# ZONE-ADAPTIVE CONTROL STRATEGY FOR A MULTIPERIODIC MODEL OF RISK

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Abstract

In this paper intended to illustrate the adaptive control approach in insurance, a zone-adaptive control strategy harmonizing the requirements of principles of solvency and equity is considered in the simplistic framework of diffusion multiperiodic risk model. The adjacent works by the author set the similar adaptive control strategies in more realistic Poisson-exponential multiperiodic risk model. The room for further generalizations is large. In particular, it is the risk theory insight into the problem of asset–liability and solvency adaptive management in insurance under deficient information. The latter means that the intensities of the successive annual claim arrival processes are the random variables which comply with a certain scenario.

Keywords: Multiperiodic insurance process, equity, solvency, zone-adaptive control, diffusion risk model

## 1. Introduction

It is generally accepted that the insurance system is a mechanism for reducing the adverse financial impact of random events that prevent fulfillment of reasonable expectations. This mechanism is fine-tuned by intensive calculations rooted in insurance ethics. The fundamental principle is that insurers should charge a premium equal to the expected value of claim payments and expenses, loaded with an amount *necessary* to provide adequate security for the insured, rather than benefit those who seek unearned profit.

Insurance management requires great care and inventiveness rather than purely formal calculations, and the merits of a skilled insurer bear analogy to those of a prudent ruler. Relevance of the course toward synthesis of risk theory and control theory was recognized by many scholars. In particular, K. Borch (see [**3**], p. 451) noticed that "general formulation of the actuary's problem leads directly to the general theory of *optimal control processes* or *adaptive control processes*" and "the theory of control processes seems to be "tailor–made" for the problems which actuaries have struggled to formulate for more than a century."

Though optimal control, e.g., optimal pay-out of dividends, is a traditional set-up of actuarial mathematics (see, e.g., [**2**], [**21**]), single-purposed objectives like "to find the policy which maximizes the expected total discounted dividend pay-outs until the time of bankruptcy" (see [**21**], p. 105) may appear deficient to practical people. That kind of objectives may impress some shareholders, let alone mathematicians, but it will be resented by other parties to the insurance business.

Practical people tend to take the theory with a grain of salt, being concerned lest academics entice them into taking a false step. C.-O. Segerdhal described this danger in his discussion of [**3**] with this grotesque paradox: "I should think that if a manager of an insurance company came to his board or to his policyholders and said something like this: "Gentlemen, I am running this company along lines proposed by modern economics. This means that the company will certainly go broke. The probability of ruin is equal to one. It will go broke, but I will try to postpone as long as possible the deplorable but inevitable moment when you lose your money. Or, alternatively, before that happens, I will try to make as much money as possible to distribute. I do not care what happens then", I think such a managing director would not need any *deus ex machina* to be relieved of the burden of his duties. His board would see to that immediately."

On the contrary (cite again C.-O. Segerdhal [**3**]), "there is one life insurance company[1] in England which has been operating since 1762, so its management does not seem to have followed these lines. I do not think they are very sorry about that, nor are, I think, its policyholders, employees, or British life insurance as a whole." Long-term steady business is the ultimate goal of insurance management, and the theoretical implement to achieve it is adaptive control.

The present paper is an attempt of a theoretical insight into the asset–liability and solvency adaptive management of the long-term insurance business. With this end in view, a multiperiodic model of risk with annual accounting and subsequent annual control interventions is set forth. The emphasis is put on relevance of the adaptive control and on feasibility of its analytical realization; in practice such problems are attacked mostly by means of simulation.

Concerning the balance of simulation and analytical methods, it is worthwhile to recall that "the classical analytical methods and simulation should not be regarded as being in competition. A general rule is that an analytical technique should always be used wherever it is tractable. On the other hand, the temptation should be resisted to manipulate the premises of the model in order to make the analytical calculations possible, if that can only be done at the cost of the applicability of the model to real-world conditions. If that is done, as is often the

[1]This company is called *Equitable Life*; though it went into severe troubles a few years after the paper [**3**] was published, it does not undermine the idea of C.-O. Segerdhal.

case in theoretically-orientated risk theory, a warning of the restricted applicability — or non-applicability — should be clearly given. The wide realm of application of simulation methods begins at the frontier where other methods become intractable" (see [**6**], Chapter 1, Section 5.5, p. 154).

