Extensions of the Wang transform for the pricing of insurance and financial risks *

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Abstract. It is well known that the Wang transform [15] for the pricing of financial and insurance risks is derived from Bühlmann's economic premium principle [1]. Recently, some extensions of the Wang transform are proposed, for example, a multivariate extension by Kijima [5] and a extended class of probability transforms by Kijima and Muromachi [9]. In this article we describe the essence of these recent extensions of the Wang transforms and show some special examples related to the Student's t distributions.

Keywords: Bühlmann's economic premium principle, Wang transform, Gaussian copula, non-central t distribution

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1 Introduction

In the actuarial literature, there have been developed many probability transforms for pricing financial and insurance risks. Such methods include the variance loading, the standard deviation loading, and the Esscher transform. Recently, Wang [13] proposed a universal pricing method based on the following transformation from $F(x)$ to $F^*(x)$:

$$F^*(x) = \Phi[\Phi^{-1}(F(x)) + \theta], \tag{1}$$

where Φ denotes the standard normal cumulative distribution function (CDF for short) and θ is a constant. The transform is now called the *Wang transform* and produces a risk-adjusted CDF $F^*(x)$. The mean value evaluated under $F^*(x)$ will define a risk-adjusted "fair value" of risk X with CDF $F(x)$ at some future time, which can be discounted to time zero using the risk-free interest rate. The parameter θ is considered to be a risk premium.

The Wang transform not only possesses various desirable properties as a pricing method, but also has a sound economic interpretation. For example, the Wang transform (as well as the Esscher transform) is the only distortion function, among the family of distortions, that can recover CAPM (the capital asset pricing model) for underlying assets and the Black-Scholes formula for options. See Wang [14] and Kijima [5] for details.

Among them, the most striking result on the transform (1) is that it is consistent with Bühlmann's economic premium principle. Wang [15] showed that the transform (1) can be derived from Bühlmann's principle under some assumptions on the aggregate risk. The result is extended to the multivariate setting by Kijima [5]. Additionally, Kijima and Muromachi [9] derived a class of transforms that are consistent with Bühlmann's principle, thereby extending the results of Wang [14]. Namely, based on the idea of Kijima [5], they obtained the transform

$$F^*(x) = E[\Phi(G^{-1}(F(x))Y + \theta)], \tag{2}$$

where Y is *any* positive random variable and the expectation is taken with respect to Y. Here, $G(x)$ denotes the CDF of random variable U/Y and U represents a standard normal random variable, independent of Y. In particular, when $Y = 1$ almost surely, we have $G(x) = \Phi(x)$, so that the transform (2) is reduced to the Wang transform (1). Kijima and Muromachi [9] extended the transform (2) to the multivariate setting by using a Gaussian copula, in order to preserve the linearity for the pricing functional.[1] A multiperiod extension

[1] The pricing functional π is said to be linear if $\pi(aX + bY) = a\pi(X) + b\pi(Y)$ for all risks X, Y and

was also discussed in their paper.

It is often said that a drawback of the Wang transform (1) is the normal distributions, that never match the fat-tailness observed in the actual markets. In fact, some empirical studies suggest to use t distributions, whose CDF is denoted by $T_\nu(x)$, with $\nu = 3$ or 4 degree of freedom for return distributions of financial and insurance assets (see, e.g., Platen and Stahl [12]). Hence, it is natural to consider the case that $Y = \sqrt{\chi_\nu^2/\nu}$, where χ_ν^2 denotes a chi-square random variable with ν degree of freedom. As Kijima and Muromachi [8] observed, this case leads to the two-parameter transformation

$$F^*(x) = P_{\nu:-\lambda}[T_\nu^{-1}(F(x))], \tag{3}$$

where $P_{\nu:\delta}$ denotes the CDF of non-central t distribution with ν degree of freedom and non-centrality parameter δ. However, contrary to our intuition, the risk-adjusted distribution (3) derived from t distributions is not fatter in tail parts than the original Wang transform (1) that is derived from normal distributions. In order to make more fatter tails, Wang [14] proposed a two-parameter transformation

$$F^*(x) \quad = \quad T_\nu[\Phi^{-1}(F(x)) + \theta],$$

which is called *two-parameter Wang transform.* Unfortunately this transform is not consistent with Bühlmann's economic premium principle, however, according to Wang [16], it can explain the market prices of financial/insurance products such as catastrophic bonds more consistently than the one-parameter Wang transform.

In this paper we describe the essence of some recent extensions of the one-parameter Wang transforms proposed by Kijima [5] and Kijima and Muromachi [9] in the univariate and multivariate settings. The present paper is organized as follows. In the next section, we review the result of Bühlmann [1] and derive the one-parameter Wang transform (1). Using the idea presented in Section 2, the general transform (2) is derived, and a special case related to the Student's t distribution is described in Section 3. The multivariate extensions are described in Section 4, and Section 5 concludes the article.

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constants a, b. If it is not linear, arbitrage opportunities are not precluded. See, e.g., Harrison and Kreps [2] and Kijima [4] for details.

