Robust Fitting of Claim Severity Distributions and the Method of Trimmed Moments

Vytaras Brazauskas*

University of Wisconsin-Milwaukee

38th ASTIN Colloquium

Manchester, UK, July 13 – 16, 2008

*In collaboration with Jones and Zitikis (University of Western Ontario)

Outline

1. **Background**
   - Motivation: Hurricane Damages
   - Introduction and Preliminaries
2. **Method of Trimmed Moments**
   - Definition
   - Asymptotic Properties
   - Examples
3. **Comparisons and Conclusions**
   - Simulations
   - Real-Data Example
   - Concluding Remarks

1. Background

Motivation: Hurricane Damages

- **Data**
  - ▷ Top 30 damaging hurricanes in the United States: 1925–1995.
  - ▷ Normalized to 1995 dollars by inflation, personal property increases, coastal county population changes.
  - ▷ Published by Pielke and Landsea (1998) in *Weather and Forecasting*.

- **Objectives**
  - ▷ STATISTICAL: Model fitting
  - ▷ ACTUARIAL: Ratemaking

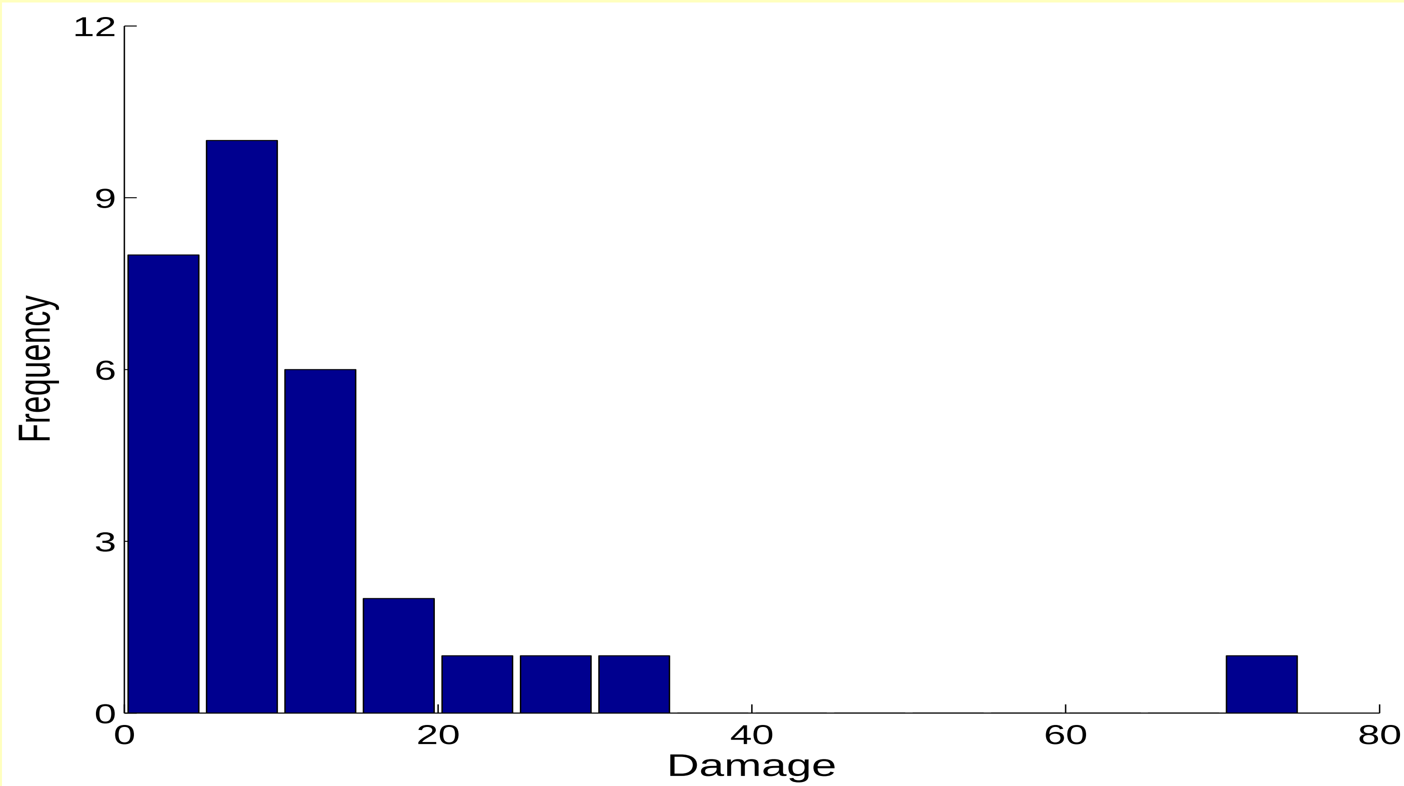


FIGURE 1: Histogram of the top 30 damaging hurricanes.

