Reinsurance Contract Valuation When the Liabilities are of Fractional Brownian Motion type

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Abstract

In this paper we study the valuation of reinsurance contracts for liabilities exhibiting long range dependence modelled by fractional Brownian motion. We examine both aggregated excess and proportional reinsurance contracts and we model the contract as an Asian type option. Specifically using fractional Itô calculus and ideas from option pricing theory we derive a partial differential equation the solution of which provides the value of the reinsurance policy. An analytical solution is found for this equation and the results obtained by this approach are compared with the results obtained by Monte-Carlo simulation.

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1 Introduction

Reinsurance is, broadly speaking, the insurance of insurance companies. If an individual risk is too big for an insurance company or the loss potential of its entire portfolio is too heavy - then the insurance company either decides, or is forced to by legal restrictions, to buy reinsurance protection. Often the reinsurance company does the same, i.e. it retrocedes part of the risk or parts of its portfolio to a third company. By passing on parts of risks, large risks particularly are finally split up into a number of portions placed with many different carriers.

Some of the most common types of reinsurance are proportional and aggregated excess reinsurance. A proportional reinsurance treaty means that the ceding company cedes to the reinsurer a fixed percentage of each risk of the covered portfolio, the reinsurer in return pays the same percentage of each claim and receives the same percentage of the underlying gross premiums. In an aggregated excess treaty the reinsurer pays the total of all claims exceeding the retention stated in the contract. For more on reinsurance one can see for example [4] and [25].

Our aim is to study the pricing of reinsurance policies for liabilities of an insurance business exhibiting long range dependence. As a suitable model for the liabilities we choose a fractional Brownian motion with drift and Hurst exponent $H > 1/2$. Fractional Brownian motion has been used to model a wide variety of stochastic data arising in engineering and physics (network traffic data, solar activity, levels of a river, turbulence in an incompressible fluid flow see e.g. [23]) as well as in financial mathematics (log returns of the stock prices, see for example [23],[21], [26], [8], the electricity price in a liberated electricity market, see for example [24], foreign exchange rates, see for example [13] and weather derivatives [3] and references therein). Furthermore, fractional Brownian motion (as a special case of self similar process) has been used recently to model the claims an insurance business may face (see eg. [16],[17],[5], [9], [10] etc).

In this paper we study the valuation of a reinsurance policy for both aggregated excess and proportional reinsurance, by modeling the contract as an Asian type option with underlying the total claim amount by the end of the contract. Using fractional Itô calculus we develop a partial differential equation, similar to the Black-Scholes equation, that provides the value of the contract. This equation is solved analytically using the technique of similarity solutions thus providing an analytic formula, in terms of special functions, for the valuation of the reinsurance contract in terms of the various parameters of the model. This analytic formula can be estimated very easily, without having to resort to time-consuming Monte-Carlo techniques. The results of the paper elucidate, among other things, the effect of long range dependence on the pricing of reinsurance contracts.

2 The model

2.1 A model for the claims process

Let us assume that the claims process is of the form

$$\begin{aligned} dC_t &= b_t dt + \sigma_t dB_t^H \\ C_0 &= c \end{aligned}$$

where with C_t we denote the claims at time t and with b_t we denote the expectations of the claims which may model seasonalities. The term B_t^H is a fractional Brownian motion with Hurst exponent H which is used to model the long range dependence often present in insurance claims. Here we assume that $H \in \left(\frac{1}{2}, 1\right)$. We furthermore assume that $\sigma_t \in L_\phi^2, t\sigma_t \in L_\phi^2$. For the definition of these spaces and for more on fractional Brownian motion one can see the appendix.