2 Bühlmann's principle and the Wang transform

Bühlmann [1] considered a single-period economy for risk exchanges among a set of agents $j = 1, 2, \ldots, n$. Each agent is characterized by an exponential utility function $u_j(x) = -\mathrm{e}^{-\lambda_j x}$, $x \geq 0$, and initial wealth w_j. Suppose that agent j faces a risk of potential loss X_j and is willing to buy/sell a risk exchange Y_j. If agent j is an insurance company, the risk exchange Y_j is thought of the sum of all insurance policies sold by j. While the original risk X_j belongs to agent j, the risk exchange Y_j can be freely bought/sold by the agents in the market. Denoting the price of Y_j by $\pi(Y_j)$, the equilibrium price for this risk exchange economy is characterized by:

1. For any j, $E[u_j(w_j - X_j + Y_j - \pi(Y_j))]$ is maximized with respect to Y_j, and
2. $\sum_{j=1}^{n} Y_j = 0$ for all possible states.

In this setting, Bühlmann [1] derived that the equilibrium price $\pi(Y)$ for the risk exchange is given by

$$\pi(X) = E[\eta X], \qquad \eta = \frac{\mathrm{e}^{-\lambda Z}}{E[\mathrm{e}^{-\lambda Z}]}, \tag{4}$$

where $Z = \sum_{j=1}^{n} X_j$ is the aggregate risk and λ is given by

$$\lambda^{-1} = \sum_{j=1}^{n} \lambda_j^{-1}, \qquad \lambda_j > 0.$$

The parameter λ is thought of the *risk aversion index* of the representative agent in the market.

Wang [15] showed that the transform (1) can be derived from Bühlmann's economic premium principle (4) under the following assumptions on the aggregate risk $Z = \sum_{j=1}^{n} X_j$:

3. There are so many individual risks X_j in the market that the aggregate risk Z can be approximated by a normal random variable, and
4. The correlation coefficient between $Z_0 = (Z - \mu_Z)/\sigma_Z$ and $U = \Phi^{-1}[F(X)]$ is ρ, [2] where $\mu_Z = E[Z]$ and $\sigma_Z^2 = V[Z]$.

[2]This assumption can be stated that the random vector $(F(X), \Phi(Z_0))$ follows a bivariate Gaussian copula with correlation coefficient ρ.

Here, $F(x)$ denotes the CDF of risk X of interest. For the sake of simplicity, it is assumed that $F(x)$ is strictly increasing in x. The inverse function of $F(x)$ is denoted by $F^{-1}(x)$.

If the random vector (Z_0, U) follows a bivariate normal distribution with correlation coefficient ρ, there exists a normal random variable ξ, independent of (Z_0, U), such that $Z_0 = \rho U + \xi$. Hence, from (4), it follows that

$$\pi(X) = \frac{E[X\mathrm{e}^{-\theta U}]}{E[\mathrm{e}^{-\theta U}]}, \qquad \theta = \lambda\sigma_Z\rho.$$

Since $U = \Phi^{-1}[F(X)]$ follows a standard normal distribution, we obtain

$$\pi(X) = \mathrm{e}^{-\theta^2/2}E[X\mathrm{e}^{-\theta U}] = \mathrm{e}^{-\theta^2/2}\int_R x\mathrm{e}^{-\theta\Phi^{-1}[F(x)]}\mathrm{d}F(x), \tag{5}$$

where R stands for the real line.

We intend to write the pricing functional $\pi(X)$ in terms of a transformed CDF $F^*(x)$ such that

$$\pi(X) = \int_R x\mathrm{d}F^*(x) = E^*[X],$$

where E^* stands for the expectation operator associated with the CDF $F^*(x)$. It follows from (5) that

$$F^*(x) = \mathrm{e}^{-\theta^2/2}\int_{-\infty}^{x}\mathrm{e}^{-\theta\Phi^{-1}[F(y)]}\mathrm{d}F(y). \tag{6}$$

For any x, let $I_x(y) = 1$ if $y \le x$ and $I_x(y) = 0$ otherwise. Then, using the function $I_x(y)$, (6) can be written as

$$\begin{aligned} F^*(x) &= \mathrm{e}^{-\theta^2/2}E\left[I_x(X)\mathrm{e}^{-\theta\Phi^{-1}[F(X)]}\right] \\ &= \mathrm{e}^{-\theta^2/2}E\left[I_x(F^{-1}(\Phi(U)))\mathrm{e}^{-\theta U}\right], \end{aligned}$$

where the second equality follows since $U = \Phi^{-1}[F(X)]$ by assumption. Applying a lemma [3] in Kijima and Muromachi [7], we obtain

$$F^*(x) = E\left[I_x(F^{-1}(\Phi(U - \theta)))\right],$$

[3]The lemma is that for any bivariate normal random vector (X, Y) and any function $h(x)$, we have

$$E[h(X)\mathrm{e}^{-Y}] = E[\mathrm{e}^{-Y}]E[h(X - Cov[X, Y])] \tag{7}$$

when the expectations exist.

since $E[\mathrm{e}^{-\theta U}] = \mathrm{e}^{\theta^2/2}$ and $Cov[U, \theta U] = \theta$. It follows from the definition of $I_x(y)$ that

$$\begin{aligned} F^*(x) &= P\{F^{-1}(\Phi(U - \theta)) \le x\} \\ &= P\{U - \theta \le \Phi^{-1}(F(x))\} \\ &= \Phi[\Phi^{-1}(F(x)) + \theta], \qquad \theta = \lambda \sigma_Z \rho, \end{aligned}$$

which is the Wang transform (1).

