ds$ and $\Sigma(T,s) = (T-s)\sigma_s$

- The price of the reinsurance policy is given in terms of the expectation of a functional of fractional Brownian motion.

- Such expectations may often be represented as solutions of deterministic pde's. This approach may be considered as a generalization of the celebrated Feynmann-Kac to non- Markovian processes.

- We show that the price of the reinsurance policy can be given by the solution of a partial differential equation.

PDE For Aggregated Excess Reinsurance Contract Price

The actuarially fair price for the aggregated excess reinsurance policy can be given by $w(T, C_0, I_0)$ where w is the classical solution of the partial differential equation

$$-\delta w - w_1 + b_s w_2 + C_s w_3 + \sigma_s A_s w_{22} + (s A_s \sigma_s - \sigma_s B_s) w_{23} = 0$$

where

$$w = w(T - t, C, I)$$

with initial condition $w(0, C_0, I) = (I - K)^+$.

Solution of the PDE for Aggregated Excess Reinsurance Contract Price

Assume, without loss of generality that $\sigma(t) = \sigma$, $b(t) = b$ are deterministic constants. The price of the aggregated excess reinsurance contract is given by the formula

$$\begin{aligned}
\bar{u}\,(\tau, C_0, I_0) &= e^{-\delta\tau} z(\tau, C_0, I_0)\Phi(k_1) + e^{-\delta\tau}\frac{\sqrt{\bar{\tau}}}{\sqrt{2\pi}}\, e^{-\frac{k_1^2}{2}} - e^{-\delta\tau} k\Phi(k_1), \\
k_1 &= \frac{z(\tau, C_0, I_0) - k}{\sqrt{\bar{\tau}}} \\
\tau &= T - t
\end{aligned}$$

where

$$\begin{aligned}
\bar{\tau} &= \sigma^2 T^2 (T^{2H} - t^{2H}) + 2\left(\frac{\sigma^2 H}{2H+1} - \frac{\sigma^2(H-0.5)}{2H+2}\right)(T^{2H+2} - t^{2H+2}) \\
&- \frac{4\sigma^2 HT}{2H+1}(T^{2H+1} - t^{2H+1}) \\
z &= \frac{b\tau^2}{2} + \tau C_0 + I_0
\end{aligned}$$

Proportional Excess Reinsurance Contract Price

The above arguments may be generalized to other types of reinsurance contracts. The pricing equation will be essentially the same but with different initial condition. The price of the proportional reinsurance contract solves the same equation with the aggregated excess reinsurance contract but with different initial condition

$$\frac{\partial \bar{u}}{\partial \bar{\tau}} = \frac{1}{2}\frac{\partial^2 \bar{u}}{\partial z^2}$$

$$u(0, I) = \alpha I$$

Thus we have that the solution will be given by

$$\bar{u}\left(\bar{\tau}, z\right) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\bar{\tau}}} \exp\left(-\frac{\left(z - z^{'}\right)^2}{2\bar{\tau}}\right) a z' dz^{'} = \alpha z.$$

The proof follows in an analogous manner as in the case of the aggregated excess reinsurance contract.

Numerical Treatment of the Problem

In this section we calculate the value of the aggregated excess reinsurance policy using the analytic solution and the Monte Carlo method. In order to implement the Monte Carlo method we simulate a large number of paths of I_T and then we compute the expected value of the discounted payoff, i.e.

$$\bar{u}(T, I_0, C_0) = E\left[e^{-\delta T}\left(I_T - K\right)^+\right].$$

The values of the parameters we use are claims expectation b=0.5, claims volatility $\sigma = 0.25$, Hurst exponent $H = 0.7$, interest rate $\delta = 0.05$, strike price $k = 10$ and expiry time $T = 10$. For the Monte Carlo method we use 2^N points for the simulation of each path where $N = 14$ and M paths with $M = 20000$.