2.2 Aggregated excess reinsurance

Consider first the case of aggregated excess reinsurance, i.e consider a reinsurance policy according to which if the total claim amount until time T is less than K, where with K we denote a certain percent of the underlying premium volume, the reinsurance company pays nothing whereas if the total claim amount is higher that K it pays the excess of K. This reinsurance scheme can be thought of as an Asian type contigent claim with underlying the total claims until time T. The payoff of this option is given by

$$\max(0, I_T - K) = (I_T - K)^+ = \left(C_0 + \int_0^T C_t dt - K \right)^+ .$$

where

$$I_T := C_0 + \int_0^T C_t dt$$

or in differential form, using the model for the claims process proposed in the previous subsection

$$\begin{aligned} dI_t &= C_t = \int_0^t b_s ds + \int_0^t \sigma_s dB_s^H + C_0 \\ I_0 &= C_0 = c \end{aligned}$$

From an actuarial perspective the fair price for the reinsurance policy will be equal to the expected value of the discounted payoff, i.e:

$$\upsilon(T, I_0) = E\left[e^{-\delta T}\left(I_T - K\right)^+\right].$$

2.3 Proportional reinsurance

In this type of reinsurance the company pays a percentage equal to α of the total claims up to time T. The payoff is given by

$$\alpha I_T = \alpha \int_0^T C_t dt.$$

The proportional reinsurance contract is an Asian type option with final payoff αI.

3 A PDE for the reinsurance option price

At this section using techniques from fractional stochastic calculus we derive a deterministic pde for the value of the reinsurance policy. The underlying can be written in more convenient form according to the next lemma.

Lemma 3.1 *The underlying I_T can be expressed in terms of a fractional Itô process as follows*

$$I_T = I_0 + \int_0^T \Theta_t dt + \int_0^T \Sigma(T, s) dB_s^H + C_0 T.$$

where $\Theta_t = \int_0^t b_s ds$ *and* $\Sigma(T, s) = (T - s)\sigma_s$

Proof: Integrating with respect to time we obtain

$$\begin{aligned} I_T &= I_0 + \int_0^T \int_0^t b_s ds dt + \int_0^T \int_0^t \sigma_s dB_s^H dt + C_0 T \\ I_0 &= C_0 = c \end{aligned}$$

We now simplify the third term on the rhs. According to the stochastic Fubini theorem, for any $\psi(s,t)\,\epsilon\, L_2(\phi(\Re))$, we have that

$$\begin{aligned}\int_0^T\int_0^t \sigma_s\psi(s,t)dB_s^H dt &= \int_0^T\int_0^t \sigma_s\psi(s,t)1_{[0,t]}(s)\diamond dW_s^H dsdt\\ &= \int_0^T\int_0^t 1_{[0,t]}(s)\psi(s,t)dt\sigma_s\diamond dW_s^H ds\\ &= \int_0^T\left[\int_s^T \psi(s,t)dt\right]\sigma_s dB_s^H\end{aligned}$$

where $\diamond$ denotes the Wick product, (see appendix). Choosing $\psi(s,t)=1$

$$\int_0^T\int_0^t \sigma_s\psi(s,t)dB_s^H dt = \int_0^T (T-s)\,\sigma_s dB_s^H$$

Thus if we denote as $\Theta_t=\int_0^t b_s ds$ and $\Sigma(T,s)=(T-s)\sigma_s$ we find that

$$I_T = I_0 + \int_0^T \Theta_t dt + \int_0^T \Sigma(T,s)dB_s^H + C_0 T.$$

and the lemma is proved. □

The price of the reinsurance policy is given in terms of the expectation of a functional of fractional Brownian motion. As was shown in [2] and [3] in the context of weather derivatives such expectations may often be represented as solutions of deterministic pde's. This approach may be considered as a generalization of the celebrated Feynmann-Kac to non- Markovian processes. Thus extending the arguments of [2] and of [3] we may show that the price of the reinsurance policy can be given by the solution of a partial differential equation as the following proposition shows.

Proposition 3.1 *The actuarially fair price for the aggregated excess reinsurance policy can be given by the formula* $v(T,I_0)=w(T,C_0,I_0)$ *where* w *is the classical solution of the partial differential equation*

$$\begin{aligned}&-\delta w - w_1 + b_s w_2 + C_s w_3 + \sigma_s A_s w_{22} + (sA_s\sigma_s - \sigma_s B_s)w_{23} = 0 \quad (1)\\ &w = w(T-t,C,I)\end{aligned}$$

with initial condition $w(0, C_0, I) = (I - K)^+$.

