Claim Counts Modeling and Stable Distributions.

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Abstract: In this communication we show how the generalized Poisson Pascal (G.P.P.) distribution may be a very good choice when trying to model claim counts. This is a very critical step in the modeling process of an experience rating scheme such as Bonus-Malus Systems (BMS), for any conclusion will depend on this early choice. Neither parametric models like the negative binomial or Poisson mixed by Inverse Gaussian, nor nonparametric ones like the good risk-bad risk model are enough to model real world. The application of the G.P.P. may be of a great help when trying to model claim counts with a high skewness, as was indicated in Panjer, Willmot (1992) and Klugman, Panjer, Willmot (1998). The difficulty to overcome when using it is the numerical handling of stable distributions.

Keywords: Claim counts, counting distributions, stable distributions, compound Poisson distributions, mixed Poisson distributions, portfolio structure, Bonus-malus systems, generalized Poisson Pascal distribution.

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1-Introduction

This short communication addresses the problem of fitting probabilistic models to claim counts arisen in heterogeneous portfolios of insurance policies. This is a very critical step when trying to model an experience rating scheme such as Bonus-Malus Systems (BMS), for any conclusion will depend on this early choice. Calculations of the collective premium, bonus-malus premium scales, evaluations of the system, and so on, may be invalidated by the adoption of an improper model supplying a mistaken portfolio structure. This is probably the reason why many efforts have been devoted in the search of a variety of counting distributions, able to provide a tool box full of resources against the bothering features that real world may shelter.

When speaking of counting distributions able to model the heterogeneity, we are mostly referring to mixed Poisson distributions. Inside this class, two kinds of models - the parametric and non parametric cases - have been traditionally distinguished.

The former consists in families where the mixing distribution belongs to some well known continuous parametric family. This is the case of the gamma distribution (producing the negative binomial as an unconditioned), the inverse Gaussian distribution and the lognormal distribution (see for instance Lemaire (1995), Denuit, Maréchal, Pitrebois, Whalin (2007)). The case of the inverse Gaussian mixing distribution has been generalized to a Generalized Inverse Gaussian (producing the Sichel's distribution as an unconditioned, see Panjer, Willmot (1992)), though this last case seems not to be of a wide use, except for some application to Exact Credibility in the area of the number of excesses over a threshold (see Vilar-Zanón, Lozano-Colomer (2007)).

The later consists in choosing a discrete mixing distribution that resumes the complexity of the risk parameter in a finite set of values to be estimated, jointly with their probabilities. The simpler example is the well known good risk bad risk model, which consists in a mixing distribution with support containing two values of the risk parameter. Some drawbacks of this subclass are theoretical and some others are practical. Firstly it is a very naïve model of the risk parameter randomness for sure, as it corresponds to descriptions that could be equally qualified (good-bad, for instance). Even in their simpler version, it is necessary to use four parameters contravening that way the elementary rule of parsimoniousness that should be always a strong guideline in any mathematical modeling process. If we tried to sophisticate the model adding subsequent values to the support we would surely deepen in this last defect. Moreover, the numerical estimation procedures would have to cope with two times as many unknowns as points in the support. This would very soon drive the numerical calculations into a dead end, making impossible any improvement of the former drawback. This could be the reason we only find in literature practical cases with at most three points in the support of the mixing distribution. Another drawback could be the lack of robustness of these models with respect to time evolution.

These are the main reasons for we have focused our research on counting models of the first case, i.e. parametric models. We where in search of models able to fit the data in a variety of real cases as wide as possible, because that was what our research on BMS was calling for (Heras, Vilar, Gil (2002), Heras, Gil, García-Pineda, Vilar (2004)). For this aim, the material found in Klugman, Panjer, Willmot (1998) and Panjer, Willmot (1992) has been very useful.

We would like to show how some parametric models that as far as we know have not been used so often, may solve the fitting step with quite good results and elegance. A more detailed version of the following development can be found in Vilar-Zanón, Heras-Martinez, Gil-Fana (2004).

2-The claim counts distribution.

The family of distribution we are interested in is included in the intersection between the classes of compound and mixed Poisson distributions. One of the main advantages when working with such an element is that all the numerical difficulties carried in the mixing side will be easily solved in the compound side, with the help of Panjer's algorithm for instance. Another advantage, very important from the practical point of view, is referred to the fact that very often our data will have a very high skewness indeed, making impossible from the beginning the fitting of the more frequently used models as are the ones with a gamma or an inverse Gaussian structures. It is known that some compound Poisson models can afford an arbitrary high skewness for given expectation and variance, as it is shown for instance in Panjer, Willmot (1992). These last models form a subclass called Generalized Poisson Pascal (GPP) distributions, where the secondary distribution is an Extended Truncated Negative Binomial (ETNB). This last is a well known discrete biparametric family belonging to the C(a,b,1) class. These models (considered as mixed Poisson distributions) were very soon studied in one or in other way, see for instance Thyrion (1960), Tripathi, Gurland, Bhalerao (1986) and Hougaard (1986).The interesting fact is that in the mixing side, the ETNB corresponds to a structure function depending on a stable distribution

$$u(\lambda)=\frac{e^{\lambda_1}}{\mu\lambda_1^{1/\alpha}}e^{-\lambda/\mu}f_\alpha\left(\frac{\lambda}{\mu\lambda_1^{1/\alpha}}\right),\lambda>0 \tag{2.1}$$

with $\lambda_1>0,\mu>0,\alpha\in(0,1)$, and f_α a stable density with stability index α, and characteristic function (c.f.)