Regarding the general multiperiodic model of risk, each trajectory may be diagrammed as

$$\mathfrak{w}_0 \underbrace{\xrightarrow{\gamma_0} \mathfrak{u}_0 \xrightarrow{\pi_1} \mathfrak{w}_1}_{\text{1-st year}} \cdots \xrightarrow{\pi_{k-1}} \mathfrak{w}_{k-1} \underbrace{\xrightarrow{\gamma_{k-1}} \mathfrak{u}_{k-1} \xrightarrow{\pi_k} \mathfrak{w}_k}_{k\text{-th year}} \cdots . \tag{1}$$

According to this diagram (for $k = 1, 2, \dots$), at the end of $(k-1)$-th year the state variable $\mathfrak{w}_{k-1}$ is observed. It describes the insurer's position at that moment. Then, at the beginning of $k$-th year the control rule $\gamma_{k-1}$ is applied to choose the control variable $\mathfrak{u}_{k-1}$. Thereupon the $k$-th year probability mechanism of insurance unfolds; the transition function of this mechanism is denoted by $\pi_k$. It defines the insurer's position at the end of the $k$-th year.

Paramount in (1) is the annual probability mechanism of insurance. It is modelled in [**16**] and [**17**] by Lundberg's collective risk model, while in this paper it is taken simplistic: the annual risk reserve of an insurance company at time $t$ is assumed to be

$$R_t(u, \tau) = u + (1 + \tau)\mu t - V(t), \quad V(t) = \mu t + \sigma \mathsf{W}_t, \quad t \geqslant 0, \tag{2}$$

where $u$ is the initial risk reserve, $\mathsf{W}_t$ is a standard Brownian motion, $\mu$ is the premium rate calculated according to the expected value principle (see e.g., [**5**], p. 85), i.e., $\mathsf{E}V(s) = \mu s$, $\tau$ is the adaptive premium loading, and $\sigma > 0$ is a constant diffusion coefficient, $\mathsf{D}V(s) = \sigma^2 s$.

The recommendation of [**6**] to supplement the analysis of a model with a clear warning of its restricted applicability is straightforward in that case. Observe that we consciously deal with the simplistic annual mechanism (2) which allows elementary computations and transparent mathematics. On the one hand, it is adequate when a telling illustration of the adaptive control approach is sought. On the other hand, simplistic model suggests the lines along which the analysis may be extended to a number of more realistic insurance risk models, where explicit formulae are no more available.

It is also noteworthy that Brownian motion is common in risk theory for the following reasons. On the one hand, application of diffusion models per se often provides insight into complicated problems, otherwise intractable (see, e.g., [**2**], [**21**]). On the other hand, when the jumping risk reserve may be duly replaced by an appropriate Brownian motion for which the probabilities of interest may be computed exactly, very useful approximations are obtained (see, e.g., [**12**], [**10**], [**11**], [**19**], [**18**], [**8**], with overview [**1**]). The path-wise convergence to Brownian motion is justified, e.g., in the large-sample case, or for large frequency and small severity ("heavy traffic" situation).

The paper is arranged as follows. Section 2 is devoted to synthesis of a zone-adaptive strategy in diffusion risk model. Starting with definitions of the target value of the risk reserve corresponding to a level $0 < \alpha < 1$, and of the basic adaptive strategy, it yields the diffusion counterpart of the results of [**15**]—[**17**]. Section 3 discusses targeting, solvency and dynamic solvency provisions of the zone-adaptive strategy and applies the concept of the general multiperiodic controlled risk model introduced in [**14**]. Section 4 touches upon some aspects of further generalization of zone-adaptive strategies in the diffusion risk model. Section 5 contains auxiliary results. The control-oriented reader may wish to start from Section 3.

## 2. Synthesis of a zone-adaptive strategy in diffusion risk model

Put $M_t(u, \tau) = \inf_{0<s\leqslant t} R_s(u, \tau)$, $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-y^2/2} dy$ the distribution function of the standard normal distribution, and $\phi(x)$ its density.

Definition 2.1. The target value $u_{\mu,\sigma}(\alpha,t)$ of the risk reserve corresponding to a level $0<\alpha<1$ is a positive solution of the equation

$$\psi_t(u;0)=\mathsf{P}\{M_t(u,0)<0\}=\alpha. \tag{3}$$

For $0<\alpha<1$, introduce $c_\alpha=\Phi^{-1}(1-\alpha/2)$.