Proof: Consider a function of three variables $f(t, x, y)$, $x = C_t$, $y = I_t$

$$f(T, C_T, I_T) = f(T, \int_0^T b_t dt + \int_0^T \sigma_t dB_t^H + C_0, I_0 + \int_0^T \Theta_t dt + T \int_0^T \sigma_t dB_t^H - \int_0^T t\sigma_t dB_t^H + C_0 T)$$

$$= f(T, \int_0^T b_t dt + X_T + C_0, I_0 + \int_0^T \Theta_t dt + T X_T - Y_T + C_0 T)$$

where

$$X(T) := \int_0^T \sigma_s dB_s^H, \quad Y(T) := \int_0^T s\sigma_s dB_s^H.$$

We now apply Itô 's lemma (see proposition A.2) on

$$f(t, X_t, Y_t) = e^{-\delta t} w \left(T - t, \int_0^t b_s ds + X_t + C_0, I_0 + \int_0^t \Theta_s ds + t X_t - Y_t + C_0 t \right)$$

By standard manipulations we find that

$$\begin{aligned}
f_t &= e^{-\delta t}(-\delta w - w_1 + b_t w_2 + C_t w_3) \\
f_x &= e^{-\delta t}\left(w_2 + t w_3\right) \\
f_y &= -e^{-\delta t} w_3 \\
f_{xx} &= e^{-\delta t}\left(w_{22} + 2t w_{23} + t^2 w_{33}\right) \\
f_{yy} &= e^{-\delta t} w_{33} \\
f_{xy} &= e^{-\delta t}\left(-w_{23} - t w_{33}\right).
\end{aligned}$$

Thus, substituting the above expressions in Itô's lemma we obtain

$$\begin{aligned}
e^{-\delta T} w\left(0, C_T, I_T\right) &= w\left(T, C_{0,} I_0\right) + \int\limits_0^T e^{-\delta s}\left(-\delta w - w_1 + b_s w_2 + C_s w_3\right) ds \\
&+ \int\limits_0^T e^{-\delta s}\left[\left(w_2 + s w_3\right)\sigma(s) - w_3 s \sigma(s)\right] dB_s^H \\
&+ \int\limits_0^T e^{-\delta s}\left[w_{22} + 2 s w_{23} + s^2 w_{33}\right]\sigma_s \left(\int\limits_0^s \phi(s,u)\sigma(u) du\right) ds \\
&+ \int\limits_0^T e^{-\delta s} w_{33} s \sigma(s) \left[\int\limits_0^s \phi(s,u)\sigma(u) u du\right] ds \\
&+ \int\limits_0^T e^{-\delta s}(-w_{23} - s w_{33}) \left(\sigma_s \int\limits_0^s \phi(s,u)\sigma(u) u du + s\sigma_s \int\limits_0^s \phi(s,u)\sigma(u) du\right) ds.
\end{aligned}$$

Let

$$A_s := \int\limits_0^s \phi(s,u)\sigma(u) du \quad B_s := \int\limits_0^s \phi(s,u) u \sigma(u) du.$$

Then

$$\begin{aligned}
e^{-\delta T} w\left(0, C_T, I_T\right) &= w\left(T, C_{0,} I_0\right) + \int\limits_0^T e^{-\delta s}\left(-\delta w - w_1 + b_s w_2 + C_s w_3\right) ds \\
&+ \int\limits_0^T e^{-\delta s}\left[\left(w_2 + s w_3\right)\sigma(s) - w_3 s \sigma(s)\right] dB_s^H \\
&+ \int\limits_0^T e^{-\delta s}\left(\sigma_s A_s w_{22} + (2 s A_s \sigma_s - \sigma_s B_s - s \sigma_s A_s) w_{23}\right) ds
\end{aligned}$$

and note that terms containing w_{33} cancel out. Taking expectations we get

$$\begin{aligned} E\left[e^{-\delta t}w(0,C_T,I_T)\right] &= w\left(T,C_0,I_0\right)+E\left[\int\limits_0^T e^{-\delta s}\left[-\delta w-w_1+b_s w_2+C_s w_3\right]ds\right] \\ &\quad +E\left[\int\limits_0^T e^{-\delta s}\left[\sigma_s A_s w_{22}+(2sA_s\sigma_s-\sigma_s B_s-s\sigma_s A_s)w_{23}\right]ds\right] \end{aligned}$$