Introduction and Preliminaries

- **Loss Severity Distributions**
  - ▷ Risk evaluations
  - ▷ Ratemaking
  - ▷ Reserve calculations

- **Standard Estimation & Fitting Techniques**
  - ▷ Empirical Nonparametric
    - + Simple approach
    - + Weak assumptions
    - – Lack of smoothness
    - – Limited to the range of observed data
  - ▷ Parametric
    - + Stretchability beyond the range of observed data
    - + Smoothness
    - + Efficiency
    - + Special distributional features (e.g., a mode at 0)
    - – Strong assumptions
    - – Outliers (e.g., a loss that receives an extensive media attention)

- **Robust Estimation & Fitting**
  - ▷ NOT ROBUST: maximum likelihood, method-of-moments
  - ▷ ROBUST: M-, L-, R-statistics
  - ▷ M ($maximum$ likelihood type)
    - \+ Most popular
    - \+ Easy to generalize
    - – Computationally complex
    - – Lack of transparency
  - ▷ L ($linear$ combinations of order statistics)
    - – Not easy to generalize
    - \+ Computer friendly
    - \+ Transparent

2. Method of Trimmed Moments

Definition

- **Assumptions & Notation**
  - ▷ DATA: $X_1, \dots, X_n$ *i.i.d.* with cdf F
  - ▷ CDF:
    - + F is *continuous*
    - + F depends on $\theta_1, \dots, \theta_k$ (*unknown* parameters)
  - ▷ ORDERED DATA: $X_{1:n} \leq \cdots \leq X_{n:n}$

- **Four-Step Procedure**

1. SAMPLE TRIMMED MOMENTS:

$$\widehat{\mu}_j = \frac{1}{n - m_n(j) - m_n^*(j)} \sum_{i=m_n(j)+1}^{n-m_n^*(j)} h_j(X_{i:n})$$

$j = 1, \ldots, k$, with $m_n(j)/n \approx a_j$, $m_n^*(j)/n \approx b_j$ chosen trimming proportions, h_j chosen function.

2. POPULATION TRIMMED MOMENTS:

$$\mu_j := \mu_j(\theta_1, \ldots, \theta_k) = \frac{1}{1 - a_j - b_j} \int_{a_j}^{1-b_j} h_j(F^{-1}(u))\, \mathrm{d}u$$

$j = 1, \ldots, k$.

3. MATCH & SOLVE:

$$\begin{cases} \mu_1(\theta_1, \ldots, \theta_k) & = & \widehat{\mu}_1, \\ & \vdots & \\ \mu_k(\theta_1, \ldots, \theta_k) & = & \widehat{\mu}_k. \end{cases}$$

4. MTM ESTIMATORS OF $\theta_1, \ldots, \theta_k$:

$$\begin{cases} \widehat{\theta}_1 & = & g_1(\widehat{\mu}_1, \ldots, \widehat{\mu}_k), \\ & \vdots & \\ \widehat{\theta}_k & = & g_k(\widehat{\mu}_1, \ldots, \widehat{\mu}_k). \end{cases}$$

Asymptotic Properties

$$\left(\widehat{\theta}_1, \ldots, \widehat{\theta}_k\right) \quad \text{is} \quad \mathcal{AN}\Big((\theta_1, \ldots, \theta_k),\, n^{-1}\, \mathbf{D\Sigma D'} \Big)$$

where $\mathbf{D}_{k \times k}$ with $d_{ij} = \frac{\partial g_i}{\partial \widehat{\mu}_j}\Big|_{(\mu_1, \ldots, \mu_k)}$ and $\mathbf{\Sigma}_{k \times k}$ with

$$\begin{aligned} \sigma_{ij}^2 \quad &= \quad \frac{1}{(1-a_i-b_i)(1-a_j-b_j)} \\ &\times \quad \int_{a_i}^{1-b_i} \int_{a_j}^{1-b_j} \big(\min\{u,v\} - uv \big)\, \mathrm{d}h_j\big(F^{-1}(v)\big)\, \mathrm{d}h_i\big(F^{-1}(u)\big) \end{aligned}$$

Examples

- **Pareto Model**

  ▷ CDF, QF:

$$F(x) = 1 - \left(\frac{x}{x_0}\right)^{-\alpha}, \qquad F^{-1}(u) = x_0(1-u)^{-1/\alpha}$$

$\alpha > 0$, $x > x_0$ (*known*), $0 < u < 1$.

  ▷ FUNCTION h_1:

$$h_1(t) = \log t$$

▷ SAMPLE TM:

$$\widehat{\mu}_1 = \frac{1}{n - m_n - m_n^*} \sum_{i=m_n+1}^{n-m_n^*} \log(X_{i:n}/x_0)$$

▷ POPULATION TM:

$$\begin{aligned} \mu_1 &= \frac{1}{1-a-b} \int_a^{1-b} \log(F^{-1}(u)/x_0)\, \mathrm{d}u \\ &= \frac{-1/\alpha}{1-a-b} \int_a^{1-b} \log(1-u)\, \mathrm{d}u \\ &= \big(1/\alpha\big) \times const_1 \end{aligned}$$

▷ MTM of α:

$$\widehat{\alpha}_{\mathrm{MTM}} = \frac{const_1}{\widehat{\mu}_1}$$

▷ ASYMPTOTICS:

$$\widehat{\alpha}_{\mathrm{MTM}} \quad \text{is} \quad \mathcal{AN}\left(\alpha, \, \frac{\alpha^2}{n}\, Const_1\right)$$

