A generalized linear discrete time model for managing the solvency interaction and singularities arising from potential regulatory constraints imposed within a portfolio of different insurance products

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Abstract

We consider a typical portfolio of different insurance products and investigate the pricing process using the framework of a generalized linear discrete time model. Moreover, we assume that, due to regulatory constraints, the resulting system is singular and calculate the solution using the tools of matrix pencil theory. Finally, we present a numerical application for two different portfolios.

Key-words: Pooling of risks, Solvency Interaction, Singularities, Matrix pencil theory, Descriptor Control Systems; Non Causality

1. Introduction

According to the actuarial literature, the pooling of risks is a very efficient risk management technique used by large employee benefit schemes of multinational companies to self-insure their obligations, see Encyclopaedia of Actuarial Science (2006). Furthermore, the issue of pooling may be also applicable for the management of a portfolio of different insurance products (lines of business) within an insurance company when there is some kind of surplus interaction.

We develop our analysis in three stages:

- We specify a model to describe the premium rating process associated with the sharing claim experience for each product in the pool,
- We formulate and examine the interaction of the surpluses among the insurance products in the pool.
- In the special case, where a target for zero surplus is required for some of the products (e.g. due to a regulatory constraint), a descriptor generalized model is derived where the solutions are more complex and the important notion of causality does not exist.

Actually, in this paper we have two main objectives.

(i) We aim, to provide a comprehensive, convenient and practical actuarial model for the management of a portfolio of different insurance products using the standard tools of control theory. Thus, with the view of control theory, claims may be regarded as the input ($\underline{u}$), the accumulated surplus as the state ($\underline{x}$) and the gross premiums as the output ($\underline{y}$) vector-variable of the designed system.

(ii) Additionally, we introduce the mathematical framework for manipulating and solving systems (1.1) by presenting some preliminary concepts and definitions from *matrix pencil theory*. Note also that for a given pair of constant matrices E and $A \in \mathbb{R}^{m\times n}$, with $\det E = 0$ (which uniquely determine the underlying matrix pencil $sE - A$ of system (1.1); the corresponding pencil can be either *regular* or *singular*, see the **Appendix** for further details.

$$\begin{aligned} E\underline{x}_k &= A\underline{x}_{k-1} + B\underline{u}_k \\ \underline{y}_k &= C\underline{x}_k + D\underline{u}_k \end{aligned} \tag{1.1}$$

2. Notations and Model Framework

We present the necessary notation used throughout the paper.

m: The total *number of different products,* participating in the portfolio of the insurance company.

e_i : The *expense* factor for the i^{th}-insurance product,

i.e. $(1-e_i)\times$ Gross Premium is the margin for expenses.

r_i : The *annual rate of investment return* for the i^{th}-insurance product.

λ_{ij} : The *interaction factor*, $i, j = 1, 2, \ldots, m$, is the proportion of accumulated surplus of the i^{th}-product transferred to the j^{th}-insurance product.

ε_i : The *profit sharing factor* (feedback factor) for the i-insurance product, which includes premium repayments and determines the percentage of accumulated surplus repaid to the policyholders.

$\{C_{i,k}\}_{k\in\mathbb{N}}$: The *actual total amount incurred claims* sequence for the i-company in year k, i.e. $(k-1, k]$.

$\{\hat{C}_{i,k}\}_{k\in\mathbb{N}}$: The *estimated total expected annual incurred claims* sequence in year k for the i^{th}-insurance product. Obviously, there is always a small (or larger) delay period of d_i years in updating information. In that case and using the latest information of the two available years, we obtain

$$\hat{C}_{i,k} = f\begin{pmatrix}\text{previous}\\ \text{years}\end{pmatrix} = w_i C_{i,k-d_i-1} + (1-w_i)\, C_{i,k-d_i-2} \qquad (2.1)$$

w_i : The weighted factor for the average claims (over the two recent years) for the i^{th}-insurance product.

d_i : The length of time delay (measured in years) for the i^{th}-insurance product. Thus, it takes about d_i years for incurred claims to be fully reported, processed and settled. Obviously, the available claim information at the beginning of the k year (or at the end of $k-1$) refers to the experience of the years $k-d_i-1$, $k-d_i-2$, $k-d_i-3$, . . ., 2, 1, 0, i.e. years prior to and inclusive of years $k-d_i-1$.

$\left\{P_{i,k}\right\}_{k\in\mathbb{N}}$: The *gross annual premium* (***GAP***) sequence paid at the end of the k^{th} year for the i^{th} -insurance product. The GAP is determined as an expense-adjusted premium $P_{i,k}^{(e)}$ less the surplus adjustment, see also Zimbidis and Haberman (2001a), where

$$P_{i,k}^{(e)} = \hat{C}_{i,k} + \left(1-e_i\right)P_{i,k}^{(e)} = \frac{\hat{C}_{i,k}}{e_i}.$$

Thus, it follows that

$$P_{i,k} = P_{i,k}^{(e)} - \sum_{j=1}^{m} \varepsilon_j \lambda_{ij}\left(S_{j,k} - S_{j,k-d_j-1}\right) = \frac{\hat{C}_{i,k}}{e_i} - \sum_{j=1}^{m} \varepsilon_j \lambda_{ij}\left(S_{j,k} - S_{j,k-d_j-1}\right), \text{ for } i = 1,2,\ldots,m. \qquad (2.2)$$

Equation (2.2) is the decision function Note that, GAP is calculated annually at the beginning of each year according to a base premium and a profit sharing scheme. The last one mandates an extra modification of the base premium through a refund (charge) to the policyholder a certain percentage of the benefit scheme's total accumulated surplus (deficit). Obviously, the manager of the portfolio should firstly consider the difference between the real and the target-surplus, see expression (2.2).

$\left\{S_{i,k}\right\}_{k\in\mathbb{N}}$: The *accumulated surplus* (**AS**) sequence at the end of the k year for the i^{th} -insurance product, where

$$S_{i,k} = \left(1+r_i\right)\sum_{j=1}^{m} \lambda_{ij} S_{j,k-1} + e_i P_{i,k} - C_{i,k} \qquad \text{for } i = 1,2,\ldots,m \qquad (2.3)$$

We further assume that all the parameters m, e_i, r_i, λ_{ij}, ε_i, w_i and d_i for $i, j = 1,2,\ldots,m$ are constants over time, see also Zimbidis and Haberman (2001a).

Now, we are ready to construct a realistic insurance model, described by the system of equations (2.1)-(2.3). The analytic solution is derived in the following sections, and the parameters are designed with respect to the solvency interaction arising from the surplus exchange process. The formulation of the problem differs form the one followed by Balzer and Benjamin (1980), Balzer (1982), Benjamin (1984), and Zimbidis and Haberman (2001 a,b), since the delay period of data updating may be varied for each of the products and the causality of the solution of the system under some very realistic circumstances may also be lost. In the next section the control model is developed.