We now add equation

$$u(T,C_0,I_0)=E\left[e^{-\delta T}\left(I_T-k\right)^+\right]$$

to the above and we have that

$$\begin{aligned} u(T,C_0,I_0)+E\left[e^{-\delta t}w(0,C_T,I_T)\right] &= w\left(T,C_0,I_0\right)+E\left[e^{-\delta T}\left(I_T-k\right)^+\right] \\ +E\left[\int\limits_0^T e^{-\delta s}\left[-\delta w-w_1+b_s w_2+C_s w_3+\sigma_s A_s w_{22}+(sA_s\sigma_s-\sigma_s B_s)w_{23}\right]ds\right] \end{aligned}$$

Therefore, if we choose $w(T-t,C,I)$ to be the solution of the equation

$$-\delta w-w_1+b_s w_2+C_s w_3+\sigma_s A_s w_{22}+(sA_s\sigma_s-\sigma_s B_s)w_{23}=0$$

with initial condition $w(0,C_0,I)=(I-K)^+$, then $w(T,C_0,I_0)$ will be the price of the reinsurance policy. $\square$

We now provide an analytic formula for the price of the aggregated excess reinsurance contract by solving the partial differential equation (1).

Proposition 3.2 *Assume, without loss of generality that $\sigma(t)=\sigma$, $b(t)=b$ are deterministic constants. The price of the aggregated excess reinsurance contract is given by the formula*

$$\begin{aligned} \bar{u}\left(\tau,C_0,I_0\right) &= e^{-\delta\tau}z(\tau,C_0,I_0)\Phi(k_1)+e^{-\delta\tau}\frac{\sqrt{\tau}}{\sqrt{2\pi}}e^{-\frac{k_1^2}{2}}-e^{-\delta\tau}k\Phi(k_1), \\ k_1 &= \frac{z(\tau,C_0,I_0)-k}{\sqrt{\tau}} \\ \tau &= T-t \end{aligned}$$

where

$$\begin{aligned}
\bar{\tau} &= \sigma^2 T^2 (T^{2H} - t^{2H}) + 2(\frac{\sigma^2 H}{2H+1} - \frac{\sigma^2 (H-0.5)}{2H+2})(T^{2H+2} - t^{2H+2}) \\
&- \frac{4\sigma^2 HT}{2H+1}(T^{2H+1} - t^{2H+1}) \\
z &= \frac{b\tau^2}{2} + \tau C_0 + I_0
\end{aligned}$$

Proof: Let us look for a similarity solution of equation (1). We have to solve the equation

$$-\delta w - w_\tau + b w_x + x w_y + \sigma A_{T-\tau} w_{xx} + \Lambda_{T-\tau} w_{xy} = 0$$

where

$$\Lambda_u = (u A_u \sigma - \sigma B_u).$$

We look for special solutions of the form

$$w(\tau, x, y) = u(\tau, z)$$

where the variable z is defined by

$$z = c_1(\tau) + c_2(\tau)x + c_3(\tau)y$$

where $c_i(\tau)$ are functions to be specified in what follows.