$$\phi_\alpha(t)=e^{-(-it)^\alpha},t\in\mathbb{R} \tag{2.2}$$

The links between these three parameter and the compounding $P(\lambda>0)$ - $ETNB(r\in(-1,0),\beta>0)$ ones, are the following

$$\begin{gathered}\alpha=-r \\ \mu=\beta \\ \lambda_1=\frac{\lambda}{(1+\beta)^\alpha-1}\end{gathered} \tag{2.3}$$

Working with stable distributions is not a trivial task at all. It is very well known that these only have a close form in three cases, namely $\alpha=2$ (normal distribution), $\alpha=1$ (Cauchy distribution), and $\alpha=1/2$ (Lévy distribution). Out of these cases, any probability calculation will rely on the skill for numerically inverting the transform of the stable distribution. This explains the utmost importance of (2.2).

The classical reference on stable distributions is Zolotarev (1986), and we also found very much help in Nolan (1997) and Nolan (1999). Some visits to Professor John Nolan’s web page on stable distributions were also very helpful,

(see http://academic2.american.edu/~jpnolan/stable/stable.html).

When using stable distributions we may proceed with care because these can be parameterized in different manners, and choosing one or another depends on the kind of work that we are facing. If we are trying to calculate numerical values of either the density (p.d.f.) or the distribution function (c.D.f.), then it is advised to use Zolotarez’s M parametrization (Zolotarev (1986) p.10-11) also known as the **S**(α,β,γ,δ;0) parametrization (Nolan (1999)). These four parameters stands respectively for stability, asymmetry ($\beta \in [-1,1]$), scale and localization. This must be carefully considered because they do not correspond with the usual parameters of expectation, standard deviation and skewness coefficient. The **S**(α,β,γ,δ;0) parametrization is useful when dealing with the numerical evaluation of stable d.f.'s and c.D.F.'s, for it opens the way to inversion formulae where the integrals are defined on bounded intervals, thus highly simplifying the subsequent numerical calculations. To valuate the γ, β, and δ parameters we have to rewrite (2.2) in such a manner that it may be compared with the expression of the same c.f. given under the **S**(α,β,γ,δ;0) parametrization. Then we must proceed cautiously because generally the location and scale parameters will not be equal to 0 and 1, so we will have to "typify" the density (for instance) in order to apply the formulae of numerical inversion. That is, we will have to consider

$$\frac{1}{\gamma} f_\alpha\left(\frac{\lambda-\delta}{\gamma}\right) \tag{2.4}$$

instead of f_α.

3-Fitting the model to a sample.

An example of fitting to a real case can be found in (Vilar-Zanón, Heras-Martinez, Gil-Fana (2004)). Nevertheless we will rather use here a data sample taken from Denuit, Dhaene (2000). This is interesting because the authors say that it also corresponds to a typical real world example, and it will make possible to compare our results with the ones obtained by them using several parametric as well as non parametric models. These are the Poisson, negative binomial, a good-bad risks one, and finally a finite Poisson mixture with three atoms.

Our estimation method was moment matching, while the referenced authors' one was Maximum Likelihood. We codified all our algorithms in Maple worksheets.

Table 1 has been entirely taken from the last reference. We have added the last column (E) where we have written down our own numerical results. We observe that our p-value is less then the one corresponding to the non parametric models. Though we think that it could be enough good to make us finally choose the GPP. What we loose in p-value could largely be rewarded by the fact that GPP is more parsimonious (less parameters) and depicts a more complete and sophisticated risk parameter randomness.

k	n_k	A	B	C	D	E
0	102,435	102,026	102,435	102,435	102,442	102,435
1	8,804	9,544	8,805	8,811	8,778	8,805
2	714	446	712	703	743	710
3	65	14	68	76	63	71
4	12	0	10	8	5	9
5	1	0	2	1	0	1
≥ 6	0	0	0	0	0	0
χ^2_{Obs}		365.67	1.25	3.78	8.18	1.52
# d.f.		5	1	3	4	1
p-value		$<10^{-6}$	0.26	0.29	0.09	0.22

COLUMN A: Expected frequencies with homogeneous Poisson

COLUMN B: Expected frequencies with 3-points

$\hat{\lambda}_1 = 0.132,\ \hat{\lambda}_2 = 0.829,\ \hat{\lambda}_3 \simeq 0.000, \hat{\pi}_1 = 0.651,\ \hat{\pi}_2 = 0.009,\ \hat{\pi}_3 \simeq 0.340$