3. The Model and System of Equations

The system which is a classical MIMO (multi input-multi output) starts from an initial point (for further mathematical details see **Appendix**) for the first year's premium, then claim data provide the input background for the development of the surplus level etc. Finally, combining both claims (directly) and surplus information through a feedback mechanism, a decision function is built for premium development. Analytically, we derive the following two equations,

For the i^{th}-insurance product, the k^{th}-year's premium and surplus proceedings are determined according to the following equations,

$$P_{i,k} = \frac{1}{e_i}\left(w_i C_{i,k-d_i-1} + (1-w_i) C_{i,k-d_i-2}\right) - \sum_{j=1}^{m} \varepsilon_j \lambda_{ij} \left(S_{j,k} - S_{j,k-d_j-1}\right), \qquad (3.1)$$

and $S_{i,k} = (1+r_i)\sum_{j=1}^{m} \lambda_{ij} S_{j,k-1} + w_i C_{i,k-d_i-1} + (1-w_i) C_{i,k-d_i-2} - e_i \sum_{j=1}^{m} \varepsilon_j \lambda_{ij} \left(S_{j,k} - S_{j,k-d_j-1}\right) - C_{i,k}$ (3.2)

for $i = 1, 2, \ldots, m$.

Each of the m insurance products generates its own system of equations. These systems can not be solved independently since the interaction factors λ_{ij} exists in them. Thus, considering expressions (3.1) and (3.2), the following systems (**S1** and **S2**) of $2m$ delay difference equations that describe the premium rating, the surplus process and the interaction within the portfolio of insurance product are respectively derived.

$$\left.\begin{aligned}
P_{1,k} &= \frac{1}{e_1}\left(w_1 C_{1,k-d_1-1} + (1-w_1) C_{1,k-d_1-2}\right) - \sum_{j=1}^{m} \varepsilon_j \lambda_{1j} \left(S_{j,k} - S_{j,k-d_j-1}\right) \\
&\vdots \\
P_{i,k} &= \frac{1}{e_i}\left(w_i C_{i,k-d_i-1} + (1-w_i) C_{i,k-d_i-2}\right) - \sum_{j=1}^{m} \varepsilon_j \lambda_{ij} \left(S_{j,k} - S_{j,k-d_j-1}\right) \\
&\vdots \\
P_{m,k} &= \frac{1}{e_m}\left(w_m C_{m,k-d_m-1} + (1-w_m) C_{m,k-d_m-2}\right) - \sum_{j=1}^{m} \varepsilon_j \lambda_{mj} \left(S_{j,k} - S_{j,k-d_j-1}\right)
\end{aligned}\right\} \text{(S1)}$$

$$\left.\begin{aligned}
&\left(1+e_1\varepsilon_1\lambda_{11}\right)S_{1,k}+e_1\sum_{j=2}^{m}\varepsilon_j\lambda_{1j}S_{j,k}=\left(1+r_1\right)\sum_{j=1}^{m}\lambda_{1j}S_{j,k-1}+e_1\sum_{j=1}^{m}\varepsilon_j\lambda_{1j}S_{j,k-d_j-1}\\
&\qquad\qquad\qquad\qquad -C_{1,k}+w_1C_{1,k-d_1-1}+\left(1-w_1\right)C_{1,k-d_1-2}\\
&\vdots\\
&e_i\sum_{j=1}^{i-1}\varepsilon_j\lambda_{ij}S_{j,k}+\left(1+e_i\varepsilon_i\lambda_{ii}\right)S_{i,k}+e_i\sum_{j=i+1}^{m}\varepsilon_j\lambda_{ij}S_{j,k}=\left(1+r_i\right)\sum_{j=1}^{m}\lambda_{ij}S_{j,k-1}+e_i\sum_{j=1}^{m}\varepsilon_j\lambda_{ij}S_{j,k-d_j-1}\\
&\qquad\qquad\qquad\qquad -C_{i,k}+w_iC_{i,k-d_i-1}+\left(1-w_i\right)C_{i,k-d_i-2}\\
&\vdots\\
&e_m\sum_{j=2}^{m-1}\varepsilon_j\lambda_{mj}S_{j,k}+\left(1+e_m\varepsilon_m\lambda_{mm}\right)S_{m,k}=\left(1+r_m\right)\sum_{j=1}^{m}\lambda_{mj}S_{j,k-1}+e_m\sum_{j=1}^{m}\varepsilon_j\lambda_{mj}S_{j,k-d_j-1}\\
&\qquad\qquad\qquad\qquad -C_{m,k}+w_mC_{m,k-d_m-1}+\left(1-w_m\right)C_{m,k-d_m-2}
\end{aligned}\right\}\text{(S2)}$$

Obviously, working with systems (S1) and (S2) is not an easy task. Thus, the matrix-vector reformulation is more appropriate. So, we may denote

$$\underline{S}=\begin{bmatrix} S_{1,k}\\ S_{1,k-1}\\ \vdots\\ S_{1,k-d_1}\\ ---\\ S_{2,k}\\ S_{2,k-1}\\ \vdots\\ S_{2,k-d_2}\\ ---\\ \vdots\\ ---\\ S_{m,k}\\ S_{m,k-1}\\ \vdots\\ S_{m,k-d_m}\end{bmatrix}\in\mathbb{R}^{\sum_{i=1}^{m}(1+d_i)}\ ,\ \underline{u}=\begin{bmatrix} C_{1,k}\\ C_{1,k-1}\\ \vdots\\ C_{1,k-d_1-2}\\ ---\\ C_{2,k}\\ C_{2,k-1}\\ \vdots\\ C_{2,k-d_2-2}\\ ---\\ \vdots\\ ---\\ C_{m,k}\\ C_{m,k-1}\\ \vdots\\ C_{m,k-d_m-2}\end{bmatrix}\in\mathbb{R}^{\sum_{i=1}^{m}(3+d_i)}$$

and

$$\underline{P}=\begin{bmatrix} P_{1,k}\\ P_{2,k}\\ \vdots\\ P_{m,k}\end{bmatrix}\in\mathbb{R}^{m}.$$

It should be mentioned that the input vector is determined by considering the actual $C_{i,k-d_i}$'s when they are available or the expectation $\hat{C}_{i,k-d_i}$'s otherwise. In other words, we obtain

$$C_{i,k-j} = \begin{cases} \text{is replaced by } \hat{C}_{i,k-j}, & \text{for } j = 0,1,2,\ldots,d_i \\ \text{remain unchanged,} & \text{for } j = d_i+1, d_i+2,\ldots \end{cases}.$$

For the use of systems S1 and S2, five super-matrices, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{E}$ are respectively introduced. We start with (S2):

$$\mathbf{E} = \begin{bmatrix} \mathbf{E_{11}} & \mathbf{E_{12}} & \cdots & \mathbf{E_{1m}} \\ \mathbf{E_{21}} & \mathbf{E_{22}} & \cdots & \mathbf{E_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{E_{m1}} & \mathbf{E_{m2}} & \cdots & \mathbf{E_{mm}} \end{bmatrix} \in \mathbb{R}^{\sum_{i=1}^{m}(1+d_i)\times\sum_{i=1}^{m}(1+d_i)},$$

where its elements are $\mathbf{E_{ii}} = \begin{bmatrix} 1+e_i\varepsilon_i\lambda_{ii} & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{(1+d_i)\times(1+d_i)}$

and $\mathbf{E_{ij}} = \begin{bmatrix} e_i\varepsilon_j\lambda_{ij} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(1+d_i)\times(1+d_j)}.$