3 An extension to the class of probability transforms

Given a CDF $F(x)$ of risk X, suppose that there exist random variables U and $Y > 0$, independent of each other, such that U follows a standard normal distribution and

$$F(X) = G(U/Y), \qquad Y > 0,$$

where $G(x)$ denotes the CDF of random variable U/Y. For example, when $Y = 1$, we have $G(x) = \Phi(x)$ as for the univariate Wang transform. When $Y = \sqrt{\chi_\nu^2/\nu}$, where χ_ν^2 denotes a chi-square random variable with ν degree of freedom, the random variable U/Y follows a t distribution with ν degree of freedom. However, at this point, we do not specify the CDF $G(x)$. We only assume that $G(x)$ is strictly increasing in x for the sake of simplicity. In summary, our assumption on the aggregate risk can be stated as

$$Z_0 = \rho U + \xi, \qquad U = Y\alpha(X), \tag{8}$$

where $\alpha(x) \equiv G^{-1}(F(x))$ and U and ξ are normally distributed random variables, independent of each other.

As for (5), we have

$$\begin{aligned} \pi(X) &= \mathrm{e}^{-\theta^2/2} E\left[E\left[X\mathrm{e}^{-\theta Y\alpha(X)}\middle|Y\right]\right] \\ &= \mathrm{e}^{-\theta^2/2} E\left[\int_R x\mathrm{e}^{-\theta Y\alpha(x)}\mathrm{d}F(x)\right]. \end{aligned}$$

It follows that (cf. (6))

$$F^*(x) = \mathrm{e}^{-\theta^2/2} E\left[\int_{-\infty}^x \mathrm{e}^{-\theta Y\alpha(z)}\mathrm{d}F(z)\right],$$

where the expectation is taken with respect to Y. Using the function $I_x(y)$, we obtain

$$\begin{aligned} F^*(x) &= \mathrm{e}^{-\theta^2/2} E\left[E\left[I_x(X)\mathrm{e}^{-\theta Y\alpha(X)}\middle|Y\right]\right] \\ &= \mathrm{e}^{-\theta^2/2} E\left[E\left[I_x(\alpha^{-1}(U/Y))\mathrm{e}^{-\theta U}\middle|Y\right]\right]. \end{aligned}$$

Since Y and U are independent of each other by assumption, we obtain from (7) that

$$E\left[I_x(\alpha^{-1}(U/Y))\mathrm{e}^{-\theta U}\Big|Y\right] = \mathrm{e}^{\theta^2/2}E\left[I_x(\alpha^{-1}((U-\theta)/Y))\Big|Y\right].$$

It follows from the definition of $I_x(y)$ that

$$\begin{aligned} F^*(x) &= E\left[E\left[I_x(\alpha^{-1}((U-\theta)/Y))\Big|Y\right]\right] \\ &= E\left[P\left\{\alpha^{-1}((U-\theta)/Y)) \le x\Big|Y\right\}\right] \\ &= E\left[P\left\{U \le \alpha(x)Y + \theta\Big|Y\right\}\right]. \end{aligned}$$

Since U follows the standard normal distribution, we finally obtain the transformation (2), i.e.

$$F^*(x) = E[\Phi(\alpha(x)Y + \theta)], \qquad \theta = \lambda\sigma_Z\rho, \quad \alpha(x) = G^{-1}(F(x)). \tag{9}$$

When the aggregated risk Z and $U = \Phi^{-1}[F(X)]$ are uncorrelated, i.e. $\rho = 0$ in (8), so that $\theta = 0$ in (9), we have $F^*(x) = F(x)$. This means that no distortion takes place for the uncorrelated case.

For the CDF $F^*(x)$ given by (9), we denote the random variable associated with $F^*(x)$ by X^*. Recall that the risk under consideration is the random variable X with CDF $F(x)$. If these random variables represent the amount of a profit/loss, we take $\theta > 0$. The other case can be treated similarly. In the next theorem, we call X greater than X^* in the sense of *first order stochastic dominance*, denoted by $X \ge_{\mathrm{FSD}} X^*$, if $F(x) < F^*(x)$ for all x.[4]

Theorem 3.1 *If $\theta > 0$, then $X \ge_{\mathrm{FSD}} X^*$. That is, the investor evaluates the loss worse than the actual loss when pricing the risk X.*

For the proof, see Kijima and Muromachi [9].

