Optimal Insurance Coverage of a Durable Consumption Good with a Premium Loading in a Continuous Time Economy

Masaaki Kijima* and Teruyoshi Suzuki†

Abstract

This article analyzes the optimal deductible level of insurance on durable consumption goods with a positive premium loading in a continuous-time economy. Assuming financial assets and durable consumption goods can be traded without transaction costs, we provide an explicit solution for the optimal insurance coverage of durable consumption goods together with optimal trading strategies for the amount of the durable consumption goods and financial assets. Using the solution, we show that an increase in premium loading decreases both demand for insured assets and their insurance coverage. We also show that an increase in premium loadings can affect optimal investment strategies through the effects on the optimal amount of durable consumption goods held. Moreover we show that there exist unique parameter values of the loss process such as when no insurance is optimal. Numerical examples help us understand how the risk of financial investment and the risk aversion measure affects an agent's optimal insurance coverage.

Keywords: Insurance, deductible, durable consumption goods, optimal consumption and investment

*Tokyo Metropolitan University, Daiwa Securities Group Chair: Kyoto University

†Corresponding author: Graduate School of Economics Hokkaido University, NISHI 7, KITA 9, KITA-KU SAPPORO 060-0809, JAPAN, e-mail:suzuki@econ.hokudai.ac.jp

1 Introduction

Insurance markets have advanced social welfare over time. However providing insurance requires positive premium loading to cover the costs linked to monitoring policies and the management of an organization. Therefore the insurance premiums received by an insurance company are always larger than the expected loss. When insurance is costly, the choice of insurance coverage is not simple. Arrow (1971) and Mossin (1968) were the first to examine optimal insurance. Mossin (1968) showed that full insurance coverage is not optimal when the premium includes a positive loading. Arrow (1971) showed that it is optimal to purchase a deductible insurance on aggregate wealth when a positive loading premium is charged. The benefit of reducing coverage comes from the reduction of the positive insurance cost. Such classic literature is concerned with the case of a single insurable asset without consumption in a static model. Therefore the agent implicitly transforms the corresponding loss into the reduction of consumption or savings, and cannot hedge against the shock of loss by reducing his consumption over time. To study the interaction between the risk of loss and consumption, we should use a continuous time model.

In a dynamic setting with consumption, an agent faces competing objectives. Insurance improves welfare by absorbing a large loss, however it is costly and reduces the utility of consumption. This allows the agent to follow a self-insurance strategy by accumulating buffer wealth. In other words, the agent can formulate a strategy that reduces consumption to provide against future loss. It follows that when the agent's wealth is diversified into risky and risk-less securities, the investment strategy of the agent can be affected by the premium loading of insurance through the reduction of consumption.

Since Merton (1969), a considerable number of studies have been conducted on intertemporal consumption and investment strategy in a continuous time economy. We note that in Merton (1969), the optimal portfolio is equal to the tangency portfolio which is given under a static model (Capital Asset Pricing Model). However Merton (1973) showed that when future investment opportunities are time-varying the optimal portfolio differs from the tangency portfolio. To our knowledge, Briys(1986) first approached optimal insurance using the methods of Merton (1969). For power utility functions and considering the full loss of wealth, the optimal insurance decision is independent of wealth and of all consumption decisions even if the premium loading is positive. The results were consistent with static models. Moore and Young (2006) extended Briys(1986) and showed numerical analysis in which optimal insurance is inconsistent with static models. They showed numerical examples where an agent has an exogenous wage which means investment opportunities varies over time. In these cases, the parameter of loss affects the optimal investment strategies. Gollier (2002) studied the agent with liquidity constraint using a numerical solution. Although studies have been made on optimal insurance in a dynamic setting, the interaction between investment and insurance decision is still controversial.

In this paper, we consider a durable consumption good as a case where insurance decisions are inconsistent with static models. The reason for inconsistency with the static model is as follows. When a durable consumption good is insured and provides utility, the loss process can affect the policy for holding durable goods just as the policy for perishable consumption goods. And when durable goods can be traded and drive time-varying investment opportunities, the change in the amount of durable goods held affects the investment strategy for financial assets. Insurable assets are often durable consumption goods such as housing and motorcars. Then it is necessary to investigate the insurance demand for durable

consumption goods. However little attention has been given to the research of optimal insurance coverage of durable consumption goods. On the other hand various literature has been published that studies optimal consumption and investment which includes durable consumption goods. These include Hindy and Huang (1993), Detemple and Giannikos (1996), Cuoco and Liu (2000), Cocco (2004), Cauley, Pavlov and Schwarts (2005) and Grossman and Laroque (1990). We follow Damgaard, Fuglsbjerg and Munk (2003) that extend Grossman and Laroque (1990) to include a perishable consumption good and a durable consumption good in the model. In our model, we take into account the following features of a durable good: (1) A durable good can be stored and provides utility to its owner over a period of time. (2) A durable good can be resold and acts as a physical asset. (3) The stock of a durable good can be changed continuously without cost. (4) The unit price of durable goods follows a geometric Brownian motion and is partly correlated with the price of a financial risky asset. (5) The stock of a durable good depreciates at a certain physical rate over time. Beside these basic features, we assume a durable good can be insured against damage or loss.

Assuming durable goods and all other assets can be traded without transaction costs, we provide an explicit Merton-type solution for the optimal strategies of financial risky and risk-less assets, perishable consumption goods, and durable consumption goods and their insurance coverage. In our model, the optimal deductible level is independent of the stock price process however actual coverage depends on the stock price process. Therefore, factor loading of insurance can affect the investment decision.

Following on from the introduction, Section II sets up our model and provides a solution. Section III presents comparative statics on premium loading, the intensity of the damage and the loss proportion of insured assets. Section IV shows numerical results. Section V concludes this article.