Standard algebraic manipulations give

$$\begin{aligned}
w_\tau &= u_\tau + u_z(c_1' + c_2' x + c_3' y) \\
w_x &= u_z c_2 \\
w_y &= u_z c_3 \\
w_{xx} &= u_{zz} c_2^2 \\
w_{xy} &= c_2 c_3 u_{zz}
\end{aligned}$$

We now substitute this special type of solution into the partial differential equation (1) and we obtain

$$-\delta u - u_\tau - u_z(c_1' + c_2' x + c_3' y) + b c_2 u_z + x c_3 u_z + \sigma A_{T-\tau} c_2^2 u_{zz} + \Lambda_{T-\tau} c_2 c_3 u_{zz} = 0$$

For the similarity solution to exist (i.e. in order to be able to have a solution of the above equation in the form $u(\tau, z)$) we need the consistency condition

$$-c_1' - c_2' x - c_3' y + b c_2 + c_3 x = 0, \forall x, y.$$

Thus we have

$$\begin{aligned} c_1' &= bc_2 \\ c_2' &= c_3 \\ c_3' &= 0 \\ c_1 &= \frac{1}{2}\lambda b\tau^2 + b\mu\tau + \nu \\ c_2 &= \lambda\tau + \mu \\ c_3 &= \lambda \end{aligned}$$

where λ, μ, ν are constants. In order to have

$$w(0, x, y) = u(0, \nu + \mu x + \lambda y)$$

we must have $\mu = 0, \lambda = 1, \nu = 0$, and thus

$$c_1 = 0.5b\tau^2, c_2 = \tau, c_3 = 1.$$

For this choice of $c_i(t)$ the PDE that $u(\tau, z)$ satisfies simplifies to

$$\begin{aligned} -\delta u - u_\tau + u_{zz}\Delta(\tau; T) &= 0 \\ u(0, z) &= (z - k)^+ \end{aligned}$$

where

$$\Delta(\tau; T) = \sigma\tau^2 A_{T-\tau} + \Lambda_{T-\tau}\tau$$

and $z(\tau) = \frac{1}{2}b\tau^2 + \tau x + y$. Now let us define $\bar{u}(t, z)$ by

$$u := e^{-\delta\tau}\bar{u}$$

By further algebraic manipulation we see that $\bar{u}$ is the solution of the following PDE

$$\begin{aligned} \bar{u}_\tau &= \Delta(\tau; T)\bar{u}_{zz} \\ \bar{u}(0, z) &= (z - k)^+ \end{aligned}$$

By a proper rescaling of time we may convert this PDE to the standard form of the heat equation. Indeed, let $\bar{\tau}$ be such that

$$\bar{\tau} = 2\int_0^\tau \Delta(\tau'; T)d\tau'$$

Then $\bar{u}(\bar{\tau}, z)$ satisfies the PDE

$$\frac{\partial \bar{u}}{\partial \bar{\tau}} = \frac{1}{2}\frac{\partial^2 \bar{u}}{\partial z^2}$$

which can be solved using the heat kernel for the heat equation.

Before proceeding with the solution of this equation we have to see how the initial condition transforms under the changes of variables performed, and whether the initial condition we have is compatible with the particular form of similarity solution we look for. In terms of $w(T-t, C, I)$ the initial condition was $w(0, C, I) = (I-k)^+$. Undergoing the transformations of variables we have performed we see that this initial condition transforms to $u(0, z) = (z-k)^+$ which is consistent with the similarity solution assumed.

The solution of the pricing equation will be given by:

$$\bar{u}\left(\bar{\tau}, z\right) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\bar{\tau}}} \exp\left(-\frac{\left(z - z^{'}\right)^2}{2\bar{\tau}}\right)\left(z^{'} - k\right)^+ dz^{'}$$

or

$$\begin{aligned} \bar{u}\left(\tau, C_0, I_0\right) &= e^{-\delta\tau} z(\tau, C_0, I_0)\Phi(k_1) + e^{-\delta\tau}\frac{\sqrt{\bar{\tau}}}{\sqrt{2\pi}}\, e^{-\frac{k_1^2}{2}} - e^{-\delta\tau} k\Phi(k_1), \\ k_1 &= \frac{z(\tau, C_0, I_0) - k}{\sqrt{\bar{\tau}}} \\ \tau &= T - t \end{aligned}$$