▷ COMPARISON with MLE:

$$\widehat{\alpha}_{\mathrm{MLE}} = \frac{n}{\sum_{i=1}^{n} \log(X_i/x_0)} \quad \text{is} \quad \mathcal{AN}\left(\alpha, \, \frac{\alpha^2}{n}\right)$$

TABLE 1: $\mathrm{ARE}(\widehat{\alpha}_{\text{MTM}},\, \widehat{\alpha}_{\text{MLE}}) = 1/Const_1$.

a	b = 0	0.05	0.10	0.15	0.25	0.49	0.70
0	1	.918	.847	.783	.666	.423	.238
0.05	1.00	.918	.848	.783	.667	.425	.242
0.10	1.00	.918	.848	.785	.669	.430	.250
0.15	.999	.919	.850	.787	.672	.437	.261
0.25	.995	.918	.851	.790	.679	.452	.285
0.49	.958	.897	.839	.786	.688	.487	–
0.70	.857	.824	.781	.738	.659	–	–

- **Lognormal Model**

▷ CDF, QF:

$$F(x) = \Phi\left(\frac{\log(x - x_0) - \theta}{\sigma}\right)$$

$$\log(F^{-1}(u) - x_0) = \theta + \sigma\,\Phi^{-1}(u)$$

$\theta \in \mathcal{R}$, $\sigma > 0$, $x > x_0$ (*known*), $0 < u < 1$,
and Φ, Φ^{-1} are CDF, QF of $N(0,1)$.

▷ FUNCTIONS h:

$$h_1(t) = \log(t - x_0), \qquad h_2(t) = \log^2(t - x_0)$$

▷ SAMPLE TMS:

$$\widehat{\mu}_j = \frac{1}{n - m_n - m_n^*} \sum_{i=m_n+1}^{n-m_n^*} \Big[\log(X_{i:n} - x_0) \Big]^j, \qquad j = 1, 2$$

▷ POPULATION TMS:

$$\begin{aligned} \mu_1 &= \frac{1}{1-a-b} \int_a^{1-b} \log(F^{-1}(u) - x_0)\, \mathrm{d}u \\ &= \theta + \sigma \times c_1 \end{aligned}$$

$$\begin{aligned} \mu_2 &= \frac{1}{1-a-b} \int_a^{1-b} \Big[\log(F^{-1}(u) - x_0) \Big]^2 \mathrm{d}u \\ &= \theta^2 + 2\theta\sigma \times c_1 + \sigma^2 \times c_2 \end{aligned}$$

▷ MTM and MLE of (θ, σ):

$$\begin{cases} \widehat{\theta}_{\text{MTM}} & = & \widehat{\mu}_1 - c_1 \, \widehat{\sigma}_{\text{MTM}} \\ \widehat{\sigma}_{\text{MTM}} & = & \sqrt{(\widehat{\mu}_2 - \widehat{\mu}_1^2) \big/ (c_2 - c_1^2)} \end{cases}$$

$$\begin{cases} \widehat{\theta}_{\text{MLE}} & = & n^{-1} \sum_{i=1}^{n} \log(X_i - x_0) \\ \widehat{\sigma}_{\text{MLE}} & = & \sqrt{n^{-1} \sum_{i=1}^{n} \left(\log(X_i - x_0) - \widehat{\theta}_{\text{MLE}} \right)^2} \end{cases}$$

▷ ASYMPTOTICS:

$$\left(\widehat{\theta}_{\text{MTM}}, \, \widehat{\sigma}_{\text{MTM}} \right) \quad \text{is} \quad \mathcal{AN} \left((\theta, \sigma), \, \frac{\sigma^2}{n} \, \mathbf{S} \right)$$

$$\left(\widehat{\theta}_{\text{MLE}},\ \widehat{\sigma}_{\text{MLE}}\right) \quad \text{is} \quad \mathcal{AN}\left((\theta,\sigma),\ \frac{\sigma^2}{n}\,\mathbf{S_0}\right)$$

TABLE 2: $\text{ARE}\big((\widehat{\theta}_{\text{MTM}},\widehat{\sigma}_{\text{MTM}}),\ (\widehat{\theta}_{\text{MLE}},\widehat{\sigma}_{\text{MLE}})\big) = \sqrt{|\mathbf{S_0}|/|\mathbf{S}|}$.

	b				
a	0	0.05	0.15	0.49	0.70
0	**1**	.932	.821	.502	.312
0.05	.932	**.872**	.771	.470	.286
0.15	.821	.771	**.676**	.390	.208
0.49	.502	.470	.390	**.074**	–
0.70	.312	.286	.208	–	–

3. Comparisons and Conclusions

Simulations

- **Study Design**
  - ▷ NUMBER OF SIMULATED SAMPLES: $M = 100,000$
  - ▷ SIZES OF SAMPLES: $n = 50,\ 100,\ 250,\ 500$
  - ▷ SELECTED MODELS:
    - + Pareto($x_0 = 1,\ \alpha = 0.50$)
    - + Lognormal($x_0 = 1,\ \theta = 5,\ \sigma = 3$)
  - ▷ METHODS OF ESTIMATION: MTM, MLE
  - ▷ REPORTING: standardized MEAN, RE

Table 3: Pareto($x_0 = 1,\ \alpha = 0.50$) model.