2 A Model

2.1 Set up

We consider an infinite-horizon, continuous-time stochastic economy with a perishable consumption good, a durable consumption good and two financial assets. One of the financial assets is a risk-free security paying a constant continuously compounded interest rate r. The other is a risky security i.e. stock whose price process follows a geometric Brownian motion

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma_S dw_1(t), \quad t \geq 0 \tag{1}$$

where $(w_1(t), w_2(t))$ is an uncorrelated two dimensional Wiener process and where μ and σ_S are constants.

We now make several assumptions about the market:

(a) Financial securities and durable goods can be bought in unlimited quantities and are infinitely divisible.

(b) Financial securities can be sold short but durable goods can not be sold short.

(c) There are no transaction costs.

The unit price of a durable good $P(t)$ also follows a geometric Brownian motion

$$\frac{dP(t)}{P(t)} = \mu_P dt + \sigma_{P1} dw_1(t) + \sigma_{P2} dw_2(t), \quad t \geq 0 \tag{2}$$

where μ_P, σ_{P1} and σ_{P2} are constants and where

$$\sigma_P^2 = \sigma_{P1}^2 + \sigma_{P2}^2.$$

We should note that the unit price of the durable good is partly correlated with the price of the financial risky asset.

We assume that the stock of the durable consumption good depreciates at a certain rate δ over time. We also assume that durable consumption goods can be damaged by an insured event represented by a Poisson process $N(t)$ which is independent of $(w_1(t), w_2(t))$. We denote by λ the intensity of the events and by ℓ the constant loss proportion of the durable good when the insured event occurs. Letting $K(t)$ be the number of units of the durable good held at time t, then $K(t)$ follows

$$\frac{dK(t)}{K(t)} = (-\delta + \lambda\ell)dt - \ell dN(t), \quad t \geq 0 \tag{3}$$

where δ, ℓ, λ are constants. We require $K(t) > 0$ from the assumption that the agent can not take a short position on durable goods.

The agent can purchase an insurance contract to cover the risk of loss. We denote by $q(t)$ the indemnity paid by the insurer at time t. The payment must be positive then the constraint is

$$q(t) \geq 0. \tag{4}$$

Assuming that the insurance premium is payable continuously and includes a positive loading represented by a factor ϕ, the premium to be paid and denoted by $p(t)$ is given by

$$p(t) = \lambda\phi q(t)$$

where $\phi \geq 1$. We assume that the premium loading is sufficiently small to satisfy the solvency condition.

We denote by $\theta_0(t)$ and $\theta(t)$ the amount held in the risk-free and risky security at time t. We define the wealth of the agent as the sum of his investments in the risk-free and risky assets and the value of his current amount of durable goods $K(t)$ multiplied by the current price of durable goods $P(t)$. Therefore the wealth $X(t)$ is given as

$$X(t) = \theta_0(t) + \theta(t) + K(t)P(t), \quad t \geq 0. \tag{5}$$

Under the assumption that the agent follows a perishable consumption strategy $C(t)$ and self-financing strategy $(\theta_0(t), \theta(t), K(t))$, the wealth process $X(t)$ evolves as

$$\begin{aligned} dX(t) \;=\; & \Big(r(X(t) - K(t)P(t)) + \theta(t)(\mu - r) + (\mu_P - \delta + \lambda\ell)K(t)P(t) - C(t) - p(t)\Big)dt \\ & + \Big(\theta(t)\sigma_S + K(t)P(t)\sigma_{P1}\Big)dw_1(t) + K(t)P(t)\sigma_{P2}dw_2(t) \\ & + \Big(q(t-) - \ell P(t)K(t-)\Big)dN(t), \qquad t \geq 0. \end{aligned} \tag{6}$$

At the time η when an insured event occurs, there is a jump in his wealth due to the damage of the durable goods. We require that the consumption and trading strategies satisfy the solvency condition of the agent and that his total wealth is always positive although an insured event has occurred:

$$X(\eta) = X(\eta-) - \ell P(\eta)K(\eta-) + q(t) > 0, \quad t \geq 0. \tag{7}$$

A policy $\mathcal{S}_t = (\theta(t), K(t), C(t), q(t))$ is admissible if the policy satisfies (4), (7) and $K(t), C(t) > 0$. We denote by $\mathcal{A}(x, k, p)$ the set of admissible policies where $x = X(0),\ k = K(0),\ p = P(0)$. We assume $\mathcal{A}(x, k, p)$ is a non-empty set.

We assume that the utility function exhibits constant relative risk aversion, i.e.:

$$U(c, k) = \frac{1}{1-\gamma}\left(c^{\beta}k^{1-\beta}\right)^{1-\gamma}, \quad 0 < \beta < 1,\ 0 < \gamma < 1$$

where c denotes the perishable consumption rate and k denotes the stock of durable goods held. The agent's objective is to find the policy $\mathcal{S}_t \in \mathcal{A}$ that maximizes his time 0 expected utility:

$$J^{\mathcal{S}}(x, p) = E\left[\int_0^{\infty} \mathrm{e}^{-\rho t} U(C(t), K(t))dt\right]$$

where ρ is time preference parameter. Therefore the value function of agents is given by

$$V(x, p) = \sup_{\mathcal{S}_t \in \mathcal{A},\ t>0} J^{\mathcal{S}}(x, p). \tag{8}$$

From the dynamic programming principle, the value function satisfies

$$V(x, p) = \sup_{\mathcal{S} \in \mathcal{A},\ t>0} E\left[\int_0^{\eta} \mathrm{e}^{-\rho t} U(C(t), K(t))dt + \mathrm{e}^{-\rho\eta} V(X(\eta), P(\eta))\right]. \tag{9}$$