3.1 A special case related to t distribution

Suppose that $Y > 0$ is a continuous random variable with probability density function (PDF for short) $g(x)$. In order to obtain the PDF of U/Y, we need to consider the joint PDF of $(U/Y, Y)$. Denoting its joint PDF by $h(x, y)$, we have

$$h(x, y) = \phi(xy)g(y)y, \qquad y > 0, \quad x \in R,$$

[4]See Kijima and Ohnishi [10] for the application of stochastic dominance relations to finance.

where $\phi(u)$ denotes the PDF of the standard normal distribution. Hence, the CDF of U/Y is obtained as

$$G(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} h(u,v)\mathrm{d}v\mathrm{d}u.$$

The risk-adjusted CDF $F^*(x)$ is then calculated by (9).

In particular, when $Y = \sqrt{\chi^2_\nu/\nu}$, it is well known that the random variable U/Y follows a t distribution with ν degree of freedom, whose CDF is denoted by $T_\nu(x)$, i.e. $G(x) = T_\nu(x)$. On the other hand, the transform (9) is written as

$$F^*(x) = E[\Phi(\alpha(x)Y + \theta)] = P\left\{\frac{U-\theta}{Y} \le \alpha(x)\right\}.$$

Recall that the random variable $(U+\delta)/Y$ follows a non-central t distribution with ν degree of freedom and non-centrality parameter δ whose PDF is given by (see, e.g., Johnson and Kotz [3])

$$p_{\nu:\delta}(t) = \left(\frac{1}{2}\right)^{\frac{\nu-1}{2}} \frac{\mathrm{e}^{-\frac{1}{2}\delta^2}}{\sqrt{\pi\nu}\Gamma(\frac{\nu}{2})} \int_0^\infty x^\nu \exp\left\{-\frac{1}{2}\left[\left(1+\frac{t^2}{\nu}\right)x^2 - 2\frac{t\delta}{\sqrt{\nu}}x\right]\right\}\mathrm{d}x.$$

Denoting the CDF of the non-central t distribution by $P_{\nu:\delta}(x)$, we arrive at the two-parameter transformation (3), i.e.

$$F^*(x) = P_{\nu:-\theta}[T_\nu^{-1}(F(x))], \qquad \theta = \lambda\sigma_Z\rho. \tag{10}$$

Note that, when $F(x) = T_\nu(x)$, the transform (10) distorts the t distribution with ν degree of freedom to a non-central t distribution with the same degree of freedom and non-centrality parameter $-\theta$, where θ represents the risk premium. A more elementary derivation of (10) is given by Kijima and Muromachi [8].

Remark 3.1 Since χ^2_ν/ν converges to unity as $\nu \to \infty$, the non-central t distribution converges to a shifted normal distribution with shift parameter δ. It follows that the new transformation (10) converges to the Wang transform (1) as $\nu \to \infty$.

Kijima and Muromachi [9] gave a proof of the following theorem, which means that the Wang transform (1) produces the fattest tail distributions among the class of probability transforms given by (9).

Theorem 3.2 *Suppose $\theta \ge 0$. Then, for all x such that $F(x) < 1/2$ and for $\theta \ge 0$, we have*

$$\Phi[\Phi^{-1}(F(x)) + \theta] \quad \ge \quad E[\Phi(G^{-1}(F(x))Y + \theta)]$$

for all $Y > 0$, where $G(x)$ is the CDF of random variable U/Y.

Some numerical comparisons of the tail distributions with different parameters are shown in Kijima and Muromachi [9]. Among them, we consider the risk-adjusted CDF $F^*(x)$ for the given CDF $F(x)$ of a risk X. Figure 1 depicts the risk-adjusted CDF $F^*(x)$ when the original risk follows a t distribution with $\nu = 3$ degrees of freedom, i.e. $F(x) = T_3(x)$, for $\theta = 0.7$, which corresponds to the case of a relatively high risk premium. Recall that empirical studies suggest to use such distributions as $F(x) = T_3(x)$ for actual asset returns. The transformed CDF calculated by (10) is denoted by

$$F_\nu^t(x) \quad = \quad P_{\nu:-\theta}[T_\nu^{-1}(T_3(x))]. \tag{11}$$

Note that, for $\nu = 3$, the transformation (11) produces a non-central t distribution, i.e. $F_3^t(x) = P_{3:-\theta}(x)$. For other ν, the transformed CDF $F_\nu^t(x)$ is not a non-central t distribution. Figure 1 shows that the tail parts of the risk-adjusted distributions $F_\nu^t(x)$ are increasing in ν. That is, the transformation (11) produces fatter risk-adjusted CDF as Y becomes less variable. This result is consistent with Theorem 3.2. Kijima and Muromachi [9] show that the higher the risk premium, the more definitely the situation becomes.