Statistic	*Estimator* a	b	*Sample Size* 50	100	250	500	∞
MEAN/α	0	0	1.02	1.01	1.00	1.00	1
	0.05	0.05	0.99	1.01	1.00	1.00	1
	0.10	0.10	1.01	1.01	1.00	1.00	1
	0.25	0.25	1.01	1.01	1.00	1.00	1
	0.49	0.49	1.03	1.01	1.01	1.00	1
	0.10	0.70	1.04	1.02	1.01	1.00	1
	0.25	0.00	1.03	1.01	1.01	1.00	1

Note: Standard errors for all entries $\leq .001$

TABLE 4: Pareto($x_0 = 1,\ \alpha = 0.50$) model.

Statistic	*Estimator* a	b	*Sample Size* 50	100	250	500	∞
RE	0	0	0.92	0.96	0.98	1.00	1
	0.05	0.05	0.90	0.92	0.92	0.92	0.918
	0.10	0.10	0.80	0.83	0.84	0.85	0.848
	0.25	0.25	0.65	0.65	0.68	0.68	0.679
	0.49	0.49	0.43	0.45	0.47	0.48	0.487
	0.10	0.70	0.21	0.23	0.24	0.25	0.250
	0.25	0.00	0.87	0.95	0.97	0.99	0.995

NOTE: Standard errors for all entries $\leq .006$

TABLE 5: Lognormal($x_0 = 1$, $\theta = 5$, $\sigma = 3$) model.

Statistic	*Estimator* a	b	*Sample Size* 50	100	250	500	∞
RE	0	0	0.99	1.00	1.00	1.00	1
	0.05	0.05	0.82	0.87	0.87	0.87	0.872
	0.10	0.10	0.77	0.77	0.77	0.77	0.769
	0.25	0.25	0.48	0.50	0.50	0.51	0.507
	0.49	0.49	0.04	0.06	0.07	0.07	0.074
	0.10	0.70	0.24	0.25	0.25	0.25	0.248
	0.25	0.00	0.73	0.72	0.72	0.72	0.722

NOTE: Standard errors for all entries $\leq .003$

Real-Data Example

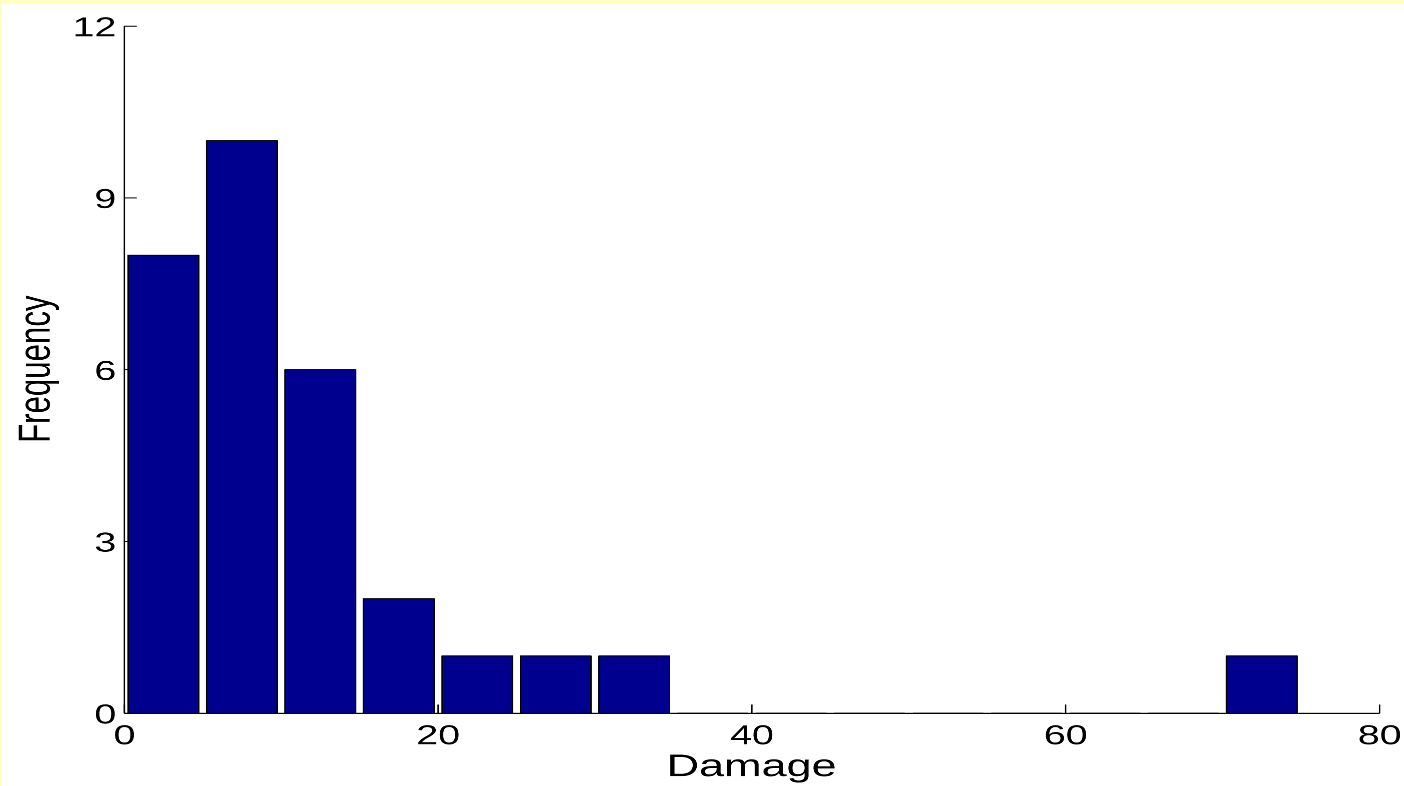


FIGURE 1: Histogram of the top 30 damaging hurricanes.