Then the Hamilton-Jacobi-Bellman (HJB) equation corresponding to this problem can be written as

$$\begin{aligned}
\rho V(x,p) \;=\; & \sup_{\mathcal{S}\in\mathcal{A}}\Big\{\frac{1}{1-\gamma}(c^{\beta}k^{1-\beta})^{1-\gamma} + \Big(r(x-pk) + \theta(\mu - r) + (\mu_P - \delta)kp - c - \lambda\phi q\Big)\frac{\partial V}{\partial x}(x,p) \\
& + \frac{1}{2}\Big(\theta^2\sigma_S^2 + k^2p^2\sigma_P^2 + 2\theta\sigma_S\sigma_{P1}kp\Big)\frac{\partial^2 V}{\partial x^2}(x,p) + \mu_P p\frac{\partial V}{\partial p}(x,p) \\
& + \frac{1}{2}\sigma_P^2 p^2\frac{\partial^2 V}{\partial p^2}(x,p) + \Big(\theta\sigma_S\sigma_{P1} + \sigma_P^2 kp\Big)p\frac{\partial^2 V}{\partial x\partial p}(x,p) \\
& + \lambda\Big(V(x - \ell kp + q, p) - V(x,p) + \ell kp\frac{\partial V}{\partial x}(x,p)\Big)\Big\}.
\end{aligned} \tag{10}$$

2.2 An explicit solution

We introduce some auxiliary parameters and a nonlinear equation. After giving an assumption and a lemma, we show a solution. Constants are defined as follows:

$$\begin{aligned}
\Lambda_0 \;=\; & -\frac{\rho}{\gamma} + \frac{1-\gamma}{\gamma}\left\{r - (1-\beta)\mu_P + \frac{1}{2}(1-\beta)[1 + (1-\beta)(1-\gamma)]\sigma_P^2\right\} \\
& + \frac{1-\gamma}{2\gamma^2\sigma_S^2}(\mu - r - (1-\gamma)(1-\beta)\sigma_S\sigma_{P1})^2
\end{aligned} \tag{11}$$

$$\Lambda_1 \;=\; (1-\gamma)\sigma_{P2}^2 + \frac{1}{1-\beta}\left(r - \mu_P + \delta + (\mu - r)\frac{\sigma_{P1}}{\sigma_S}\right) \tag{12}$$

$$\Lambda_2 \;=\; \left(\frac{\gamma}{1-\beta} + \frac{1-\gamma}{2}\right)\sigma_{P2}^2 \tag{13}$$

as in Damgaard, Fuglsbjerg and Munk (2003). Define the nonlinear equation

$$F(\alpha_k) = 0 \tag{14}$$

where

$$F(\alpha_k) = \begin{cases} \Lambda_0 + \Lambda_1\alpha_k + \Lambda_2\alpha_k^2 + \dfrac{\lambda}{\gamma}\left\{(1-\ell\alpha_k)^{-\gamma}\left(1+\dfrac{\beta\gamma\ell}{1-\beta}\alpha_k\right) - \left(1+\dfrac{\gamma\ell}{1-\beta}\alpha_k\right)\right\}, & \alpha_k < \hat{\alpha}_k \\ \Lambda_0' + \left(\Lambda_1 + \dfrac{\lambda(\phi-1)\ell}{1-\beta}\right)\alpha_k + \Lambda_2\alpha_k^2, & \alpha_k \geq \hat{\alpha}_k, \end{cases}$$

and where

$$\hat{\alpha}_k = \frac{1-\phi^{-\frac{1}{\gamma}}}{\ell}, \tag{15}$$

$$\Lambda_0' = \Lambda_0 + \frac{\lambda(\phi-1)}{\gamma} + \lambda\phi(\phi^{-\frac{1}{\gamma}}-1). \tag{16}$$

It is noted that the argument α_k will represent the optimal holding policy for durable consumption goods.

Under the assumption below, the nonlinear equation (14) has a unique positive root, which is stated as Lemma 1.

Assumption 1 *If $\alpha_k < \hat{\alpha}_k$ then*

$$\Lambda_0 < -\frac{1}{2}(1-\gamma)\sigma_{P2}^2\alpha_k^2 + \frac{\lambda}{\gamma}\left\{1+(1-\ell\alpha_k)^{-\gamma}(-1+\ell\gamma\alpha_k)\right\},$$

and if $\alpha_k \geq \hat{\alpha}_k$ then

$$\Lambda_0' < -\frac{1}{2}(1-\gamma)\sigma_{P2}^2\alpha_k^2.$$

Lemma 1 *The nonlinear equation (14) has a unique positive root under Assumption 1.*

The optimal solution for problem (9) is given as follows.

Theorem 1 *Under Assumption 1, the value function for problem (9) is given by*

$$\bar{V}(x,p) = \frac{1}{1-\gamma}\alpha_v p^{-(1-\beta)(1-\gamma)}x^{1-\gamma} \tag{17}$$

and the controls are given in feedback form as

$$\bar{\theta}(t) = \alpha_\theta\bar{X}(t), \quad \bar{\theta}_0(t) = \alpha_0\bar{X}(t), \quad \bar{K}(t) = \alpha_k\bar{X}(t)/P(t), \quad \bar{C}(t) = \alpha_c\bar{X}(t), \quad \bar{q}(t) = \alpha_q\bar{X}(t) \tag{18}$$

where $\bar{X}(t)$ is the wealth process generated by these controls and where constants α_v, α_θ are written by

$$\alpha_v = {\alpha_c}^{\beta(1-\gamma)-1}{\alpha_k}^{(\beta-1)(\gamma-1)}\beta \tag{19}$$

$$\alpha_\theta = \frac{\mu-r}{\gamma\sigma_S^2} + \Big(\beta-(\alpha_k+\beta-1)\gamma-1\Big)\frac{\sigma_{P1}}{\gamma\sigma_S} \tag{20}$$

$$\alpha_0 = 1-\alpha_\theta-\alpha_k \tag{21}$$

and where α_k is a root of the equation $F(\alpha_k)=0$ and where constants α_q, α_c are given by as follows:

$$\alpha_q = \begin{cases} \ell\alpha_k - \left(1-\phi^{-\frac{1}{\gamma}}\right), & \hat{\alpha}_k \leq \alpha_k < \infty \\ 0, & 0 < \alpha_k < \hat{\alpha}_k, \end{cases} \tag{22}$$

$$\alpha_c = \begin{cases} -\beta\Lambda_0' - \frac{1}{2}\beta(1-\gamma)\sigma_{P2}^2\alpha_k^2, & \hat{\alpha}_k \leq \alpha_k < \infty \\ -\beta\Lambda_0 - \frac{1}{2}\beta(1-\gamma)\sigma_{P2}^2\alpha_k^2 - \frac{\lambda\beta}{\gamma}\left\{(1-\ell\alpha_k)^{-\gamma}(1-\ell\gamma\alpha_k)-1\right\}, & 0 < \alpha_k < \hat{\alpha}_k. \end{cases} \tag{23}$$