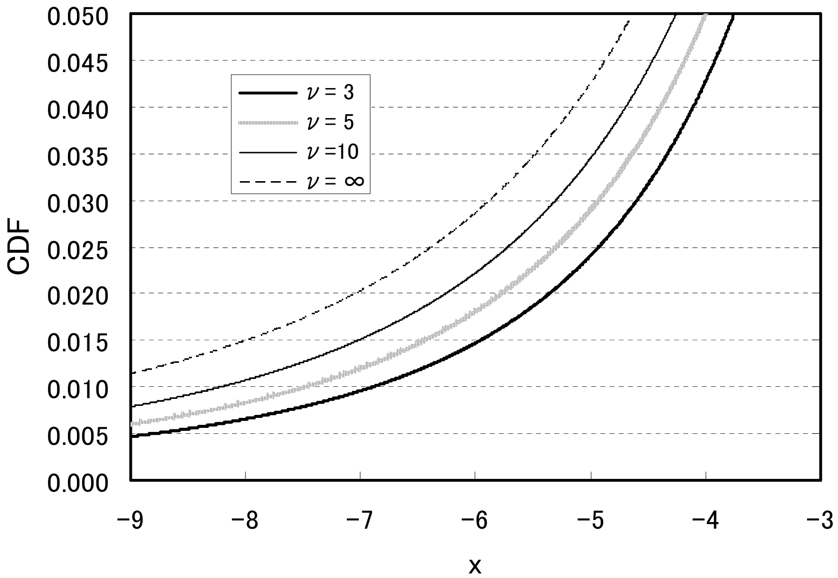

Figure 1: Risk adjusted CDFs when risk X follows T_3 for $\theta = 0.7$.

Now, we know that the Wang transform (1) produces the fattest tail distributions among the class of t distributions. However, as Wang [14] reported, the one-parameter transform

(1) is not flexible enough to match the actual market data. Thus, Wang [14] proposed the two-parameter transformation

$$F^*(x) = T_\nu[\Phi^{-1}(F(x)) + \theta], \tag{12}$$

and reported that (12) is much better to fit to the market prices, although the two-parameter transform is not consistent with the economic premium principle (4), thereby lacking a sound economic interpretation.

Figure 2 compares the tail distributions of the one-parameter and the two-parameter Wang transforms with $\nu = 5$ and $\theta = 0.2$. As expected, the two-parameter transformation (12) has fatter tails than the one-parameter counterpart.

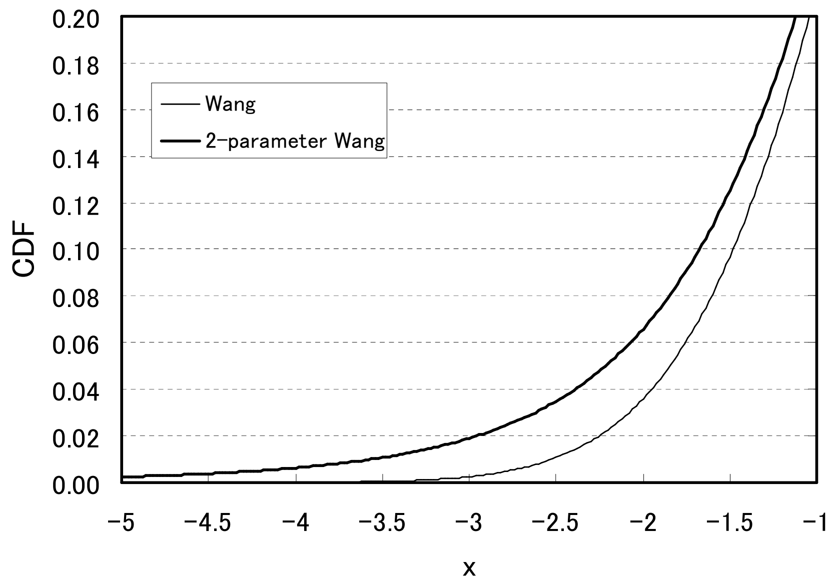

Figure 2: One-parameter and two-parameter Wang Transforms.

4 A multivariate extension

In this section, we consider a multivariate extension of the new transformation (9). As in Kijima [5], suppose that the underlying risks are described by a multivariate random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ and the σ-algebra is given by $\mathcal{F} = \sigma(X_1, X_2, \ldots, X_n)$. Note that a particular risk, Y say, is an $\mathcal{F}$-measurable random variable. That is, there exists an n-variate function $h(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, such that $Y = h(\boldsymbol{X})$ in general. Then, Bühlmann's economic premium principle is given by

$$\pi(Y) = \frac{E[h(\boldsymbol{X})\mathrm{e}^{-\lambda Z}]}{E[\mathrm{e}^{-\lambda Z}]}, \qquad Z = \sum_{j=1}^{n} X_j. \tag{13}$$

It is clear that this pricing functional $\pi(X)$ is linear.

The pricing formula (13) can be described in terms of the probability distortion (or the *change of measures* in other words). For the sake of simplicity, suppose that the joint PDF of $\boldsymbol{X}$ exists, which is denoted by $f(\boldsymbol{x})$. The PDF defined by

$$f^*(\boldsymbol{x}) \quad = \quad \frac{\mathrm{e}^{-\lambda z}}{E[\mathrm{e}^{-\lambda Z}]} f(\boldsymbol{x}), \qquad z = \sum_{j=1}^{n} x_j,$$

provides the risk-adjusted distortion. That is, Bühlmann's equilibrium price is obtained as

$$\pi(Y) = \int_{R^n} \frac{h(\boldsymbol{x})\mathrm{e}^{-\lambda z}}{E[\mathrm{e}^{-\lambda Z}]} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_{R^n} h(\boldsymbol{x}) f^*(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = E^*[h(\boldsymbol{X})],$$

where E^* denotes the expectation operator associated with $f^*(\boldsymbol{x})$.