The matrix $\mathbf{A} = \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} & \cdots & \mathbf{A_{1m}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} & \cdots & \mathbf{A_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A_{m1}} & \mathbf{A_{m2}} & \cdots & \mathbf{A_{mm}} \end{bmatrix} \in \mathbb{R}^{\sum_{i=1}^{m}(1+d_i)\times\sum_{i=1}^{m}(1+d_i)},$

where its elements are $\mathbf{A_{ii}} = \begin{bmatrix} (1+r_i)\lambda_{ii} & 0 & 0 & \cdots & 0 & e_i\varepsilon_i\lambda_{ii} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{(1+d_i)\times(1+d_i)}$

and $\mathbf{A_{ij}} = \begin{bmatrix} (1+r_i)\lambda_{ij} & 0 & 0 & \cdots & 0 & e_i\varepsilon_j\lambda_{ij} \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{(1+d_i)\times(1+d_j)}.$

The matrix $$\mathbf{B}=\begin{bmatrix}\mathbf{B_{11}} & \mathbf{B_{12}} & \cdots & \mathbf{B_{1m}}\\ \mathbf{B_{21}} & \mathbf{B_{22}} & \cdots & \mathbf{B_{2m}}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{B_{m1}} & \mathbf{B_{m2}} & \cdots & \mathbf{B_{mm}}\end{bmatrix}\in\mathbb{R}^{\sum_{i=1}^{m}(1+d_i)\times\sum_{i=1}^{m}(3+d_i)},$$

where its elements are

$$\mathbf{B_{ii}}=\begin{bmatrix}-1 & 0 & \cdots & 0 & w_i & 1-w_i\\ 0 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0\end{bmatrix}\in\mathbb{R}^{(1+d_i)\times(3+d_i)} \text{ and } \mathbf{B_{ij}}=\mathbb{O}\in\mathbb{R}^{(1+d_i)\times(3+d_j)} \text{ for } i\neq j.$$

Finally, for the system (S1), we define

$$\mathbf{C}=\begin{bmatrix}\mathbf{C_1} & \mathbf{C_2} & \cdots & \mathbf{C_m}\end{bmatrix}\in\mathbb{R}^{m\times\sum_{i=1}^{m}(1+d_i)},$$

where its elements are $$\mathbf{C_i}=\begin{bmatrix}-\varepsilon_{i1}\lambda_{i1} & 0 & 0 & \cdots & 0 & \varepsilon_{i1}\lambda_{i1}\\ -\varepsilon_{i2}\lambda_{i2} & 0 & 0 & \cdots & 0 & \varepsilon_{i2}\lambda_{i2}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -\varepsilon_{im}\lambda_{im} & 0 & 0 & \cdots & 0 & \varepsilon_{im}\lambda_{im}\end{bmatrix}\in\mathbb{R}^{m\times(1+d_i)},$$

and $$\mathbf{D}=\begin{bmatrix}\mathbf{D_1} & \mathbf{D_2} & \cdots & \mathbf{D_m}\end{bmatrix}\in\mathbb{R}^{m\times\sum_{i=1}^{m}(3+d_i)},$$

where its elements are $$\mathbf{D_i}=\begin{bmatrix}0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & \cdots & 0 & w_i & 1-w_i\\ 0 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0\end{bmatrix}\in\mathbb{R}^{m\times\sum_{i=1}^{m}(3+d_i)}.$$

Thus, the complex input-output systems (S1) and (S2) can be combined and expressed as a linear generalized difference system of type (1.1), i.e.

$$\begin{cases}\mathbf{E}\underline{S}_k=\mathbf{A}\underline{S}_{k-1}+\mathbf{B}\underline{u}_k\\ \underline{P}_k=\mathbf{C}\underline{S}_k+\mathbf{D}\underline{u}_k\end{cases} \tag{3.3}$$

It is clear that system (3.3) is a generalized difference system of first order, since the (super) matrix $\mathbf{E}$ holds. Following now, the classical theory of difference equations, see for instance Bellmann and Cooke (1963), the analytical solution to equation (3.3) is given by:

$$\begin{aligned} \underline{S}_k &= \mathbf{E}^{-1}\mathbf{A}\underline{S}_{k-1} + \mathbf{E}^{-1}\mathbf{B}\underline{u}_k \\ &\Rightarrow \underline{S}_k = \left(\mathbf{E}^{-1}\mathbf{A}\right)^k \underline{S}_o + \sum_{j=0}^{k-1}\left(\mathbf{E}^{-1}\mathbf{B}\right)^{k-1-j}\underline{u}_j \end{aligned}, \tag{3.4}$$

and
$$\underline{P}_k = \mathbf{C}\left[\left(\mathbf{E}^{-1}\mathbf{A}\right)^k \underline{S}_o + \sum_{j=0}^{k-1}\left(\mathbf{E}^{-1}\mathbf{B}\right)^{k-1-j}\underline{u}_j\right] + \mathbf{D}\underline{u}_k \tag{3.5}$$

However, the matrix $\mathbf{E}$ can easily be singular (or $\det\mathbf{E} \to 0$) e.g. assuming that some of the different insurance products do not accumulate a surplus due to a forced regulatory constraint i.e.

$$\begin{aligned} 0 = &\left(1+r_i\right)\sum_{j=1}^{m}\lambda_{ij}S_{j,k-1} \\ &+ e_i\sum_{j=1}^{i-1}\varepsilon_j\lambda_{ij}\left(0 - S_{j,k-d_j-1}\right) - e_i\varepsilon_i\lambda_{ii}S_{i,k-d_i-1} + e_i\sum_{j=i+1}^{m}\varepsilon_j\lambda_{ij}\left(0 - S_{j,k-d_j-1}\right) \\ &- C_{i,k} + w_iC_{i,k-d_1-1} + \left(1-w_i\right)C_{i,k-d_i-2} \end{aligned}$$

or equivalently,

$$0 = \left(1+r_i\right)\sum_{j=1}^{m}\lambda_{ij}S_{j,k-1} + e_i\sum_{j=1}^{m}\varepsilon_j\lambda_{ij}S_{j,k-d_j-1} - C_{i,k} + w_iC_{i,k-d_i-1} + \left(1-w_i\right)C_{i,k-d_i-2}.$$

If we consider the strategy above, we manipulate our system as follows

$$\mathbf{E_{ii}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{(1+d_i)\times(1+d_i)},\ \mathbf{E_{ij}} = \mathbb{O} \in \mathbb{R}^{(1+d_i)\times(1+d_j)}$$

and, obviously the $\det\mathbf{E} = 0$, so the system (3.3) becomes a descriptor.