The proof is presented in the Appendix. We show it is optimal to have either no coverage or partial coverage insurance. The threshold level $\hat{\alpha}_k$ does not depend on state variables $(X(t), P(t), K(t))$. Therefore when no coverage is optimal, the policy of no insurance will not be changed in the future. And when positive coverage is optimal, the coverage level (22) is given by the feed back form as shown in Briys (1986) and Moore and Young (2006). It is noted that optimal deductible level $\left(1-\phi^{-\frac{1}{\gamma}}\right)\bar{X}(t)$ is independent of the stock price process and durable goods price process. However, the actual amount of coverage depends on the holding strategy of durable goods. It follows that investment decisions are affected by the loading factor of insurance premium ϕ as presented in Section 3.

Remark 1 (a) The constant proportion α_θ given by (20) is the same form as shown in Damgaard et al. (2003) although we introduced the risk of damage and its insurance. The first part of (20) represents a mean-variance tangency portfolio and the second part represents a hedged portfolio used for holding durable goods. (b) When $\phi = 1$ or, $\lambda = 0$ or $\ell = 0$, our model is reduced to thato of Damgarrd et al. (2003)

Remark 2 The constant α_c given by (23) can be written as:

$$\begin{aligned} \alpha_c &= -\beta\Lambda_0 - \frac{1}{2}\beta(1-\gamma)\sigma_{P2}^2\alpha_k^2 \\ &\quad - \min\left\{\frac{\lambda\beta(\phi-1)}{\gamma} + \lambda\beta\phi(\phi^{-\frac{1}{\gamma}}-1), \frac{\lambda\beta}{\gamma}\left\{(1-\ell\alpha_k)^{-\gamma}(1-\ell\gamma\alpha_k)-1\right\}\right\}. \end{aligned}$$

The first term of the equation above is implied in Merton (1969). The second term is intertemporal protection for an undiversified risk of durable goods. This term is given by Damgaard, et al. (2003). The last term is given by the minimum of two components: the first is the reduction of consumption caused by using insurance with positive loading, the second is precautionary saving for self-insurance.

3 Analytical Results

In this section, we first analyze the effects on durable consumption goods from the loading factor ϕ, the intensity of damage λ and the loss proportion ℓ. We then investigate the effect on investment, insurance and consumption policies.

Effects on durable consumption goods

The optimal solution for α_k is given by a root of (14). This provides us with the following proposition from the theorem of implicit function.

Proposition 1 *Assume $\phi > 1$ then*
(i) $\hat{\alpha}_k \leq \forall\alpha_k < \infty$

$$\frac{\partial\alpha_k}{\partial\phi} < 0, \quad \frac{\partial\alpha_k}{\partial\lambda} < 0, \quad \frac{\partial\alpha_k}{\partial\ell} < 0,$$

(ii) $0 < \forall \alpha_k < \hat{\alpha}_k$

$$\frac{\partial \alpha_k}{\partial \phi} = 0, \quad \frac{\partial \alpha_k}{\partial \lambda} < 0, \quad \frac{\partial \alpha_k}{\partial \ell} < 0.$$

The proof is given in the Appendix. When partial coverage is optimal i.e. $\alpha_k \geq \hat{\alpha}_k$, an increase in factor loading ϕ decreases demand for durable consumption goods, while ϕ does not affect the holding policy of durable goods when no insurance is optimal. We can see that an increase in λ and ℓ decreases demand for durable consumption goods where partial insurance and no insurance is optimal. It is noted that the threshold $\hat{\alpha}_k$ is an increasing function of ϕ.

Effects on investment policies

Using Proposition 1, the effects on investment policies are given in the following:

Proposition 2 *Assume $\phi > 1$ then*
(i) $\hat{\alpha}_k \leq \alpha_k < \infty$

$$\frac{\partial \alpha_\theta}{\partial \phi} = -\frac{\sigma_{P1}}{\sigma_S}\frac{\partial \alpha_k}{\partial \phi} = \begin{cases} (+), & \sigma_{P1} > 0, \\ (-), & \sigma_{P1} \leq 0, \end{cases}, \quad \frac{\partial \alpha_0}{\partial \phi} = \frac{\sigma_{P1} - \sigma_S}{\sigma_S}\frac{\partial \alpha_k}{\partial \phi} = \begin{cases} (-), & \sigma_{P1} - \sigma_S > 0, \\ (+), & \sigma_{P1} - \sigma_S \leq 0. \end{cases}$$

(ii) $0 < \forall \alpha_k < \hat{\alpha}_k$.

$$\frac{\partial \alpha_\theta}{\partial \phi} = 0, \quad \frac{\partial \alpha_0}{\partial \phi} = 0.$$

The proof is given in the Appendix. The optimal investment policies are influenced by loading factor ϕ. This is because ϕ affects the amount of durable consumption goods held and the fact that durable consumption goods act as physical assets. From the derivative $\partial \alpha_\theta / \partial \phi$ when $\alpha_k \geq \hat{\alpha}_k$, we can see the effect of correlation between the processes $S(t)$ and $P(t)$. When the stock price process and the durable good price process are positively correlated an increase in premium loading increases investment in stock. This is caused by hedging against durable goods as referred in Remark 1 (a). From the derivative α_0, we can see that the effect on holding risk-less securities from an increase in ϕ depends on the difference between σ_{P1} and σ_S. When the volatility of a durable good's price is larger than the volatility of the stock price, an increase in premium loading decreases investment in risk-less securities. When no insurance is optimal, ϕ does not affect the investment policies at all.