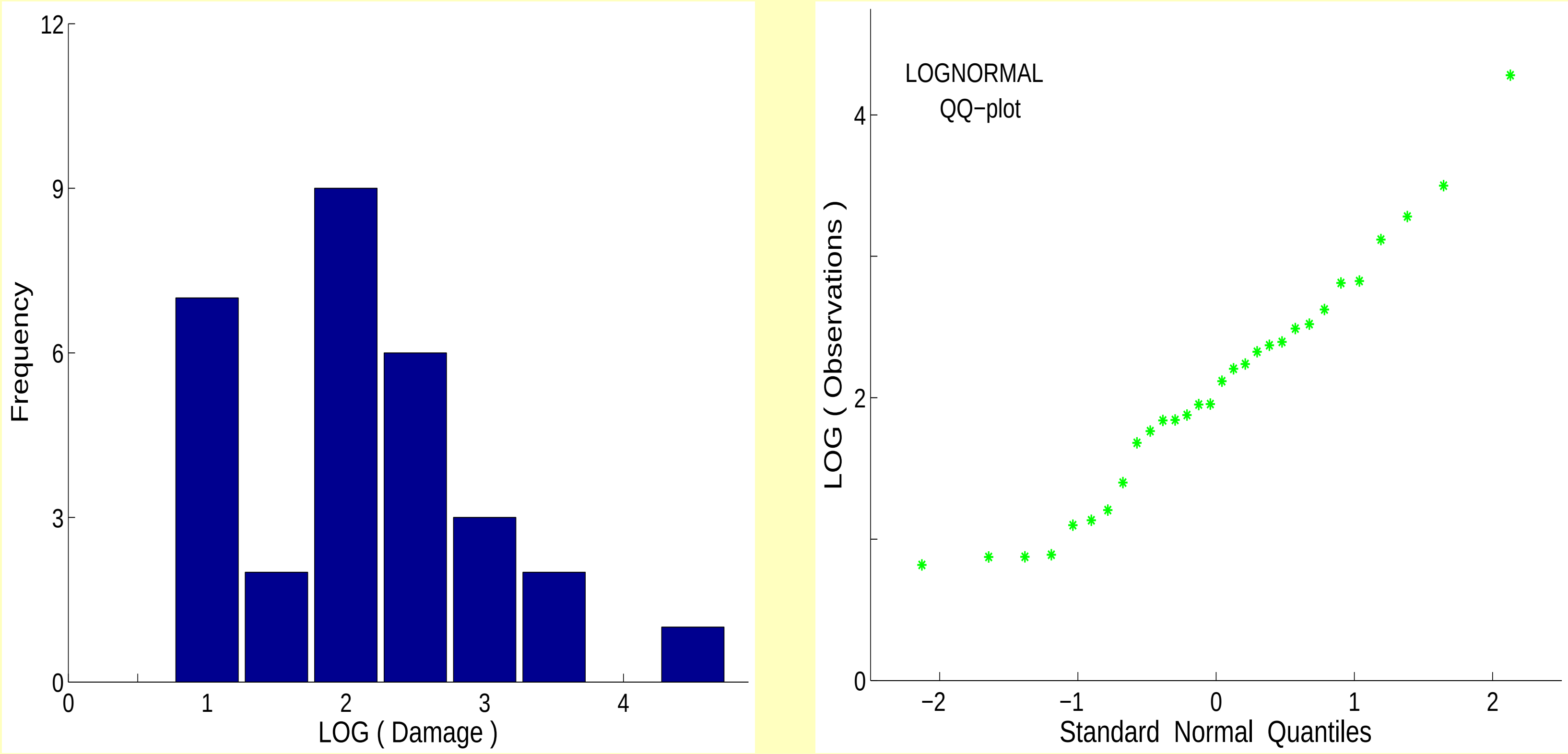
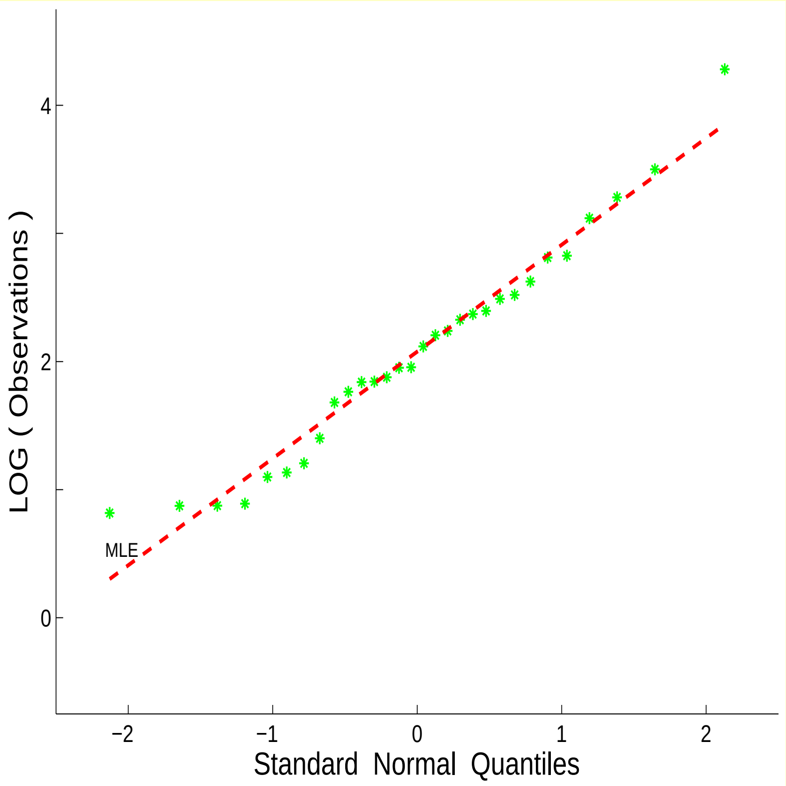
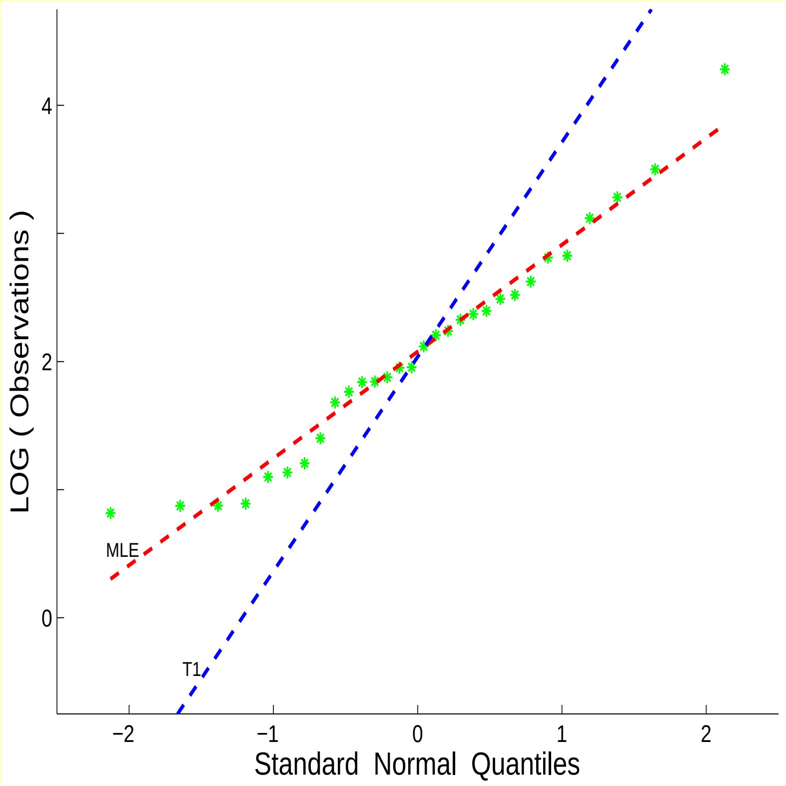