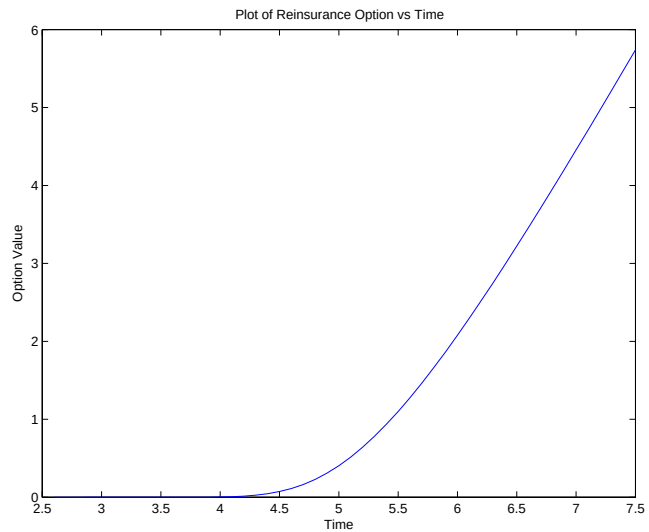
As we expect reinsurance policy value increases as these parameters take higher values. It is natural for the reinsurance policy value to increase when time to expiry, claims expectation and volatility of liabilities increase since all these parameters give higher values of I_T.

When Hurst exponent increase we see that the long-range dependence that characterizes the liabilities leads to have high liabilities being followed by high liabilities and thus making the value of I_T higher and thus the value of the option higher.

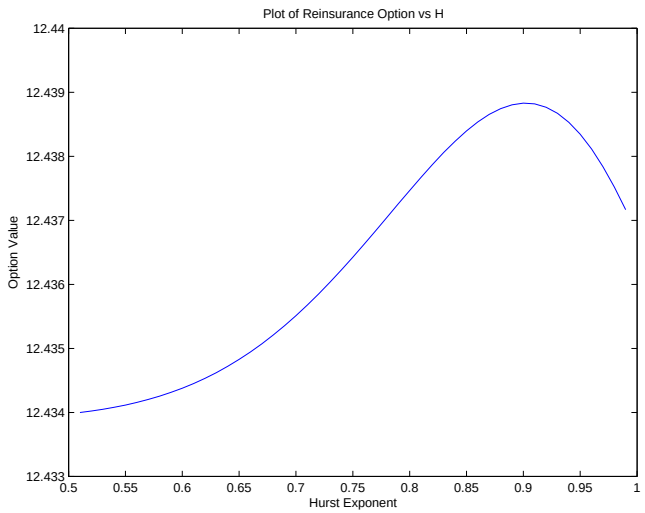
This is a result which agrees with previous results.

- Frangos, Vrontos and Yannacopoulos (2005) show that the probability of ruin of an insurance company is higher as Hurst exponent increases
- Frangos Vrontos and Yannacopoulos (2007) show that as Hurst exponent increases the demand for reinsurance is higher.

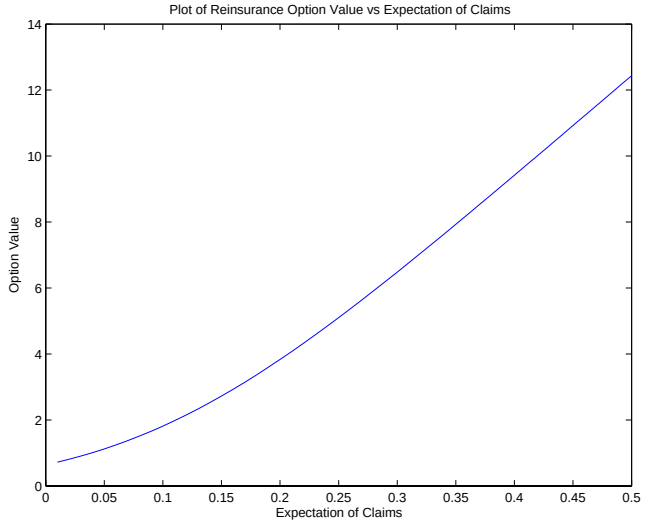
For extreme values of the Hurst exponent, i.e higher than 0.9, we see a strange behavior for the reinsurance option value, i.e. it decreases. This could be the result of the degeneracy of fractional Brownian motion for high values of H.