4.1 The multivariate Wang transform

We first derive the multivariate Wang transform. Consider a Gaussian copula [5] for the underlying risks $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$. That is, define

$$U_j \equiv \Phi^{-1}[F_j(X_j)], \qquad j = 1, 2, \ldots, n, \tag{14}$$

where $F_j(x)$ is the marginal CDF of X_j. For the sake of simplicity, it is assumed that $F_j(x)$ is strictly increasing in x for all j. A Gaussian copula assumes that $\boldsymbol{U} = (U_1, U_2, \ldots, U_n)$ follows an n-variate standard normal distribution with correlation matrix $\boldsymbol{\Sigma}_\rho = (\rho_{ij})$.

As in the univariate case, suppose that the standardized aggregate risk $Z_0 = (Z - \mu_Z)/\sigma_Z$ is expressed as

$$Z_0 = \xi + \rho \sum_{j=1}^{n} w_j U_j, \qquad U_j = \Phi^{-1}[F_j(X_j)], \tag{15}$$

where ξ is a normal random variable, independent of the other random variables. It follows from (13) and (15) that

$$\begin{aligned} \pi(Y) \quad &= \quad \frac{E[h(\boldsymbol{X})\mathrm{e}^{-\sum_{j=1}^{n}\theta_j U_j}]}{E[\mathrm{e}^{-\sum_{j=1}^{n}\theta_j U_j}]} \\ &= \quad \mathrm{e}^{-\sigma_U^2/2} E\left[h(\boldsymbol{X})\mathrm{e}^{-\sum_{j=1}^{n}\theta_j U_j}\right], \qquad \theta_j = \lambda\sigma_Z \rho w_j, \end{aligned} \tag{16}$$

[5] See Nelsen [11] for details of copulas.

where the second equation follows since $U \equiv \sum_{j=1}^n \theta_j U_j$ is normally distributed with mean 0 and variance $\sigma_U^2 \equiv \sum_{i,j} \theta_i \theta_j \rho_{ij}$ by assumption. It follows from (14) and (16) that the risk-adjusted PDF is given by

$$f^*(\boldsymbol{x}) = \mathrm{e}^{-\sigma_U^2/2} \mathrm{e}^{-\sum_{j=1}^n \theta_j \alpha_j(x_j)} f(\boldsymbol{x}), \qquad \alpha_j(x_j) = \Phi^{-1}[F_j(x_j)]. \tag{17}$$

As for the univariate case, let $I_{\boldsymbol{x}}(\boldsymbol{y}) = 1$ if $y_j \leq x_j$ for all $j = 1, 2, \ldots, n$ and $I_{\boldsymbol{x}}(\boldsymbol{y}) = 0$ otherwise. Then, integrating (17) from $-\infty$ to x_j for each component j and using the function $I_{\boldsymbol{x}}(\boldsymbol{y})$, the risk-adjusted CDF $F^*(\boldsymbol{x})$ can be written as

$$\begin{aligned} F^*(\boldsymbol{x}) &= \mathrm{e}^{-\sigma_U^2/2} E\left[I_{\boldsymbol{x}}(\boldsymbol{X}) \mathrm{e}^{-\sum_{j=1}^n \theta_j \alpha_j(X_j)}\right] \\ &= \mathrm{e}^{-\sigma_U^2/2} E\left[I_{\boldsymbol{x}}(\beta(\boldsymbol{U})) \mathrm{e}^{-U}\right], \end{aligned} \tag{18}$$

where $\beta(\boldsymbol{U}) = (F_1^{-1}(\Phi(U_1)), \ldots, F_n^{-1}(\Phi(U_n)))$.

Here, we use the multivariate version of (7), that is, for any multivariate normal random vector $(\boldsymbol{X}, Y)$ and any function $h(\boldsymbol{x})$, we have

$$E[h(\boldsymbol{X})\mathrm{e}^{-Y}] = E[\mathrm{e}^{-Y}] E[h(\boldsymbol{X} - Cov[\boldsymbol{X}, Y])] \tag{19}$$

for which the expectations exist. $Cov[\boldsymbol{X}, Y]$ means the vector with components $Cov[X_j, Y]$ [6]. From (18) and (19), we obtain

$$F^*(x) = E\left[I_{\boldsymbol{x}}(\beta(\boldsymbol{U} - \boldsymbol{\theta}_\rho))\right], \qquad \boldsymbol{\theta}_\rho = (\theta_1^\rho, \theta_2^\rho, \ldots, \theta_n^\rho),$$

where $\theta_j^\rho \equiv Cov[U_j, U] = \sum_{i=1}^n \theta_i \rho_{ij}$. It follows from the definition of $I_{\boldsymbol{x}}(\boldsymbol{y})$ that the risk-adjusted CDF $F^*(\boldsymbol{x})$ is obtained as

$$\begin{aligned} F^*(\boldsymbol{x}) &= P\left\{U_1 \leq \Phi^{-1}[F_1(x_1)] + \theta_1^\rho, \ldots, U_n \leq \Phi^{-1}[F_n(x_n)] + \theta_n^\rho\right\} \\ &= \Phi_n\left(\Phi^{-1}[F_1(x_1)] + \theta_1^\rho, \ldots, \Phi^{-1}[F_n(x_n)] + \theta_n^\rho\right), \qquad \theta_j^\rho = \sum_{i=1}^n \theta_i \rho_{ij}. \end{aligned} \tag{20}$$

Note that, when $n = 1$, (20) coincides with the Wang transform (1) since $\rho_{11} = 1$. Thus, Kijima [5] called (20) the *multivariate Wang transform*.