FIGURE 2: Preliminary diagnostics for the hurricane data.



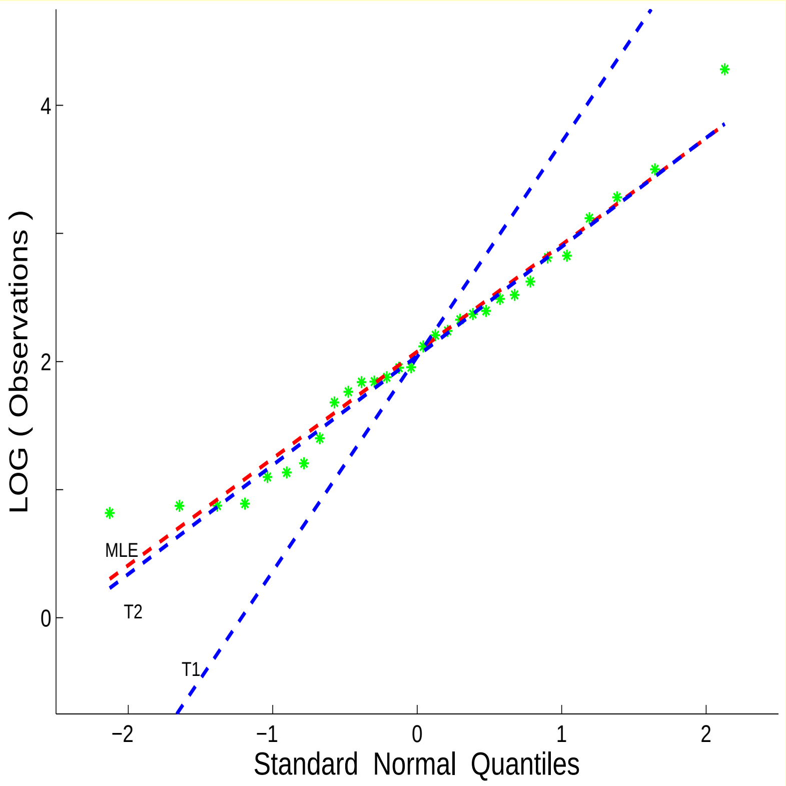
Estimator	$\widehat{\theta}$	$\widehat{\sigma}$	*Fit*
MLE	2.077	0.834	0.104

FIGURE 3: Lognormal fits to the hurricane data.