Now, by considering the regular matrix pencil approach, (see the **Appendix**) we decompose system (3.3) in two subsystems, whose solutions are provided below

$$\underline{S}_k = Q_{n\times p}\left[A_p^k\underline{\psi}_{p,o} + \sum_{j=0}^{k-1}A_p^{k-1-j}\left[PB\right]_{p\times n}\underline{u}_j\right] - Q_{n\times q}\left[\sum_{j=0}^{q-1}H_q^j\left[PB\right]_{q\times n}\underline{u}_{k+j}\right] \tag{3.6}$$

where the consistent initial condition satisfies the following important expression

$$\underline{S}_o \in C_o = \left\{ \underline{S}_o \in \mathbb{F} : \underline{S}_o = Q_{n\times p}\underline{\psi}_p - Q_{n\times q}\left[\sum_{j=0}^{q-1} H_q^j \left[PB\right]_{q\times n} \underline{u}_j\right]\right\},$$

$$\text{and } \underline{\psi}_o = Q^{-1}\underline{S}_o = \begin{bmatrix} \underline{\psi}_{p,o} \\ \underline{\psi}_{q,o} \end{bmatrix}.$$

Moreover,

$$\underline{P}_k = CQ_{n\times p}\left[A_p^k \underline{\psi}_{p,o} + \sum_{j=0}^{k-1} A_p^{k-1-j}\left[PB\right]_{p\times n} \underline{u}_j \right] + D\underline{u}_k - CQ_{n\times q}\left[\sum_{j=0}^{q-1} H_q^j \left[PB\right]_{q\times n} \underline{u}_{k+j}\right] \qquad (3.7)$$

Closing this section, we should stress that the non-causality occurs in many physical phenomena and certainly non-causal systems are by no means useless. For instance, the property of causality in image-processing applications is not of fundamental importance, even though the time is not an independent variable. However, even if the time is an independent variable, non-causal systems play a very important role. For instance, one can consider several cases of data processing that have been recorded, such as the speech, the meteorological data, demo-graphic data, stock market fluctuations etc, where their collection is not constrained causally.

The last decades, many researchers have investigated causal relationships among multi-dimensional economic variables. We should notice that the causality in economic systems is defined in Granger's (1969) sense. Characteristically, one of the most popular example for such analyses is the money-income relationship, see for instance Sims (1972), Barth and Bennett (1974), Williams, Gogdhart and Gowland (1976), Ciccolo (1978), Feige and Pearce (1979), Hsiao (1979, 1981) etc. However, it should be mentioned that whenever the money's models are not caused by income in Granger's sense, their forecast ability cannot be improved by using only the information in past income data (this is an essential definition of Granger causality), see Lütkepohl (1982). Consequently, it is not easy to conclude that in any economic system that obviously consists of many other variables, the variation in income will not have an impact on the money variable, unless assumed that not all other variables influence this relationship. This remark is a straightforward consequence of the conceptualization that a low dimensional sub-process contains little information about the structure of a higher dimensional system.

4. A numerical application for the special case of two insurance products

In this numerical application, we consider a simple situation with two insurance products, $m = 2$, with $d_1 = 2$ and $d_2 = 3$ years delay, respectively. Before we go further, it is important to determine the values of the variables which are taken into consideration in section 2.

- Firstly, we assume that the *expense* factor for the first insurance product is $e_1 = 80\%$ and for the second $e_2 = 90\%$.
- The *annual rate of investment returns* for both the first and the second insurance product is the same, i.e. $r_1 = r_2 = 4\%$.
- Moreover, we suppose the *interaction factors* $\lambda_{11} = 90\%$, $\lambda_{12} = 10\%$ and $\lambda_{22} = 95\%$, $\lambda_{21} = 5\%$. That means a greater proportion of accumulated surplus of the first (more profitable product) is transferred to the second.
- In order to obtain a faster response, we assume that the *profit sharing factor* (feedback factor) for the first product is $\varepsilon_1 = 0.3$, and for the second is almost the same, $\varepsilon_2 = 0.35$.
- the weighted factor for the average claims (over the two recent years) both for the first and the second insurance product is equal to 1/2.

In this numerical application, we are going to investigate the special case, where the *second* product accumulates zero surpluses. As already mentioned, this strategy can be realistic, since the product managers of the insurance company may hope a further success and development in the other, most profitable, product. Consequently, mathematically speaking, the expressions are

$$\underline{S} = \begin{bmatrix} S_{1,k} \\ S_{1,k-1} \\ S_{1,k-2} \\ S_{2,k} \\ S_{2,k-1} \\ S_{2,k-2} \\ S_{2,k-3} \end{bmatrix} \in \mathbb{R}^7 \ , \ \underline{u} = \begin{bmatrix} C_{1,k} \\ C_{1,k-1} \\ C_{1,k-2} \\ C_{1,k-3} \\ C_{1,k-4} \\ C_{2,k} \\ C_{2,k-1} \\ C_{2,k-2} \\ C_{2,k-3} \\ C_{2,k-4} \\ C_{2,k-5} \end{bmatrix} \in \mathbb{R}^{11}$$

and

$$\underline{P} = \begin{bmatrix} P_{1,k} \\ P_{2,k} \end{bmatrix} \in \mathbb{R}^2 .$$

According to S1 and S2, five super-matrices, **C**, **D** and **A, B**, **E** are respectively introduced. We start with (S2):

$$\mathbf{E} = \begin{bmatrix} \mathbf{E_{11}} & \mathbf{E_{11}} \\ \mathbf{E_{21}} & \mathbf{E_{22}} \end{bmatrix} = \begin{bmatrix} 1+e_1\varepsilon_1\lambda_{11} & 0 & 0 & e_1\varepsilon_2\lambda_{12} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{7\times 7},$$

the matrix $$\mathbf{A} = \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{bmatrix} = \begin{bmatrix} (1+r_1)\lambda_{11} & 0 & e_1\varepsilon_1\lambda_{11} & (1+r_1)\lambda_{12} & 0 & 0 & e_1\varepsilon_2\lambda_{12} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ (1+r_2)\lambda_{21} & 0 & e_2\varepsilon_1\lambda_{21} & (1+r_2)\lambda_{22} & 0 & 0 & e_2\varepsilon_2\lambda_{22} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{7\times 7},$$

and the matrix

$$\mathbf{B}=\begin{bmatrix}\mathbf{B}_{11} & \mathbf{B}_{12}\\ \mathbf{B}_{21} & \mathbf{B}_{22}\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0 & w_1 & 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & w_2 & 1-w_2\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\in\mathbb{R}^{7\times 11}.$$

Finally, for the system (S1), we obtain

$$\mathbf{C}=\begin{bmatrix}\mathbf{C}_1 & \mathbf{C}_2\end{bmatrix}=\begin{bmatrix}-\varepsilon_1\lambda_{11} & 0 & \varepsilon_1\lambda_{11} & -\varepsilon_2\lambda_{21} & 0 & 0 & \varepsilon_2\lambda_{21}\\ -\varepsilon_1\lambda_{12} & 0 & \varepsilon_1\lambda_{12} & -\varepsilon_2\lambda_{22} & 0 & 0 & \varepsilon_2\lambda_{22}\end{bmatrix}\in\mathbb{R}^{2\times 7},$$

and

$$\mathbf{D}=\begin{bmatrix}\mathbf{D}_1 & \mathbf{D}_2\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & w_1 & 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_2 & 1-w_2\end{bmatrix}\in\mathbb{R}^{2\times 11}$$