where

$$\begin{aligned} \bar{\tau} &= \sigma^2 T^2 (T^{2H} - t^{2H}) + 2\left(\frac{\sigma^2 H}{2H+1} - \frac{\sigma^2 (H-0.5)}{2H+2}\right)(T^{2H+2} - t^{2H+2}) - \\ &- \frac{4\sigma^2 HT}{2H+1}(T^{2H+1} - t^{2H+1}) \\ z &= \frac{b\tau^2}{2} + \tau C_0 + I_0 \end{aligned}$$

and the proposition is proved. □

The above arguments may be generalized to other types of reinsurance contracts. The pricing equation will be essentially the same but with different initial condition. The price of the proportional reinsurance contract solves the same equation with the aggregated excess reinsurance contract but with different initial condition

$$\frac{\partial \bar{u}}{\partial \bar{\tau}} = \frac{1}{2}\frac{\partial^2 \bar{u}}{\partial z^2}$$

$$u(0, I) = \alpha I$$

Thus we have that the solution will be given by

$$\bar{u}\left(\bar{\tau}, z\right) = \int\limits_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\bar{\tau}}} \exp\left(-\frac{\left(z - z^{'}\right)^2}{2\bar{\tau}}\right) az' dz^{'} = \alpha z.$$

The proof follows in an analogous manner as in the case of the aggregated excess reinsurance contract.

4 Numerical Treatment of the Problem

In this section we calculate the value of the aggregated excess reinsurance policy using the analytic solution and the Monte Carlo method. In order to implement the Monte Carlo method we simulate a large number of paths of I_T and then we compute the expected value of the discounted payoff, i.e.

$$\bar{u}(T, I_0, C_0) = E\left[e^{-\delta T}\left(I_T - K\right)^+\right].$$

The values of the parameters we use are claims expectation b=0.5, claims volatility $\sigma = 0.25$, Hurst exponent $H = 0.7$, interest rate $\delta = 0.05$, strike price $k = 10$ and expiry time $T = 10$. For the Monte Carlo method we use 2^N points for the simulation of each path where $N = 14$ and M paths with $M = 20000$. We see that the two methods give results that are close enough, i.e the explicit method gives 12.435, and the Monte Carlo gives 12.444. Using the analytic method we also plot the graphs of the option values vs time, Hurst exponent, claims expectation and volatility of liabilities. As we expect reinsurance policy value increases as these parameters take higher values. It is natural for the reinsurance policy value to increase when time to expiry, claims expectation and volatility of liabilities increase since all these parameters give higher values of I_T. When Hurst exponent increase we see that the long-range dependence that characterizes the liabilities leads to have high liabilities being followed by high liabilities and thus making the value of I_T higher and thus the value of the option higher. This is a result which agrees with previous results obtained from [9] which show that the probability of ruin at of an insurance company is higher as Hurst exponent increases and with

Figure 1: Reinsurance Contract Value as a function of time for $H = 0.7$.

Figure 2: Reinsurance Contract Value as a function of Hurst exponent.

[10] which show that as Hurst exponent increases the demand for reinsurance is higher. For extreme values of the Hurst exponent, i.e higher than 0.9, we see a strange behavior for the reinsurance option value, i.e. it decreases. This could be the result of the degeneracy of fractional Brownian motion for high values of H.

4.1 Calculation of the Reinsurance Option Sensitivities

The simple analytic formula for the value of the contract allows us to calculate the sensitivities of the reinsurance contract, known also as the ”Greeks”, i.e. the partial derivatives of the reinsurance contract value with respect to various parameters of interest such as the interest rate and the claims volatility. For the partial derivative of the reinsurance contract value with respect to δ.

Figure 3: Reinsurance Contract Value as a function of claims expectation.

Figure 4: Reinsurance Option Value as a function of claims volatility.