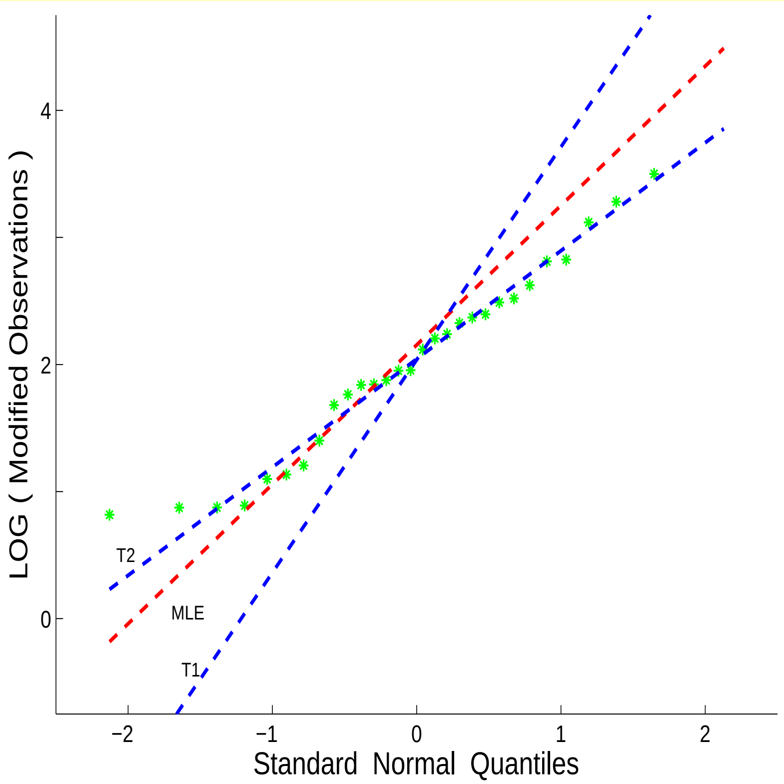
Estimator	$\widehat{\theta}$	$\widehat{\sigma}$	*Fit*
MLE	2.077	0.834	0.104
$T1\left(\frac{14}{30}, \frac{14}{30}\right)$	2.037	1.675	0.662

FIGURE 3: Lognormal fits to the hurricane data.



Estimator	$\widehat{\theta}$	$\widehat{\sigma}$	*Fit*
MLE	2.077	0.834	0.104
$T1\left(\frac{14}{30}, \frac{14}{30}\right)$	2.037	1.675	0.662
$T2\left(\frac{1}{30}, \frac{1}{30}\right)$	2.043	0.852	0.101

FIGURE 3: Lognormal fits to the hurricane data.



Estimator	$\widehat{\theta}$	$\widehat{\sigma}$	*Fit*
MLE	2.154	1.098	0.293
T1$\left(\frac{14}{30}, \frac{14}{30}\right)$	2.037	1.675	0.651
T2$\left(\frac{1}{30}, \frac{1}{30}\right)$	2.043	0.852	0.178

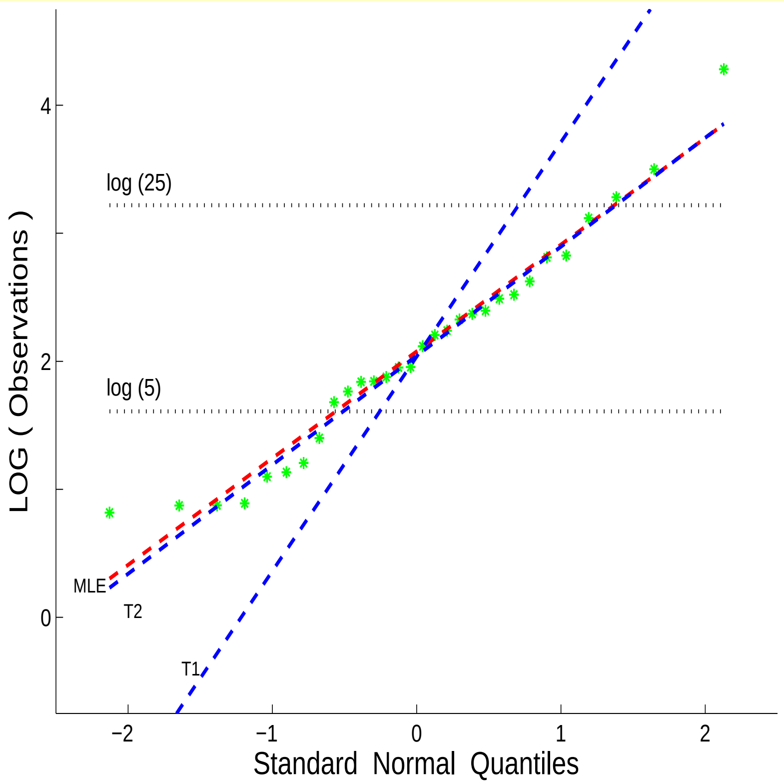
FIGURE 3: Lognormal fits to the *modified* hurricane data. (*Largest observation* 72.303 *is replaced with* 723.03)

- **Insurance Contract**

  Insurance benefit equal to the amount by which a hurricane's damage exceeds 5 (billion) with a maximum benefit of 20.