[6]See Kijima and Miyake [6] for the trivariate case. Their proof can be extended to the multivariate case with ease.

4.2 An extension to the class of probability transforms

Given the underlying risks $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, suppose that there exist a Gaussian copula $\boldsymbol{U} = (U_1, U_2, \ldots, U_n)$ with correlation matrix $\boldsymbol{\Sigma}_\rho = (\rho_{ij})$ and a positive random variable Y, independent of $\boldsymbol{U}$, such that

$$F_j(X_j) = G_j(U_j/Y), \qquad j = 1, 2, \ldots, n, \tag{21}$$

where $F_j(x_j)$ is the marginal CDF of risk X_j and $G_j(x)$ denotes the CDF of random variable U_j/Y. It is assumed that each $G_j(x)$ is strictly increasing in x for the sake of simplicity. Note that, when $Y = 1$ almost surely, the assumption (21) agrees with (14).

Suppose that the standardized aggregate risk $Z_0 = (Z - \mu_Z)/\sigma_Z$ is expressed as

$$Z_0 = \xi + \rho \sum_{j=1}^{n} w_j U_j, \qquad U_j = Y\alpha_j(X_j),$$

where $\alpha_j(x) = G_j^{-1}(F_j(x))$ and ξ is a normal random variable, independent of the other random variables. Since nothing has been changed except the definition of U_j, the risk-adjusted CDF $F^*(\boldsymbol{x})$ is expressed as

$$F^*(\boldsymbol{x}) = \mathrm{e}^{-\sigma_U^2/2} E\left[I_{\boldsymbol{x}}(\beta(\boldsymbol{U}, Y))\mathrm{e}^{-U} \right], \tag{22}$$

where $U = \sum_{j=1}^n \theta_j U_j$ and $\beta(\boldsymbol{U}, Y) \equiv (\alpha_1^{-1}(U_1/Y), \ldots, \alpha_n^{-1}(U_n/Y))$ for this case.

Here, we apply the result (19) to (22). Since Y and $\boldsymbol{U} = (U_1, U_2, \ldots, U_n)$ are mutually independent by assumption, we obtain from (19) that

$$E\left[I_{\boldsymbol{x}}(\beta(\boldsymbol{U}, Y)\mathrm{e}^{-U} \right] \;=\; \mathrm{e}^{\sigma_U^2/2} E\left[E\left[I_{\boldsymbol{x}}(\beta(\boldsymbol{U} - \boldsymbol{\theta}_\rho, Y)) \middle| Y \right] \right], \qquad \boldsymbol{\theta}_\rho = (\theta_1^\rho,\, \theta_2^\rho, \ldots,\, \theta_n^\rho).$$

Then, it follows from the definition of $I_{\boldsymbol{x}}(\boldsymbol{y})$ that

$$\begin{aligned} F^*(\boldsymbol{x}) &= E\left[E\left[I_{\boldsymbol{x}}(\beta(\boldsymbol{U} - \boldsymbol{\theta}_\rho, Y)) \middle| Y \right] \right] \\ &= E\left[P\left\{ \beta(\boldsymbol{U} - \boldsymbol{\theta}_\rho, Y) \le \boldsymbol{x} \middle| Y \right\} \right] \\ &= E\left[P\left\{ U_1 \le \alpha_1(x_1)Y + \theta_1^\rho,\, \ldots,\, U_n \le \alpha_n(x_n)Y + \theta_n^\rho \middle| Y \right\} \right]. \end{aligned}$$

Therefore, we obtain the multivariate extension of the generalized transform as

$$F^*(\boldsymbol{x}) \;=\; E\left[\Phi_n\left(\alpha_1(x_1)Y + \theta_1^\rho, \ldots, \alpha_n(x_n)Y + \theta_n^\rho \right) \right],$$

where $\alpha_j(x_j) \equiv G_j^{-1}(F_j(x_j))$ and $\theta_j^\rho = \sum_{i=1}^n \theta_i \rho_{ij}$. Note that, when $Y = 1$, we obtain (20).