Theorem 2.1. *For $0<\alpha<1$, the solution of equation* (3) *in the diffusion model* (2) *may be written as $u_{\mu,\sigma}(\alpha,t)=\sigma\sqrt{t}\,c_\alpha$.*

Proof. Note (see Theorem 5.1) that

$$\psi_t(u;0)=\mathsf{P}\{M_t(u,0)<0\}=\mathsf{P}\{\sup_{0<s\leqslant t}\mathsf{W}_s>u/\sigma\}=2\mathsf{P}\{\mathsf{W}_t>u/\sigma\}.$$

Evidently, $2\mathsf{P}\{\mathsf{W}_t>u/\sigma\}=2\Big(1-\Phi\Big(\dfrac{u}{\sigma\sqrt{t}}\Big)\Big)$. The equation $2\Big(1-\Phi\Big(\dfrac{u}{\sigma\sqrt{t}}\Big)\Big)=\alpha$ has to be solved for $u$. The solution is

$$u=\sigma\sqrt{t}\,c_\alpha,\quad c_\alpha=\Phi^{-1}(1-\alpha/2).$$

The proof is complete. □

Remark 2.1. The equation (3) may be called "neutral–loading" or "equitable–reserving". It defines the initial risk reserve sufficient to make the probability of ruin equal to $\alpha$ without resort to premium loading, which may be voted fair by customers. In the diffusion model the "probability of ruin" solvency criterion is as simple to deal with as a short-cut "ultimate risk reserve" solvency criterion (see [**6**], p. 16–17): solution of the equation $\mathsf{P}\{R_t(u,0)<0\}=\alpha$ evidently is $\hat{u}_{\mu,\sigma}(\alpha,t)=\sigma\sqrt{t}\,\hat{c}_\alpha$, where $\hat{c}_\alpha=\Phi^{-1}(1-\alpha)$.

Assuming that duration of the incoming year is $t$, we mean by $z$ a deviation, either positive or negative, of the past-year-end risk reserve from $u_{\mu,\sigma}(\alpha,t)$. Case $z<0$ means deficit, case $z>0$ means surplus.

Definition 2.2. The strategy[2]

$$u_{z,t}=u_{\mu,\sigma}(\alpha,t)+z\quad\text{and}\quad\tau_{z,t}=-\frac{z}{\mu t},\quad z\in\mathsf{R}, \tag{4}$$

is called the basic adaptive strategy.

Remark 2.2. Applying the basic adaptive strategy (4), the risk reserve is

$$R_s(u_{z,t};\tau_{z,t})=u_{\mu,\sigma}(\alpha,t)+z-\frac{z}{t}\,s-\sigma\mathsf{W}_s.$$

In particular, $R_t(u_{z,t};\tau_{z,t})=u_{\mu,\sigma}(\alpha,t)-\sigma\mathsf{W}_t$. Therefore

$$\mathsf{E}R_t(u_{z,t},\tau_{z,t})=u_{\mu,\sigma}(\alpha,t)\quad\text{for any } z\in\mathsf{R}, \tag{5}$$

and control (4) makes the capital at the time $t$ (i.e., at the year-end of a single period) equal "in the average" to the target value $u_{\mu,\sigma}(\alpha,t)$. That observation justifies the name of the target capital value.

Theorem 2.2. *For $z\in(-\sigma\sqrt{t}\,c_\alpha,\infty)$ and for the control strategy* (4) *in the diffusion model* (2)*, one has*

$$\begin{aligned}\psi_t(u_{z,t};\tau_{z,t})&=\mathsf{P}\big\{M_t(u_{z,t},\tau_{z,t})<0\big\}\\&=1-\Phi(c_\alpha)+\exp\Big\{2\frac{z}{\sigma\sqrt{t}}\Big(c_\alpha+\frac{z}{\sigma\sqrt{t}}\Big)\Big\}\Phi\Big(-2\frac{z}{\sigma\sqrt{t}}-c_\alpha\Big).\end{aligned} \tag{6}$$

[2]Since the initial capital $u_{z,t}$ may not be negative, $z>-u_{\mu,\sigma}(\alpha,t)=-\sigma\sqrt{t}\,c_\alpha$. Bear it in mind when put $z\in\mathsf{R}$ for simplicity.

Proof. Bearing in mind Theorem 2.1,

$$\boldsymbol{\psi}_t(u_{z,t};\tau_{z,t}) = \mathsf{P}\{\inf_{0<s\leqslant t}\Big(\big(1-\frac{s}{t}\big)\,z-\sigma\mathsf{W}_s\Big) < -u_{\mu,\sigma}(\alpha,t)\}$$
$$= \mathsf{P}\{\sup_{0<s\leqslant t}\Big(\sigma\mathsf{W}_s+\frac{z}{t}\,s\Big) > \sigma\sqrt{t}\,c_\alpha+z\}. \quad (7)$$