Now, in order to consider the regular matrix pencil approach, see for further details the **Appendix**, and to decompose system (1.1) into two subsystems, we should firstly find the appropriate matrices $P,Q\in\mathbb{R}^{7\times 7}$.

$$\mathbf{E}=\begin{bmatrix}1.216 & 0 & 0 & 0.252 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix},\quad \mathbf{A}=\begin{bmatrix}0.936 & 0 & 0.216 & 0.104 & 0 & 0 & 0.028\\ 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0.052 & 0 & 0.0135 & 0.988 & 0 & 0 & 0.2992\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix},$$

In this particular numerical application, if we obtain the inverse matrices

$$P=\begin{bmatrix}-0.0399 & -0.1683 & -0.0003 & -0.8706 & 0.8123 & 0 & -0.0079\\ -0.1613 & 0.0032 & -0.0456 & 0.0016 & -0.0154 & 0 & -1.0121\\ 0 & -0.7285 & 0.0437 & 0 & -0.1509 & -0.6666 & -0.0019\\ 0 & -0.6450 & -0.2139 & 0 & -0.1336 & 0.7211 & 0.0096\\ 0 & -0.1088 & 0.9748 & 0 & -0.0225 & 0.1881 & -0.0439\\ -0.8957 & 0.0124 & 0.0064 & -0.0185 & -0.0599 & 0 & 0.1433\\ 0 & 0 & 0 & 1.0222 & 0 & 0 & 0\end{bmatrix}$$

and

$$Q = \begin{bmatrix} -0.0693 & -0.1883 & 0 & 0 & 0 & -0.8912 & -0.2074 \\ -0.2437 & 0.0033 & -0.7285 & -0.6450 & -0.1088 & 0.0102 & 0 \\ -0.0002 & -0.0433 & 0.0437 & -0.2139 & 0.9748 & 0.0078 & 0 \\ 0.0053 & 0.3016 & 0 & 0 & 0 & -0.0057 & 0 \\ 1.1763 & -0.0162 & -0.1509 & -0.1336 & -0.0225 & -0.0495 & 1.0010 \\ 0 & 0 & -0.6666 & 0.7211 & 0.1881 & 0 & 0 \\ -0.0054 & -0.9613 & -0.0019 & 0.0096 & -0.0439 & 0.1733 & 0 \end{bmatrix}$$

we succeed in constructing the Weierstrass canonical form, where

$$PEQ = E_w = \begin{bmatrix} I_6 & \underline{0} \\ \underline{0}^t & 0 \end{bmatrix} \quad \text{and} \quad PAQ = A_w = \begin{bmatrix} A_6 & \underline{0} \\ \underline{0}^t & 1 \end{bmatrix} \tag{4.1}$$

where $$A_6 = \begin{bmatrix} 0.0186 & 0.2840 & 0.0051 & -0.0036 & -0.0098 & 0.1785 \\ 0.0212 & 0.0238 & 0.7065 & -0.6931 & -0.2192 & 0.1303 \\ -0.7451 & 0.1027 & 0.0701 & 0.0594 & 0.0099 & 0.6836 \\ 0.9445 & 0.0687 & 0.0405 & 0.0485 & 0.0088 & 0.5377 \\ -0.0088 & 0.0139 & -0.7093 & -0.6857 & -0.1186 & 0.0978 \\ 0.0550 & 0.1419 & -0.1087 & 0.1403 & -0.1612 & 0.7311 \end{bmatrix}.$$

Furthermore, we need to calculate the matrix $PB = \begin{bmatrix} PB_{6,7} \\ PB_{1,7} \end{bmatrix}$. Thus,

$$\mathbf{B} = \begin{bmatrix} -1 & 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{7\times 11},$$

and $$\mathbf{PB} = \begin{bmatrix} 0.0399 & 0 & 0 & -0.0199 & -0.0199 & 0.8706 & 0 & 0 & 0 & -0.4353 & -0.4353 \\ 0.1613 & 0 & 0 & -0.0806 & -0.0806 & -0.0016 & 0 & 0 & 0 & 0.0008 & 0.0008 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8957 & 0 & 0 & -0.4478 & -0.4478 & 0.0185 & 0 & 0 & 0 & -0.0092 & -0.0092 \\ 0 & 0 & 0 & 0 & 0 & -1.0222 & 0 & 0 & 0 & 0.5111 & 0.5111 \end{bmatrix}.$$

Moreover,

$$\mathbf{C} = \begin{bmatrix} -0.27 & 0 & 0.27 & -0.3325 & 0 & 0 & 0.3325 \\ -0.03 & 0 & 0.03 & -0.0175 & 0 & 0 & 0.0175 \end{bmatrix} \in \mathbb{R}^{2\times 7},$$

and

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix} \in \mathbb{R}^{2\times 11}$$

Now, we are going to examine the behaviour of the system with respect to the *spike signal*. This kind of input corresponds to the appearance of an unexpected claim into the system. Moreover, we suppose that a spike signal appears as the input of the first subsystem while the second subsystem has a zero input, i.e.

$$C_{1,k} = \begin{cases} 1, & \text{for } k = 0 \\ 0, & \text{for } k = 1, 2, \ldots \end{cases},$$

and $C_{2,k} = 0$ for $k = 0, 1, 2, \ldots$.

a) The spike signal appears only to the 1st product.

The input vectors are the following

$$\underline{u}_o = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $\underline{u}_k = 0$ for $k = 5, 6, \ldots$.

So, according to (3.6) and (3.7) expressions we obtain (4.2) and (4.3), respectively. Moreover, we assume that $\underline{\psi}_o = Q^{-1}\underline{S}_o = \underline{0} \Rightarrow \underline{S}_o = \underline{0}$.

Consequently,

$$\underline{S}_k = Q_{7\times6}\Big[A_6^{k-1}\left[PB\right]_{6\times11}\underline{u}_o + A_6^{k-2}\left[PB\right]_{6\times11}\underline{u}_1 + A_6^{k-3}\left[PB\right]_{6\times11}\underline{u}_2 + A_6^{k-4}\left[PB\right]_{6\times11}\underline{u}_3 + A_6^{k-5}\left[PB\right]_{6\times11}\underline{u}_4\Big] - Q_{7\times1}\left[PB\right]_{1\times11}\underline{u}_k \tag{4.2}$$

and

$$\underline{P}_k = CQ_{7\times6}\Big[A_6^{k-1}\left[PB\right]_{6\times11}\underline{u}_o + A_6^{k-2}\left[PB\right]_{6\times11}\underline{u}_1 + A_6^{k-3}\left[PB\right]_{6\times11}\underline{u}_2 + A_6^{k-4}\left[PB\right]_{6\times11}\underline{u}_3 + A_6^{k-5}\left[PB\right]_{6\times11}\underline{u}_4\Big] + \Big[D - CQ_{7\times1}\left[PB\right]_{1\times11}\Big]\underline{u}_k \tag{4.3}$$