Figure 5: Reinsurance Contract Vega Value as a function of volatility.

$$\frac{\partial \bar{u}}{\partial \delta} = -\tau \bar{u}$$

For the partial derivative with respect to σ we have that

$$\frac{\partial \bar{u}}{\partial \sigma} = e^{-\delta\tau} z \frac{\partial \Phi(k_1)}{\partial \sigma} + \frac{e^{-\delta\tau}}{\sqrt{2\pi}} \left[e^{-\frac{k_1^2}{2}} \frac{\partial \sqrt{\bar{\tau}}}{\partial \sigma} + \sqrt{\bar{\tau}} \frac{\partial e^{-\frac{k_1^2}{2}}}{\partial \sigma} \right] + e^{-\delta\tau} k \frac{\partial \Phi(k_1)}{\partial \sigma}$$

where

$$\begin{aligned}
\frac{\partial \Phi(k_1)}{\partial \sigma} &= \frac{-\frac{1}{4} e^{-\frac{(z-k)^2}{2\bar{\tau}}} (z-k)\sqrt{2}\frac{\partial \bar{\tau}}{\partial \sigma}}{\sqrt{\pi}\bar{\tau}^{\frac{3}{2}}} \\
\frac{\partial \bar{\tau}}{\partial \sigma} &= 2\sigma T^2 (T^{2H} - t^{2H}) + 2\left(\frac{2\sigma H}{2H+1} - \frac{2\sigma(H-\frac{1}{2})}{2H+2}\right)(T^{2H+2} - t^{2H+2}) \\
&- \frac{8\sigma H T}{2H+1}(T^{2H+1} - t^{2H+1})
\end{aligned}$$

$$\frac{\partial e^{-\frac{k_1^2}{2}}}{\partial \sigma} = \frac{\frac{1}{2}(z-k)^2 \frac{\partial \sqrt{\bar{\tau}}}{\partial \sigma} e^{-\frac{\frac{1}{2}(z-k)^2}{\bar{\tau}}}}{\bar{\tau}^2}$$

In figure 5 we see the value of vega as a function of volatility.

These greeks can also be particularly useful for the design of portfolios of reinsurance contracts. For instance, in complete analogy with the situation in financial portfolios, a reinsurer may design a reinsurance portfolio to be neutral with respect to one of the greeks of our choice, for instance with respect to the claims volatility σ. One has to be careful in the interpretation of this portfolio. For instance the different options included in the portfolio may be interpreted as different types of reinsurance contracts, possibly with more than one insurer and with different contract specifications. Thus a reinsurer by choosing appropriate reinsurance contracts with a number of insurance companies, may achieve neutrality of its position with respect to the volatility of the claims process. In analogy, an insurer may choose a number of reinsurance contracts, possibly with more than one reinsurers, to neutralize the effect of the volatility of the claims on its position.

A Basic facts on fractional stochastic calculus

In this section we review some fundamental results in fractional stochastic calculus for the convenience of the reader. The approach is based on the approach in [7]

The fractional Brownian motion (FBM) is a self affine stochastic process displaying long term correlation. FBM is characterized by the Hurst exponent H. Let us denote by W_t^H the FBM with Hurst exponent $H \in (0.5, 1)$. The process W_t^H is defined with respect to some probability space $(\Omega, \mathcal{F}, P^H)$, has continuous sample paths, is a zero-mean Gaussian random variable for all $t \geq 0$ and has autocorrelation function

$$E[W_t^H W_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - \mid t - s \mid^{2H})$$

for all $t, s \geq 0$, where by E we denote the expectation with respect to the probability measure P^H. For $H = 1/2$ we recover the usual Brownian motion. For $H > 1/2$ the FBM has a long range dependence.

Fractional Brownian motion does not have the ‘nice’ properties of Brownian motion. In particular, it is not a Markov process and it is not a semi-martingale. This presents problems in the definition of a stochastic integral and a stochastic calculus with respect to fractional Brownian motion, as we may not apply the standard theory of stochastic integration over a semimartingale to define a stochastic integral over fractional Brownian motion. Several approaches to this subject has been proposed (see e.g. [20], [6] etc). We will adopt here the theory of Duncan, Hu and Pasik-Duncan [7] and Benth [2] in which a stochastic integral over fractional Brownian motion of Hurst exponent $1/2 < H < 1$ has been defined, having some properties that have similarities with the corresponding properties of the stochastic integral over the usual Brownian motion.

We summarize here the basic results of [7] that we will use in this paper.