The loss intensity λ and the loss proportion ℓ affect the optimal financial investment strategies in the same way as ϕ shown as follows:

$$\begin{pmatrix} \dfrac{\partial \alpha_\theta}{\partial \lambda} \\ \dfrac{\partial \alpha_\theta}{\partial \ell} \end{pmatrix} = -\frac{\sigma_{P1}}{\sigma_S}\begin{pmatrix} \dfrac{\partial \alpha_k}{\partial \lambda} \\ \dfrac{\partial \alpha_k}{\partial \ell} \end{pmatrix}, \quad \begin{pmatrix} \dfrac{\partial \alpha_\theta}{\partial \lambda} \\ \dfrac{\partial \alpha_\theta}{\partial \ell} \end{pmatrix} = -\frac{\sigma_{P1} - \sigma_S}{\sigma_S}\begin{pmatrix} \dfrac{\partial \alpha_k}{\partial \lambda} \\ \dfrac{\partial \alpha_k}{\partial \ell} \end{pmatrix}, \quad 0 < \alpha_k < \infty.$$

Effects on insurance policies

We present comparative statics on insurance policies with respect to ϕ, λ and ℓ by the following proposition.

Proposition 3 *Assume $\phi > 1$, then*
(i) $\hat{\alpha}_k \leq \forall \alpha_k < \infty$

$$\begin{aligned}
\frac{\partial \alpha_q}{\partial \phi} &= \ell \frac{\partial \alpha_k}{\partial \phi} - \frac{1}{\gamma}\phi^{-\frac{1}{\gamma}-1} < 0, \\
\frac{\partial \alpha_q}{\partial \lambda} &= \ell \frac{\partial \alpha_k}{\partial \lambda} < 0, \\
\frac{\partial \alpha_q}{\partial \ell} &= \alpha_k + \ell \frac{\partial \alpha_k}{\partial \lambda} > 0,
\end{aligned}$$

(ii) $0 < \forall \alpha_k < \hat{\alpha}_k$

$$\frac{\partial \alpha_q}{\partial \phi} = \frac{\partial \alpha_q}{\partial \lambda} = \frac{\partial \alpha_q}{\partial \ell} = 0.$$

The proof is given in the Appendix. When partial coverage is optimal, we can see that an increase in ϕ decreases α_q in two way: (1) indirect effect represented by the first term; this is caused by a decrease in demand for durable goods, (2) direct effect represented by the second term; this is consistent with static models. We can also see that an increase in λ decreases demand for insurance with a decrease in durable goods held. We see that an increase in ℓ increases demand for insurance although demand for insured assets itself decreases.

From the Proposition 1, we can show that there exist unique $\hat{\lambda}$, $\hat{\phi}$ and $\hat{\ell}$ such that

$$\hat{\phi} = \inf\{\phi : \alpha_k \leq \hat{\alpha_k}\}, \quad \hat{\lambda} = \inf\{\lambda : \alpha_k \leq \hat{\alpha_k}\} \quad \text{and} \quad \hat{\ell} = \sup\{\ell : \alpha_k \leq \hat{\alpha_k}\}$$

where α_k is given by the root of (14). We state this fact as a proposition.

Proposition 4 *There exist critical values to factor loading ϕ, loss intensity λ and loss proportion ℓ such that no insurance coverage is optimal.*

The proof is given in the Appendix. An increase in ϕ and a decrease in ℓ can terminate insurance demand. These results are consistent with classic models. However the results in λ are not consistent with classic models. This is because, in our model, the increase in λ decreases demand for insured assets. Then an increase in λ can terminate insurance demand.

Effects on consumption policies

For $\alpha_k \geq \hat{\alpha}_k$, under the assumption $\phi > 1$, we can derive following comparative statics on α_c:

$$\frac{\partial \alpha_c}{\partial \phi} = -\beta(1-\gamma)\sigma_{P2}^2 \alpha_k \frac{\partial \alpha_k}{\partial \phi} - \beta\lambda\left(\frac{1}{\gamma} - 1\right)\left(1 - \phi^{-\frac{1}{\gamma}}\right) = (+) - (+) = (\pm). \tag{24}$$

$$\frac{\partial \alpha_c}{\partial \lambda} = -\beta(1-\gamma)\sigma_{P2}^2 \alpha_k \frac{\partial \alpha_k}{\partial \lambda} - \beta\left\{\frac{\phi - 1}{\gamma} + \phi\left(\phi^{-\frac{1}{\gamma}} - 1\right)\right\} = (+) - (+) = (\pm). \tag{25}$$

$$\frac{\partial \alpha_c}{\partial \ell} = -\beta(1-\gamma)\sigma_{P2}^2 \alpha_k \frac{\partial \alpha_k}{\partial \ell} = (+). \tag{26}$$

When partial insurance is optimal, each of the first term in derivatives α_c is given for intertemporal protection for holding undiversified durable goods as in Remark 2. An increase in the parameters ϕ, λ and ℓ decreases demand for protection with an decrease in demand for durable consumption goods. Then increases in ϕ, λ and ℓ partly increase perishable

consumption. However, from the second term in $\partial\alpha_c/\partial\phi$ and $\partial\alpha_c/\partial\lambda$ we see that increases in ϕ, λ decrease perishable consumption. This is because the second component represents consumption saving in order to use costly insurance. An increase in loss proportion ℓ dose not reduce consumption despite positive premium loading.

For $\alpha_k < \hat{\alpha_k}$, the marginal effects on α_c depend on several parameters. We will show the effects by way of numerical examples in Section 4.