As a special case, we consider the case when $Y = \sqrt{\chi_\nu^2/\nu}$ as in Kijima and Muromachi [8]. Then, we obtain the following multivariate transform with t copula:

$$F^*(\boldsymbol{x}) = P(\boldsymbol{\beta}; n, \nu, -\boldsymbol{\theta}, \boldsymbol{\Sigma}_\rho) = \int^{\boldsymbol{\beta}} p(\boldsymbol{u}; n, \nu, -\boldsymbol{\theta}, \boldsymbol{\Sigma}_\rho)\mathrm{d}\boldsymbol{u},$$

where $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_n)$ with $\beta_j = T_\nu^{-1}(F_j(x_j))$, $j = 1, \cdots, n$. Here,

$$\begin{aligned} p(\boldsymbol{u}; n, \nu, \boldsymbol{\delta}, \boldsymbol{\Sigma}_\rho) \quad \equiv \quad & \left(\frac{1}{2}\right)^{\frac{\nu+n}{2}-1} \frac{\exp\left\{-\frac{1}{2}\boldsymbol{\delta}^\top \boldsymbol{\Sigma}_\rho \boldsymbol{\delta}\right\}}{(\pi\nu)^{n/2}\Gamma(\frac{\nu}{2})} \frac{1}{\sqrt{|\boldsymbol{\Sigma}_\rho|}} \\ & \times \int_0^\infty z^{\nu+n-1} \exp\left\{-\frac{1}{2}\left[\left(1 + \frac{\boldsymbol{u}^\top \boldsymbol{\Sigma}_\rho^{-1} \boldsymbol{u}}{\nu}\right) z^2 - 2\frac{\boldsymbol{u}^\top \boldsymbol{\delta}}{\sqrt{\nu}} z\right]\right\} \mathrm{d}z, \end{aligned}$$

where $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_n)$ and $\boldsymbol{\Sigma}_\rho = (\rho_{ij})$ is a correlation matrix.

5 Concluding remarks

In this paper, we describe some recent extensions of the Wang transform. These extensions are derived from Bühlmann's economic premium principle, therefore, they have sound economic interpretations. However, in order to use them in practice, it is necessary to develop a robust parameter estimation method, especially for the multivariate settings.

On the Wang transform, an important future research theme is to develop probability transforms that can produce fatter tail distributions and has a sound economic interpretation. Wang [14] proposed the two-parameter transformation (12) and reported that the transformation is much better to fit the theoretical prices to the market prices than the one-parameter one, for example, Wang [16] showed that the two-parameter transformation could explain the market prices of catastrophic bonds more consistently. However, the transformation (12) is not consistent with Bühlmann's economic premium principle, thereby lacking a sound economic interpretation. We hope that our results will give a good hint for overcoming this problem.

References

[1] Bühlmann, H. (1980), "An economic premium principle," *ASTIN Bulletin*, **11**, 52–60.

[2] Harrison, M.J. and Kreps, D. (1979), "Martingales and arbitrage in multiperiod securities market," *Journal of Economic Theory*, **20**, 381–408.

[3] Johnson, N.L. and Kotz, S. (1970), *Distributions in Statistics: Continuous Univariate Distributions*, **2**, Houghton Miffin Company.

[4] Kijima, M. (2002), *Stochastic Processes with Applications to Finance*, Chapman & Hall, London.

[5] Kijima, M. (2006), "A multivariate extension of equilibrium pricing transforms: The multivariate Esscher and Wang transforms for pricing financial and insurance risks," *ASTIN Bulletin*, **36**, 269–283.

[6] Kijima, M. and Miyake, Y. (2004), "On the term structure of lending interest rates when a fraction of collateral is recovered upon default," *Japan Journal of Industrial and Applied Mathematics*, **21**, 35–56.

[7] Kijima, M. and Muromachi, Y. (2001), "Pricing of equity swaps in a stochastic interest rate economy," *Journal of Derivatives*, **8**, 19–35.

[8] Kijima, M. and Muromachi, Y. (2006), "On the Wang transform with fat-tail distributions," *Proceedings of Stanford–Tsukuba Workshop*, Stanford University.

[9] Kijima, M. and Muromachi, Y., "An Extension of the Wang Transform Derived from Bühlmann's Economic Premium Principle for Insurance Risk," *Insurance: Mathematics and Economics*, *to appear*.

[10] Kijima, M. and Ohnishi, M. (1996), "Portfolio selection problems via the bivariate characterization of stochastic dominance relations," *Mathematical Finance*, **6**, 237–277.

[11] Nelsen, R.B. (1999), *An Introduction to Copulas*, Springer, New York.

[12] Platen, E. and Stahl, G. (2003), "A structure of general and specific market risk," *Working paper*, University of Technology, Sydney.

[13] Wang, S.S. (2000), "A class of distortion operators for pricing financial and insurance risks," *Journal of Risk and Insurance*, **67**, 15–36.

[14] Wang, S.S. (2002), "A universal framework for pricing financial and insurance risks," *ASTIN Bulletin*, **32**, 213–234.

[15] Wang, S.S. (2003), "Equilibrium pricing transforms: New results using Bühlmann's 1980 economic model," *ASTIN Bulletin*, **33**, 57–73.

[16] Wang, S.S. (2004), "Cat bond Pricing Using Probability Transforms," in Special Issues on Insurance and the State of the Art in Cat Bond Pricing, **278**, 19–29.