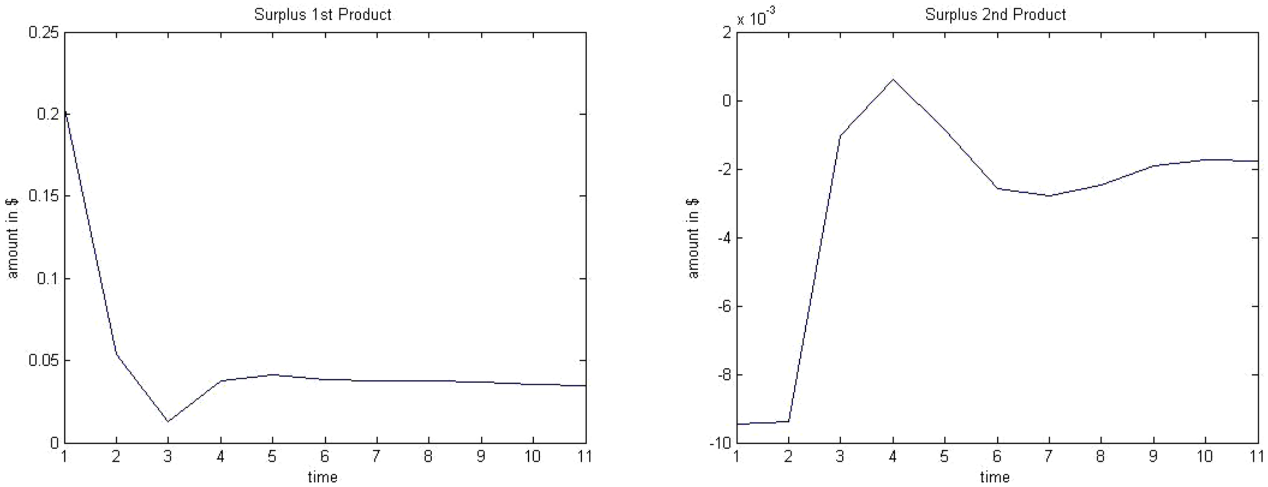

figure 1 (a), (b): The surplus for the 1st and the 2nd insurance product, respectively.

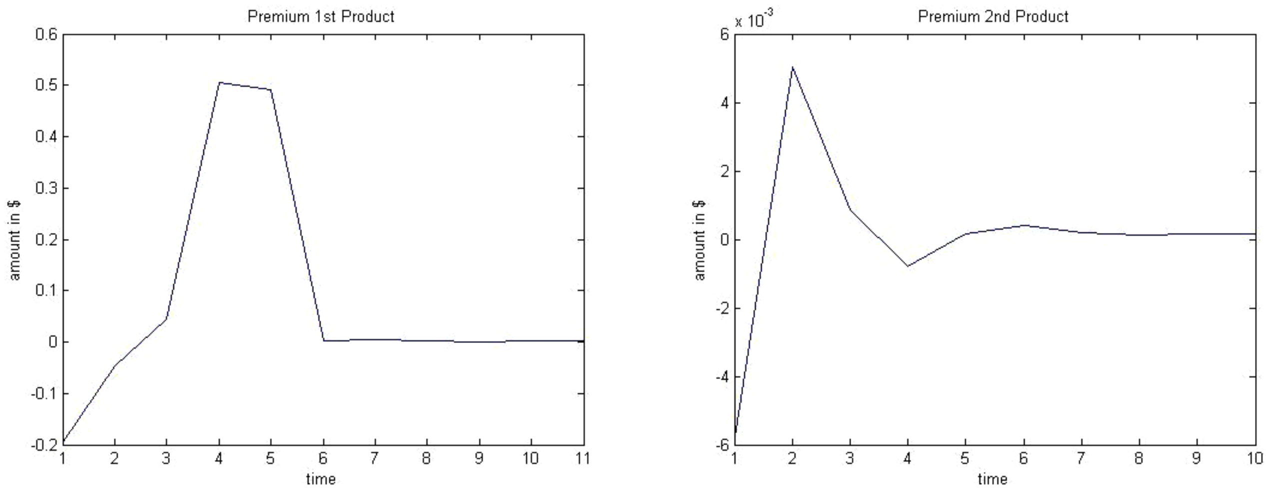

figure 2 (a), (b): The premium for the 1st and the 2nd insurance product, respectively.

b) The spike signal appears only to the 2nd product.

The input vectors are the following

$$\underline{u}_o = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \underline{u}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and $\underline{u}_k = 0$ for $k = 6, 7, \ldots$.

Again, according to (3.6) and (3.7) we obtain (4.4) and (4.5), respe-ctively. Note that we also assume that $\underline{\psi}_o = Q^{-1}\underline{S}_o = \underline{0} \Rightarrow \underline{S}_o = \underline{0}$, So,

$$\begin{aligned} \underline{S}_k = Q_{7\times6}\Big[& A_6^{k-1}[PB]_{6\times11}\underline{u}_o + A_6^{k-2}[PB]_{6\times11}\underline{u}_1 + A_6^{k-3}[PB]_{6\times11}\underline{u}_2 \\ & + A_6^{k-4}[PB]_{6\times11}\underline{u}_3 + A_6^{k-5}[PB]_{6\times11}\underline{u}_4 + A_6^{k-6}[PB]_{6\times11}\underline{u}_5\Big] - Q_{7\times1}[PB]_{1\times11}\underline{u}_k \end{aligned}, \qquad (4.4)$$

and

$$\begin{aligned} \underline{P}_k = CQ_{7\times6}\Big[& A_6^{k-1}[PB]_{6\times11}\underline{u}_o + A_6^{k-2}[PB]_{6\times11}\underline{u}_1 + A_6^{k-3}[PB]_{6\times11}\underline{u}_2 \\ & + A_6^{k-4}[PB]_{6\times11}\underline{u}_3 + A_6^{k-5}[PB]_{6\times11}\underline{u}_4 + A_6^{k-6}[PB]_{6\times11}\underline{u}_5\Big] + \Big[D - CQ_{7\times1}[PB]_{1\times11}\Big]\underline{u}_k \end{aligned}$$

(4.5)