4 Numerical Results

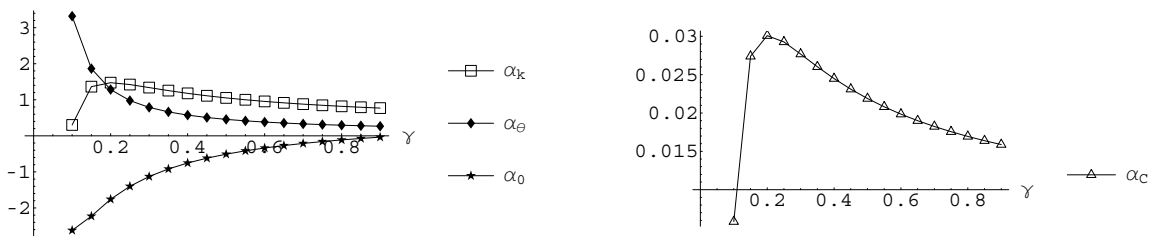
To further investigate optimal behavior in our model, we present numerical examples. We assume the following parameter values throughout the numerical part as the base scenario:

$$\begin{array}{llllll} r = 0.02, & \mu = 0.04, & \sigma_S = 0.20, & \mu_P = 0.03, & \delta = 0.02, & \sigma_{P1} = 0.07, \quad \sigma_{P2} = 0.07, \\ \gamma = 0.5, & \beta = 0.5, & \rho = 0.03, & \lambda = 1/50, & \ell = 0.8, & \phi = 1.2. \end{array} \tag{27}$$

Impact of risk aversion

We first set $\phi = 1$ to study the impact of γ when the insurance premium is fair. In this case, the optimal insurance policy provides full cover and other policies are consistent with Damgaard et al. (2003). From Figure 1, we see that the effects of risk aversion measure γ on perishable consumption is not uniform as implied in Merton (1969). We can see that γ also does not uniformly have an affect on holding policies for durable goods. This is because durable goods act as both physical assets and as consumption goods. Figure 2 shows the effects of $\phi > 1$ varying from $\gamma = 0.2, 0.5$ and $\gamma = 0.8$. For simplicity we restrict these variations within the interval while an increase in γ will uniformly reduce the amount of durable goods held (Figure 2 (a)). However, we showed that an increase in γ increases the optimal deductible level by (22). As a result, the impact on the optimal insurance coverage of γ is different from the value of ϕ as shown in Figure 2 (b). An increase in risk aversion measure can decrease insurance demand with the reduction of demand for insured assets itself. Such a result can be seen in the changing of λ and ℓ as shown in Figure 3 and Figure 4.

Figure 1: Impact of changing γ when $\phi = 1$



Impact of financial risk on insurance policies

From (22), we see that the optimal deductible level cannot be affected by financial risk although actual insurance coverage is influenced by financial risk through the amount of durable goods held. We examined the impact on the actual value of insurance coverage varying with σ_{P1} and σ_S from the base scenario (27). To study the analytical results presented in the previous section in detail, we show numerical examples of changing ϕ, λ and ℓ.

Figure 2: Impact of changing ϕ and variation in γ

(a) (b)

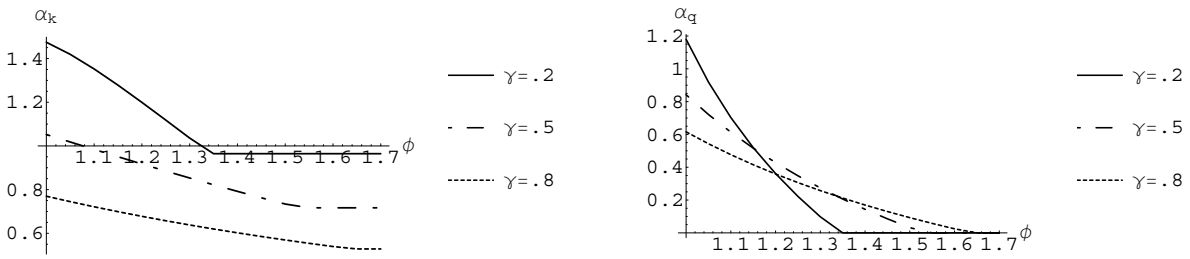


Figure 3: Impact of changing λ and variation in γ

(a) (b)

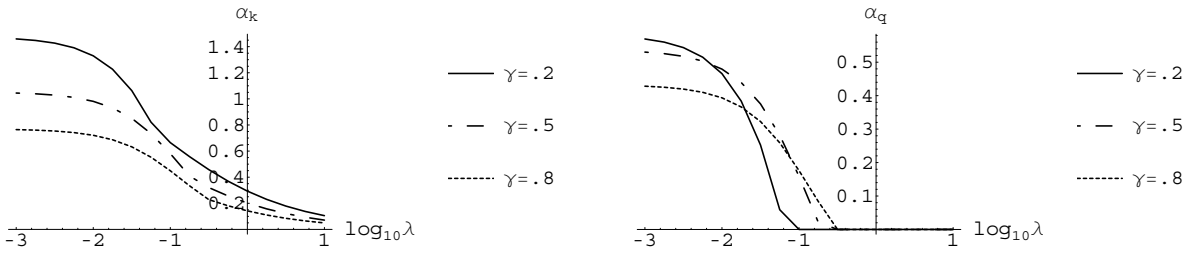
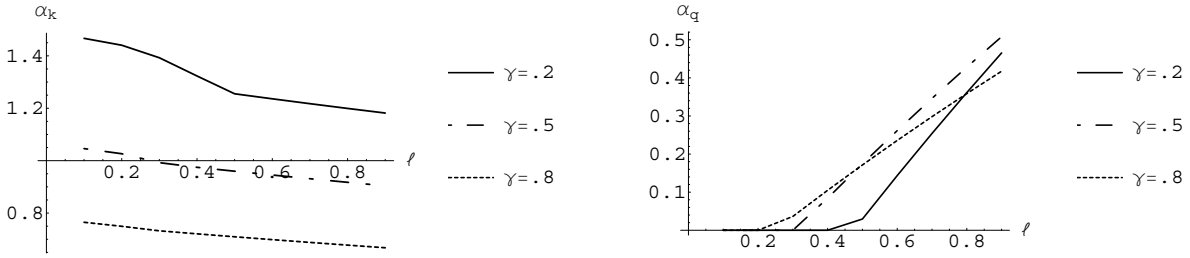