The stochastic integral I_t=$\int_0^t f_s dW_s^H$ over deterministic functions f is defined easily to provide a zero mean, Gaussian random variable with variance $var(I_t) = \int_0^\infty \int_0^\infty f_s f_t \phi(s,t) dsdt$ where $\phi(s,t) = H(2H-1) \mid s-t \mid^{2H-2}$ and let the space $L^2_\phi(R^n)$ be the set of functions $f : R^n \rightarrow R$ such that

$$|f|^2_{\phi,n} :=< f, f, >_{\phi,n} < \infty.$$

The inner product $< .,. >_{\phi,n}$ is defined as

$$< f, g >_{\phi,n} = \int\limits_{R^n \times R^n} f(s_{1,...,}s_n) g(t_{1,...,}t_n) \phi(s_1, t_1) ... \phi(s_{n,...,}t_n) dsdt.$$

The stochastic integral $\int_0^t F_s dW_s^H$ can be defined over stochastic processes F as the limit

$$\int_0^t F_s dW_s^H = \lim_{\Delta \rightarrow 0} \sum_{i=0}^{n-1} F_{t_i} \diamond (W^H_{t_{i+1}} - W^H_{t_i})$$

where $\{t_i\}$ is some partition of the interval $(0,t)$, $\Delta = \sup_i \mid t_{i+1} - t_i \mid$. By $\diamond$ we denote the Wick product which is defined by

$$\varepsilon(f) \diamond \varepsilon(g) = \varepsilon(f+g)$$

where

$$\varepsilon(f) := exp \left\{ \int_0^\infty f_t dW_t^H - \frac{1}{2} \int_0^\infty \int_0^\infty f_s f_t \phi(s,t) dsdt \right\}$$

is the stochastic exponential of the deterministic function f which is such that $\mid \int_0^\infty \int_0^\infty f_s f_t \phi(s,t) dsdt \mid < \infty$.

Duncan et al [7] provide the following generalization of Itô 's lemma in the case of fractional Brownian motion. For a proof of this result and generalizations to more complicated integrands, we refer to [7].

Proposition A.1 *Let* $\eta_t = \int_0^t a_s dW_s^H$ *where* a_t *is some deterministic function such that* $\mid \int_0^\infty \int_0^\infty a_s a_t \phi(s,t) dsdt \mid < \infty$. *Let* $f \in C^{1,2}$ *and assume that* $\frac{\partial f}{\partial x}(s, \eta_s) a_s \in \mathcal{L}(0,T)$. *Then,*

$$f(t, \eta_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, \eta_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, \eta_s) a_s dW_s^H \\ + \int_0^t \frac{\partial^2 f}{\partial x^2}(s, \eta_s) a_s \int_0^s \phi(s,v) a_v dvds, \quad a.s.$$

The following higher dimensional generalization of Itô's lemma due to Benth, [2], will be of use in this work

Proposition A.2 *Assume* $X_t = \int_0^t a(s)dB_s^H$ *and* $Y_t = \int_0^t b(s)dB_s^H$ *where* $a, b \in L_\phi^2(R)$. *For a function* $f \in C^{1,2}(R_+ \times R^2)$ *with bounded derivatives we have*

$$\begin{aligned} f(T, X(T), Y(T)) \quad &= \quad f(0,0,0) + \int_0^T f_t(s, X(s), Y(s))ds + \\ &+ \int_0^T \left[f_x(s, X(s), Y(s))a(s) + f_y(s, X(s), Y(s))b(s)\right] dB_s^H + \\ &+ \int_0^T f_{xx}(s, X(s), Y(s)) \left[a(s) \int_0^s \phi(s,u)a(u)du\right] ds + \\ &+ \int_0^T f_{yy}(s, X(s), Y(s)) \left[b(s) \int_0^s \phi(s,u)b(u)du\right] ds + \\ &+ \int_0^T f_{xy}(s, X(s), Y(s)) \left[a(s) \int_0^s \phi(s,u)b(u)du + b(s) \int_0^s \phi(s,u)a(u)ds\right] \end{aligned}$$

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