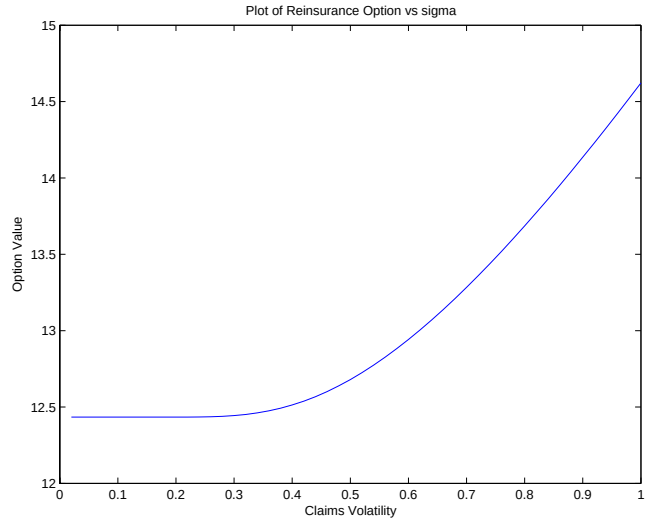
Reinsurance Contract Value as a function of time for $H = 0.7$.



Reinsurance Contract Value as a function of Hurst exponent.



Reinsurance Contract Value as a function of claims expectation.



Reinsurance Option Value as a function of claims volatility.

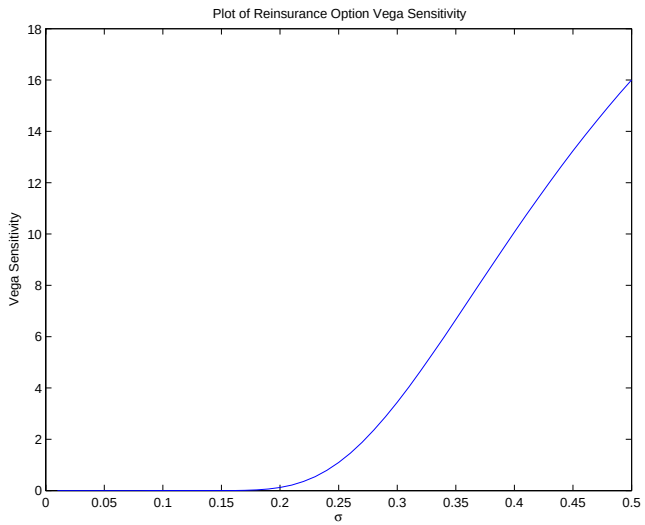
Calculation of the Reinsurance Option Sensitivities

The simple analytic formula for the value of the contract allows us to calculate the sensitivities of the reinsurance contract, known also as the "Greeks", i.e. the partial derivatives of the reinsurance contract value with respect to various parameters of interest such as the interest rate and the claims volatility. For the partial derivative of the reinsurance contract value with respect to δ.

$$\frac{\partial \bar{u}}{\partial \delta} = -\tau \bar{u}$$

For example for the partial derivative with respect to σ we have that

$$\frac{\partial \bar{u}}{\partial \sigma} = e^{-\delta\tau} z \frac{\partial \Phi(k_1)}{\partial \sigma} + \frac{e^{-\delta\tau}}{\sqrt{2\pi}} \left[e^{-\frac{k_1^2}{2}} \frac{\partial \sqrt{\bar{\tau}}}{\partial \sigma} + \sqrt{\bar{\tau}} \frac{\partial e^{-\frac{k_1^2}{2}}}{\partial \sigma} \right] + e^{-\delta\tau} k \frac{\partial \Phi(k_1)}{\partial \sigma}$$



Reinsurance Contract Vega Value as a function of volatility.

- These greeks can also be particularly useful for the design of portfolios of reinsurance contracts.
- A reinsurer may design a reinsurance portfolio to be neutral with respect to one of the greeks of our choice, for instance with respect to the claims volatility σ.
- Thus a reinsurer by choosing appropriate reinsurance contracts with a number of insurance companies, may achieve neutrality of its position with respect to the volatility of the claims process. In analogy, an insurer may choose a number of reinsurance contracts, possibly with more than one reinsurers, to neutralize the effect of the volatility of the claims on its position.