Figure 4: Impact of changing ℓ and variation in γ

(a) (b)



From Figure 5, we see that an increase in premium loading decreases demand for both durable consumption goods and insurance when partial insurance is optimal as shown by Proposition 1 and Proposition 2. We also see that both an increase in the volatility σ_{P1} and a decrease in the volatility σ_S decreases demand for durable consumption goods. Such a decrease causes a reduction of the actual insurance coverage. We can also see that when $\sigma_S = 0.1$, no insurance is optimal even when $\phi = 1.2$. This is because a low volatility of financial security decreases demand for relatively volatile durable consumption goods. From Figure 6, we see that both an increase in σ_{P1} and a decrease in σ_S also decreases demand for durable consumption goods. In this example, no insurance is optimal for all $\alpha_k < \hat{\alpha}_k = 0.381944$. We see that changes in financial risk can terminate insurance demand. Figure 7 shows the impact of loss proportion ℓ. Proposition 1 predicts an increase in ℓ decreases the amount of durable goods held however we can see that the impact is not large. We also see that change in financial risks can terminate the demand for insurance.

Effects on preishable consumption

From Proposition 3 and Proposition 4, it is satisfied that $\partial\alpha_c/\partial\lambda < 0$ for $\lambda < \hat{\lambda}$ (Figure 8 (a)). As in Remark 2, for $\lambda > \hat{\lambda}$, the increase in λ affects consumption in two ways: the first is protection demand for undiversified risk of durable consumption goods, the second is precautionary saving for self-insurance. We see that, from Figure 8 (a), $\partial\alpha_c/\partial\lambda$ is also negative for $\lambda > \hat{\lambda}$. It follows that, in the base scenario, effects on precautionary saving for self-insurance of an increase in λ is larger than the effects on the demand for protection.

From Proposition 4 and (26), it is satisfied that $\partial\alpha_c/\partial\ell > 0$ for $\ell > \hat{\ell}$ (Figure 8 (b)). For $\ell < \hat{\ell}$, an increase in ℓ also has an effect on consumption in two ways: the first is on demand for protection, the second is on precautionary saving. We see that the effect on precautionary savings is larger than the effects on demand for protection under the base scenario. We note that an increase in ℓ when partial insurance is optimal does not affect consumption savings due to the use of insurance. The reason for this is that in our model loading premium is added on the intensity λ but not on loss proportion ℓ.

5 Conclusions

We show the optimal insurance policy for durable consumption goods with positive premium loading in a framework of Merton (1969). Durable consumption goods can provide utility and can be traded as physical assets without transaction costs. The durable consumption goods' price process is assumed to be correlated with stock price process. We then show that a change in premium loading can affect investment policy through a change in the optimal amount of durable consumption goods held. We also show that financial risks affect the optimal insurance policy. We prove that there exist the critical parameter values where insurance demand will terminate. Numerical examples show that an increase in the risk aversion can decrease demand for insurance because of a decrease in demand for insured assets itself. Numerical examples also show that a change in the volatility of both durable goods price and of stock price has a large impact on the insurance policy.

In future work, we intend to explore the transaction cost problem on durable trading. Where the durable good is indivisible, in the sense that durable goods trading imply transition costs, Damgaard, et al. (2003) showed that the optimal policy for durable goods trading