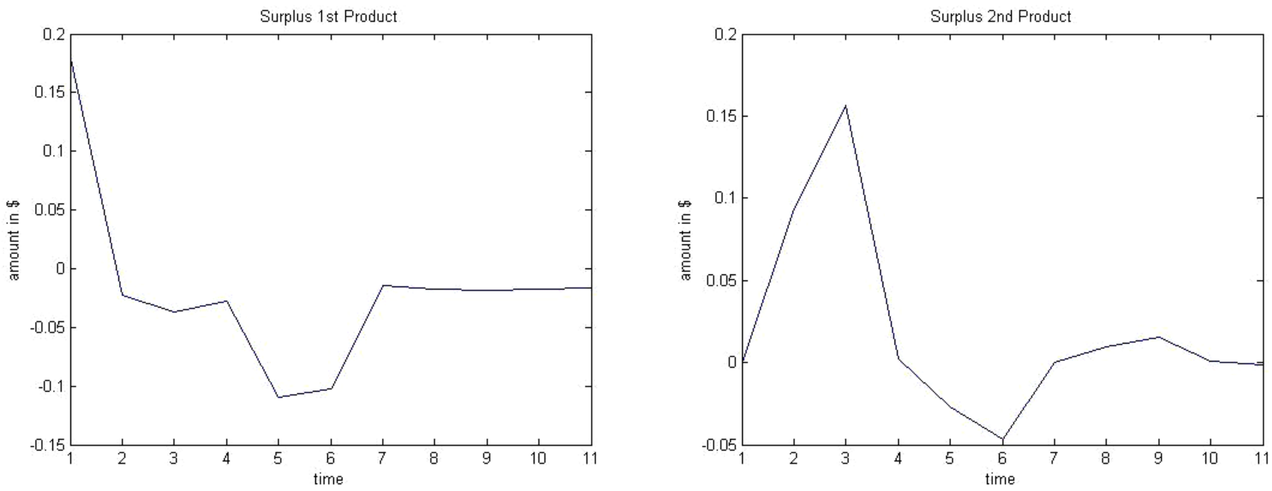

figure 3 (a), (b): The surplus for the 1st and the 2nd insurance product, respectively.

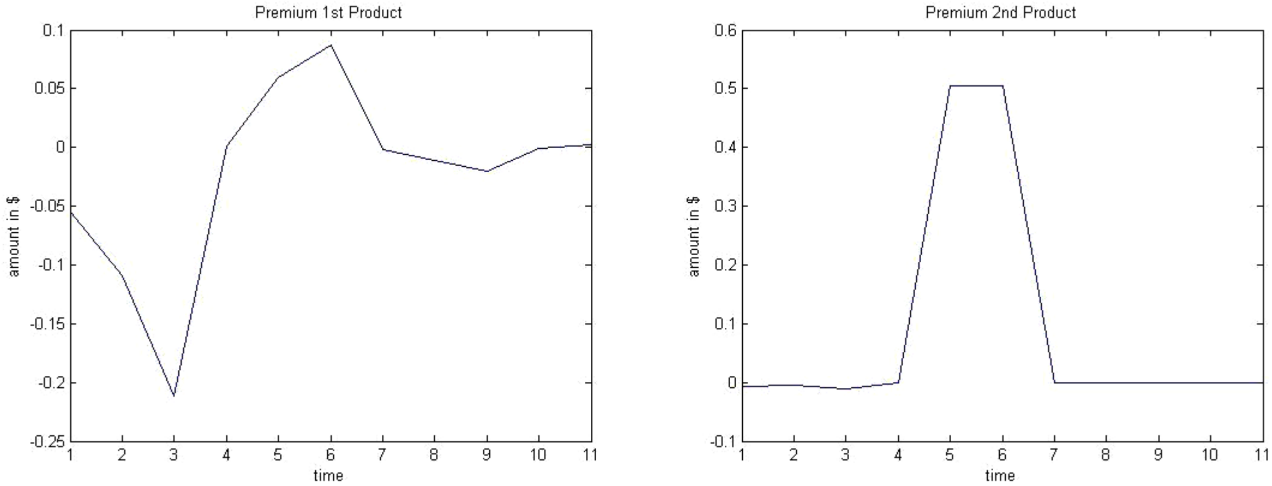

figure 4 (a), (b): The surplus for the 1st and the 2nd insurance product, respectively.

We can extend our investigation for other different signals and obtain insight into the behaviour of the system

.

5. Conclusions

In this paper, we introduce and use matrix pencil theory for the analysis and solution of an insurance system. The use of matrix pencil theory appears to be unavoidable in order to manipulate the singularities of a multiple input-output system. In the numerical application, the diagrams of the surplus and premium response with respect to the spike input signal are quite interesting. On the other hand, we could follow the opposite direction in our analysis. Define the pattern for the surplus or premium response and go back to the optimal choices for the basic controlled parameters, as the loading or interaction factors. The research in this area is being continued.

Appendix

Preliminary Concepts of Matrix Pencil Theory

for Generalized Discrete Time Systems

The solution of generalized difference control systems of type (1.1) is based on matrix pencil theory. The results are obtained using the Weierstrass canonical form's theorem and its strict equivalence invariants of the associated pencil. In the vast financial literature, (see for instance Kendrick (1972), Livesey (1972), Kreijger and Neudecker (1976), Luenberger and Arbel (1977), Campbell (1980, 1982), King and Watson (1998) etc.) such kind of systems may appear in multi-sector economies. One of the most famous generalized economic model is the Leontief input-output dynamic model, see Leontief (1954, 1977, 1986).

Now, we proceed with the respective theory. First of all, $\mathbb{F}$ is a field, while $\mathbb{F}^{m\times n}$ denotes the set of $m\times n$ matrices with elements from $\mathbb{F}$. Given $E,\ \mathrm{A}\in\mathbb{F}^{m\times n}$ and an indeterminate s, the matrix pencil $sE-A$ is called *regular* when $m=n$ and $\det\left(sE-A\right)\neq\mathbb{O}$, where $\mathbb{O}$ is the zero polynomial. In any other case, the pencil is called *singular*. We focus on regular pencils (i.e. square matrices). Let $L^{r}_{n,n}$ be the *set* on $n\times n$ *regular pencils*, i.e.

$$L^{r}_{n,n}\triangleq\left\{sE-A: E, A\in\mathbb{F}^{n\times n} and\ sE-A\ regular\right\} \tag{A1}$$

The pencil $sE-A$ is said to be *strictly equivalent* to the pencil sE_1-A_1 if and only if $P\left(sE-A\right)Q=sE_1-A_1$, where $P\in\mathbb{F}^{m\times m}$, $Q\in\mathbb{F}^{n\times n}$, and $\det P\neq 0$, $\det Q\neq 0$.

Theorem 1 (Gantmacher, 1977) (**Weierstrass canonical form**)

For a regular matrix pencil $sE-A$, there exist non singular $\mathbb{F}^{n\times n}$ matrices P and Q such that.

$$PEQ=E_w=\begin{bmatrix} I_p & O_{p,q}\\ O_{q,p} & H_q\end{bmatrix} \tag{A2}$$

$$PAQ=A_w=\begin{bmatrix} A_p & O_{p,q}\\ O_{q,p} & I_q\end{bmatrix} \tag{A3}$$

where, the H_q is a nilpotent matrix of index q, that means $H_q^q=\mathbb{O}$, see (A4)

$$H_q = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{q\times q} \qquad \text{and} \qquad I_p = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{p\times p} \quad \text{(A4)}$$

Moreover, $A_p \in \mathbb{F}^{p\times p}$ is a full rank matrix, where $p+q=n$.

For further details about matrix pencil theory see Gantmacher (1977), Karcanias (1979), Karcanias and Hayton (1981), Campbell (1980, 1982), Kalogeropoulos (1985), Dai (1989).