- **Net Premium**

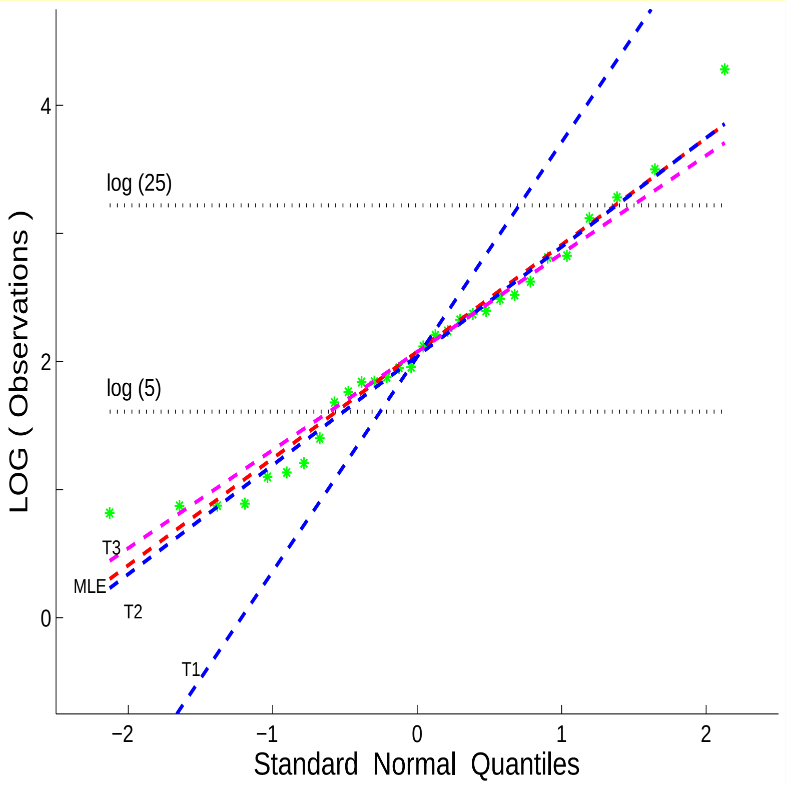
$$\text{PREMIUM} = \int_5^{25} (x-5)\, \mathrm{d}F(x) + 20[1 - F(25)]$$



Estimator	$\widehat{\theta}$	$\widehat{\sigma}$	*R-Fit*
MLE	2.077	0.834	0.054
	2.154	1.098	0.202
$T1\left(\frac{14}{30}, \frac{14}{30}\right)$	2.037	1.675	0.413
	2.037	1.675	0.413
$T2\left(\frac{1}{30}, \frac{1}{30}\right)$	2.043	0.852	0.057
	2.043	0.852	0.057

PREMIUM (MLE)	5.604 (6.896)
PREMIUM (T1)	7.347 (7.347)
PREMIUM (T2)	5.437 (5.437)
PREMIUM (EMPIRICAL)	5.416 (5.416)

FIGURE 4: Lognormal fits to the hurricane data.



Estimator	$\widehat{\theta}$	$\widehat{\sigma}$	*R-Fit*
MLE	2.077	0.834	0.054
	2.154	1.098	0.202
$T1\left(\frac{14}{30}, \frac{14}{30}\right)$	2.037	1.675	0.413
	2.037	1.675	0.413
$T2\left(\frac{1}{30}, \frac{1}{30}\right)$	2.043	0.852	0.057
	2.043	0.852	0.057
$T3\left(\frac{8}{30}, \frac{3}{30}\right)$	2.075	0.766	0.042
	2.075	0.766	0.042

PREMIUM (MLE)	5.604 (6.896)
PREMIUM (T1)	7.347 (7.347)
PREMIUM (T2)	5.437 (5.437)
PREMIUM (T3)	5.336 (5.336)
PREMIUM (EMPIRICAL)	5.416 (5.416)

FIGURE 4: Lognormal fits to the hurricane data.

Concluding Remarks

- **Summary**
  - ▷ Introduced and developed a new general method (called MTM) for estimating the parameters of claim severity distributions.
  - ▷ MTM: easy to understand; can achieve various (easily controlled by the user) degrees of robustness.
  - ▷ Established asymptotic properties of MTMs; applicable for constructing confidence intervals, or sets, and for testing hypotheses.
  - ▷ Investigated small-sample properties.
  - ▷ Real-data illustration; calculation of premiums for a layer of insurance coverage is a task for which MTM estimates are a natural choice.

- **Challenges**
  - ▷ Continuous but Non-identically distributed data?
    (e.g., covariates)
  - ▷ Continuous but dependent data?
    (e.g., copulas)
  - ▷ Discrete models?
    (e.g., claim frequencies)
  - ▷ Generalized Linear Models?