Figure 5: Impact of changing ϕ and variation in σ_{P1} and σ_S

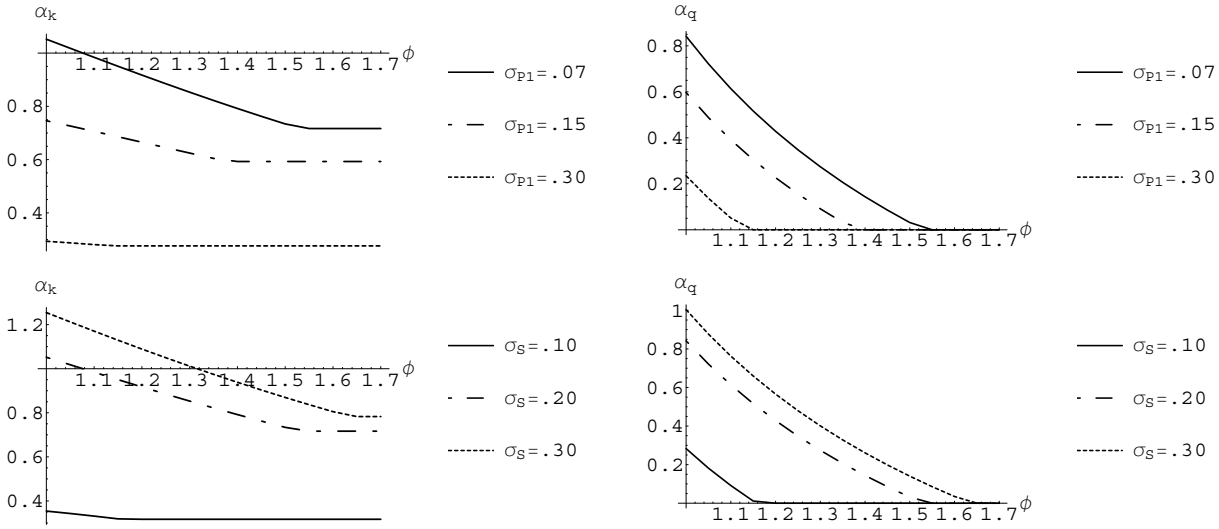


Figure 6: Impact of changing λ and variation in σ_{P1} and σ_S

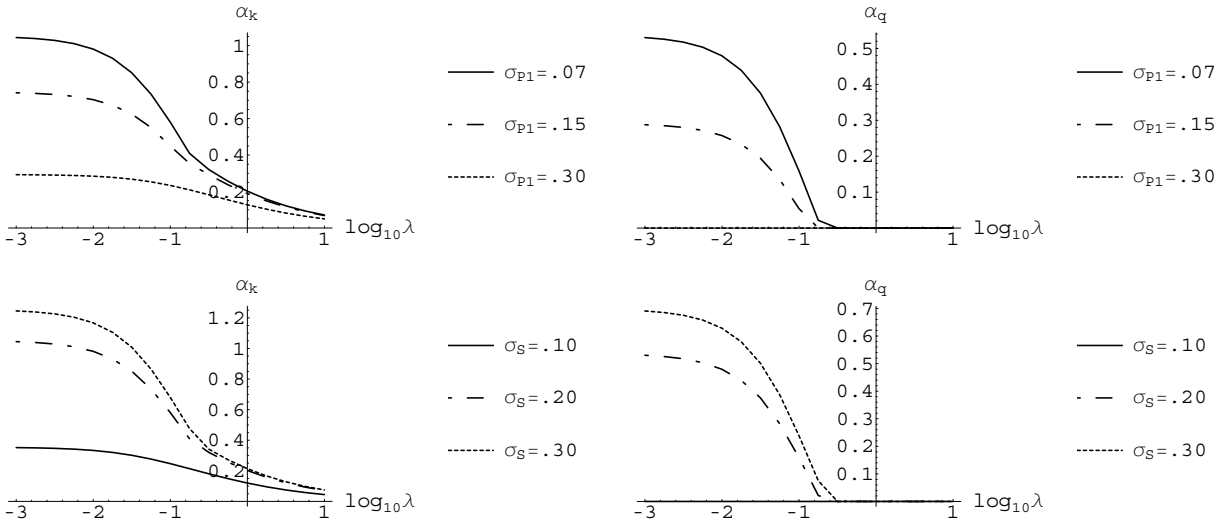


Figure 7: Impact of changing ℓ and variation in σ_{P1} and σ_S

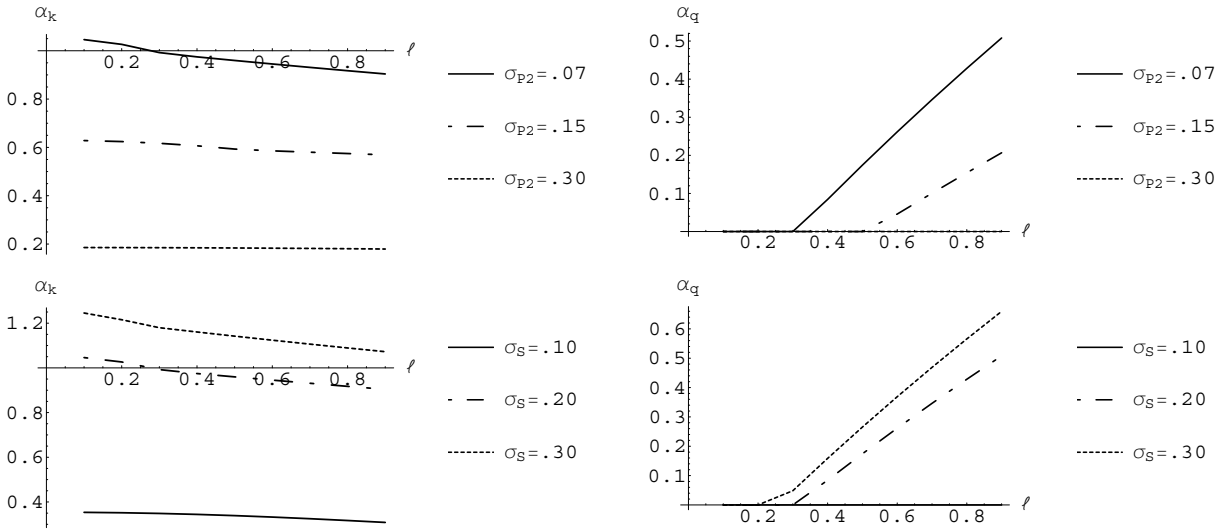
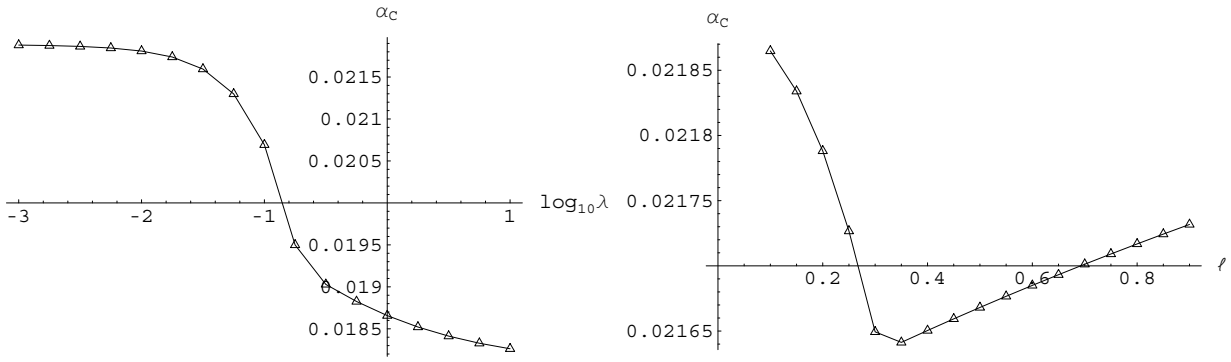


Figure 8: Impact on perishable consumption

(a) (b)



is represented by impulse control.

This paper was motivated by Gollier (1994, 2002, 2003) that investigated precautionary saving and time-diversification on the decision to insure the risk or not. In our model, precautionary saving is proportional to the wealth and the decision to follow whether self-insurance or external insurance is given by a static form. An agent who has dynamic constraint such as liquidity and leverage will have a dynamic policy for precautionary savings.

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