Proposition 1 The system $$E\underline{x}_k = A\underline{x}_{k-1} + B\underline{u}_k \quad \text{(A5)}$$

may be rewritten in the following format:

$$\underline{\psi}_{p,k} = A_p \underline{\psi}_{p,k-1} + \left[PB\right]_{p\times n} \underline{u}_k\,, \quad \text{(A6)}$$

and $$H_q \underline{\psi}_{q,k} = \underline{\psi}_{q,k-1} + \left[PB\right]_{q\times n} \underline{u}_k\,, \quad \text{(A7)}$$

where $\underline{\psi} = Q^{-1}\underline{x} = \begin{bmatrix} \underline{\psi}_p \\ \underline{\psi}_q \end{bmatrix}$, and $PB = \begin{bmatrix} PB_{p\times n} \\ PB_{q\times n} \end{bmatrix}$; $PB_{p\times n}$ and $PB_{q\times n}$ are $p\times n$ and $q\times n$ constant matrices, respectively.

Note that for square matrices $E, A \in \mathbb{F}$, the column state vector $\underline{x}_o \in \mathbb{F}^n$ is said to be *consistent initial state* associated with $k=0$ for the system (A5) if it possesses at least one solution.

Proposition 2 The solution of system (A6) is given by expression (A8),

$$\underline{\psi}_{p,k} = A_p^k \underline{\psi}_{p,o} + \sum_{j=0}^{k-1} A_p^{k-1-j} \left[PB\right]_{p\times n} \underline{u}_j\,, \quad \text{(A8)}$$

where $\underline{\psi}_{p,o} \in \mathbb{F}^p$ is the initial condition,

and the solution of system (A7) is given by expression (A9),

$$\underline{\psi}_{q,k} = -\sum_{j=0}^{q-1} H_q^j \left[PB\right]_{q\times n} \underline{u}_{k+j}\,. \quad \text{(A9)}$$

Combining expression (A8), (A9) and $\underline{x} = Q\underline{\psi}$, where $Q = \left[Q_{n\times p} \;\vdots\; Q_{n\times q} \right]$, we derive the solution for system (A5), thus

$$\underline{x}_k = Q\begin{bmatrix}\underline{y}_p \\ \underline{y}_q\end{bmatrix} = \begin{bmatrix}Q_{n\times p} & \vdots & Q_{n\times q}\end{bmatrix} = \begin{bmatrix} A_p^k \underline{\psi}_{p,o} + \sum_{j=0}^{k-1} A_p^{k-1-j}\left[PB\right]_{p\times n}\underline{u}_j \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ -\sum_{j=0}^{q^*-1} H_q^j \left[PB\right]_{q\times n}\underline{u}_{k+j}\end{bmatrix}$$

$$= Q_{n\times p}\left[A_p^k \underline{\psi}_{p,o} + \sum_{j=0}^{k-1} A_p^{k-1-j}\left[PB\right]_{p\times n}\underline{u}_j\right] - Q_{n\times q}\left[\sum_{j=0}^{q-1} H_q^j\left[PB\right]_{q\times n}\underline{u}_{k+j}\right]. \quad \text{(A10)}$$

From expression (A10), it is obvious that, in order to calculate the solution of (A5), we need also the “future” inputs $\underline{u}_j$, for $j = k, k+1, \ldots + k + q - 1$. So we obtain lemma.

Lemma 1 The expression (A10) is the solution of system (A5) if the *consistent initial state* $\underline{x}_o$ *belongs* to the set

$$C_o = \left\{\underline{x}_o \in \mathbb{F} : \underline{x}_o = Q_{n\times p}\underline{\omega}_p - Q_{n\times q}\left[\sum_{j=0}^{q-1} H_q^j\left[PB\right]_{q\times n}\underline{u}_j\right]\right\} \quad \text{(A11)}$$

Proof: For the initial condition $\underline{x}_o = Q\underline{\psi}_o$, we define $\underline{\psi}_o = \begin{bmatrix}\underline{\psi}_{p,o} \\ \underline{\psi}_{q,o}\end{bmatrix}$. Considering now the solutions (A8) and (A9), we obtain that vector $\underline{\psi}_{p,o}$ can be any arbitrary vector $\underline{\omega}_p \in \mathbb{F}^p$, since the input $\underline{u}_o$ is already known. However, for the vector $\underline{\psi}_{q,o}$, we should consider the expression $-\sum_{j=0}^{q-1} H_q^j\left[PB\right]_{q\times n}\underline{u}_j$. (Note, this expression takes into account the future inputs for $j = 0,1,\ldots,q-1$.) Consequently, the set of the solution is

$$C_o = \left\{\underline{x}_o \in \mathbb{F} : \underline{x}_o = Q\begin{bmatrix}\underline{\omega}_p \\ -\sum_{j=0}^{q^*-1} H_q^j\left[PB\right]_{q\times n}\underline{u}_j\end{bmatrix}\right\}$$
$$= \left\{\underline{x}_o \in \mathbb{F} : \underline{x}_o = Q_{n\times p}\underline{\omega}_p - Q_{n\times q}\left[\sum_{j=0}^{q-1} H_q^j\left[PB\right]_{q\times n}\underline{u}_j\right]\right\}. \quad \square$$

Furthermore, by combining the result above with the other system

$$\underline{y}_k = C\underline{x}_k + D\underline{u}_k, \quad \text{(A12)}$$

we obtain the solution of the following format

$$\underline{y}_k = CQ_{n\times p}\left[A_p^k \underline{\psi}_{p,o} + \sum_{j=0}^{k-1} A_p^{k-1-j}\left[PB\right]_{p\times n}\underline{u}_j\right] + D\underline{u}_k - CQ_{n\times q}\left[\sum_{j=0}^{q-1} H_q^j\left[PB\right]_{q\times n}\underline{u}_{k+j}\right]. \quad \text{(A13)}$$

Remark 1. The system (1.1) (or A5-A12) is *not causal*, since the state $\underline{x}_k$, and the output vector $\underline{y}_k$ for any $k = 0,1,2,\ldots$ are not fully determined by the given initial condition $\underline{x}_o$ and the past values of input variables $\underline{u}_j$ for $j = 0,1,\ldots,k$, as we need to consider the future (still unknown) inputs $\underline{u}_j$ parameters for $j = k, k+1,\ldots + k + q - 1$. In order to obtain the causality, the following two important necessary and sufficient conditions should hold. Their proofs are based on the work of Grispos and Kalogeropoulos (1998).

Proposition 3 The system (A5) has causal solution, over an interval of arbitrary length, for any consistent initial condition $\underline{x}_o$ and for any input sequence $\underline{u}_j$ for $j = 0,1,\ldots,k$ if and only if $H_q^j\left[PB\right]_{q\times n} = \mathbb{O}_{q\times n}$ for any $j = 0,1,\ldots,q-1$.

Proposition 4 The system (A11) has causal solution, over an interval of arbitrary length, for any consistent initial condition $\underline{x}_o$ and for any input sequence $\underline{u}_j$ for $j = 0,1,\ldots,k$ if and only if the matrix

$$\left[Q_{n\times q}H_q\left[PB\right]_{q\times n} \quad Q_{n\times q}H_q^2\left[PB\right]_{q\times n} \quad \cdots \quad Q_{n\times q}H_q^{q-1}\left[PB\right]_{q\times n}\right]$$

is the right annihilation of $m\times n$ constant matrix C.

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