COLUMN C: Expected frequencies with 2-points

$\hat{\lambda}_1 = 0.068,\ \hat{\lambda}_2 = 0.446,\ \hat{\pi}_1 = 0.933,\ \hat{\pi}_2 = 0.067$

COLUMN D: Expected frequencies with Negative Binomial

$\hat{r} = 1.0255, \hat{\beta} = 10.9672$

COLUMN E: Expected frequencies with Generalized Poisson Pascal (moments matching method): $\overline{\lambda} = 0.0895, \overline{r} = -0.6447, \overline{\beta} = 0.269$

Table 1: Reproduced from Denuit, Dhaene (2000). The last column (E) has been added to write down our results when fitting a GPP distribution.

Moreover, its first three characteristics (expectation, standard deviation and skewness) equates the sample ones. These are:

$$E\{N\}=0.0935,\sigma_N=0.3201,\gamma_N=3.768 \tag{3.1}$$

Describing the mixing distribution (2.1) in this particular case is straightforward. The estimated parameters are

$$\overline{\alpha}=0.6447,\overline{\mu}=0.269,\overline{\lambda}_1=0.5393 \tag{3.2}$$

Figure 1 is a plot of the resulting mixing density. Its characteristics are the following:

$$E\{\Lambda\}=0.0935,\sigma_\Lambda=0.3201,\gamma_\Lambda=4561.02 \tag{3.3}$$

If this mixing distribution had to be used in any BMS analysis, then there would still remain the task of adequately discretize it while conserving some of its characteristics, for instance the first two as is advised in Panjer, Willmot (1992). Achieving this task by means of linear goal programming would imply the technique explained in Vilar (1998).

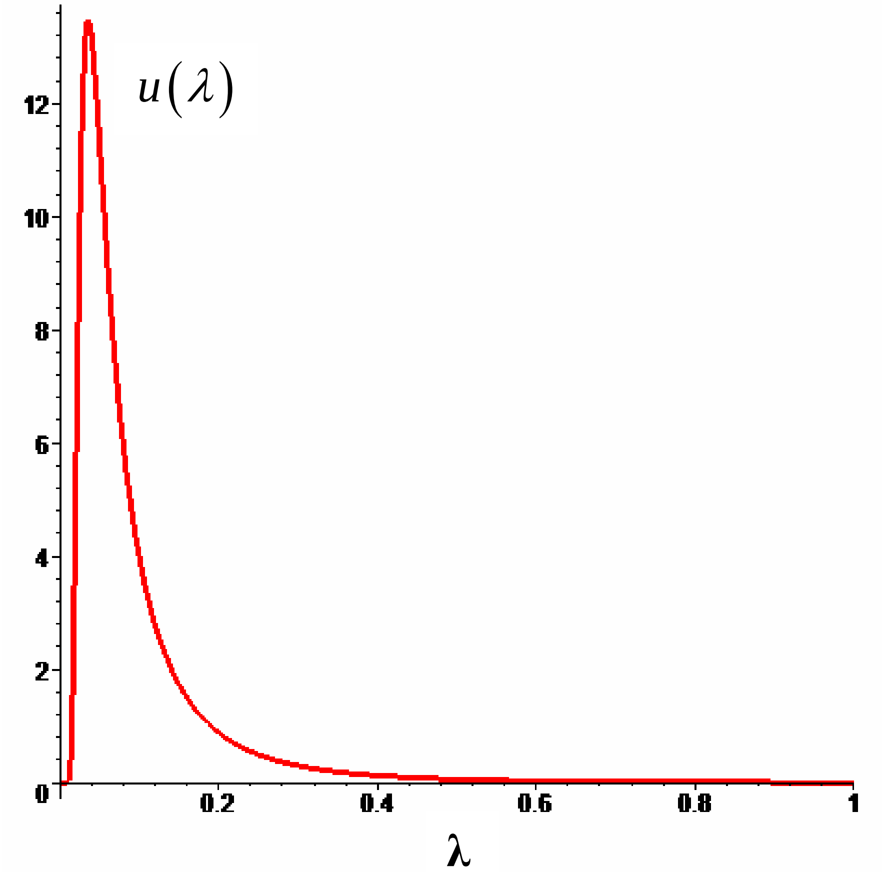

Figure 1: Plot of the mixing distribution (2.1) corresponding to the fitted GPP.

4-Conclusions.

In this short communication we have studied the use of the Generalized Poisson Pascal distribution in the claim counts modeling, and we have fitted it to a sample. Our aim was to show how is it still possible to obtain a success with this model when the other parametric models do not provide good fits. Moreover, we think that the use of stable distributions in Actuarial Science modeling problems will be increasing in the future and we have tried to provide one example of its beneficial use. The principal difficulties in using them will be of a Numerical Analysis kind, but tools are easy to obtain for coping with this obstacle.

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