Apply Theorem 5.2 which yields for $z\in\mathsf{R}$

$$\boldsymbol{\psi}_t(u_{z,t};\tau_{z,t}) = \mathsf{P}\{\sup_{0<s\leqslant t}\Big(\sigma\mathsf{W}_s+\frac{z}{t}\,s\Big) > \sigma\sqrt{t}\,c_\alpha+z\}$$
$$= 1-\Phi(c_\alpha)+\exp\Big\{2\frac{z}{\sigma\sqrt{t}}\Big(c_\alpha+\frac{z}{\sigma\sqrt{t}}\Big)\Big\}\Phi\Big(-2\frac{z}{\sigma\sqrt{t}}-c_\alpha\Big).$$

The proof is complete. □

Theorem 2.3. *For $z\in(-\sigma\sqrt{t}\,c_\alpha,\infty)$ and for the control strategy* (4) *in the diffusion model* (2)*, the probability*

$$\boldsymbol{\psi}_t(u_{z,t};\tau_{z,t}) = \mathsf{P}\big\{M_t(u_{z,t},\tau_{z,t})<0\big\},$$

*regarded as a function of $z$, is monotone decreasing, as $z$ increases.*

Proof. From the first equality (7), it is straightforward since $1-\frac{s}{t}\geqslant 0$. For the alternative proof, set $F(x)=1-\Phi(c_\alpha)+\exp\{2x(c_\alpha+x)\}\Phi(-2x-c_\alpha)$, $x\in\mathsf{R}$. One has

$$\frac{\partial}{\partial z}\boldsymbol{\psi}_t(u_{z,t};\tau_{z,t}) = \frac{1}{\sigma\sqrt{t}}\frac{dF(x)}{dx}\Big|_{x=\frac{z}{\sigma\sqrt{t}}}.$$

Direct calculus yields

$$\frac{dF(x)}{dx} = -2\exp\{2x(c_\alpha+x)\}\phi(c_\alpha+2x)\big(1-(c_\alpha+2x)M(c_\alpha+2x)\big), \quad (8)$$

where $M(c_\alpha+2x)=\dfrac{1-\Phi(c_\alpha+2x)}{\phi(c_\alpha+2x)}$ is the Mills ratio. Since for any $v\in\mathsf{R}$

$$1-vM(v) = 1-ve^{v^2/2}\int_v^\infty e^{-t^2/2}dt > 0,$$

which follows from $v^{-1}e^{-v^2/2}-\int_v^\infty e^{-t^2/2}dt = -\int_v^\infty e^{-t^2/2}d(t^{-1}) = \int_0^{1/v} e^{-1/2w^2}dw$, derivative (8) is negative and the proof is complete. □

Definition 2.3. The lower alarm barrier of a zone with target value $u_{\mu,\sigma}(\alpha,t)$ and with level $\beta$ of probability of ruin, $0<\alpha<\beta<1$, is $\underline{u}_{\mu,\sigma}(\alpha,\beta,t)=u_{\mu,\sigma}(\alpha,t)+z_{\mu,\sigma}(\alpha,\beta,t)$, where $z_{\mu,\sigma}(\alpha,\beta,t)<0$ is a solution of the equation

$$\boldsymbol{\psi}_t(u_{z,t};\tau_{z,t}) = \mathsf{P}\big\{M_t(u_{z,t},\tau_{z,t})<0\big\} = \beta. \quad (9)$$

Theorem 2.4. *For $0<\alpha<1$, the solution of equation* (9) *in the diffusion model* (2) *may be written as*

$$z_{\mu,\sigma}(\alpha,\beta,t) = -\sigma\sqrt{t}\,x_{\alpha,\beta},$$

*where $x_{\alpha,\beta}>0$ is a unique root of the equation*

$$1-\Phi(c_\alpha)+\exp\{-2x(c_\alpha-x)\}\Phi(2x-c_\alpha) = \beta. \quad (10)$$

Proof of Theorem 2.4. It is evident from (6) and Theorem 2.3. □

Definition 2.4. Assume that $0 < \alpha < \beta < 1$. The strategy

$$\widehat{u}_{z,t} = \begin{cases} \underline{u}_{\mu,\sigma}(\alpha,\beta,t), & z < z_{\mu,\sigma}(\alpha,\beta,t), \\ u_{\mu,\sigma}(\alpha,t) + z, & z_{\mu,\sigma}(\alpha,\beta,t) \leqslant z \leqslant 0, \\ u_{\mu,\sigma}(\alpha,t), & z > 0 \end{cases} \tag{11}$$

and

$$\widehat{\tau}_{z,t} = \begin{cases} \overline{\tau}_{\mu,\sigma}(\alpha,\beta,t), & z < z_{\mu,\sigma}(\alpha,\beta,t), \\ -\dfrac{z}{\mu t}, & z_{\mu,\sigma}(\alpha,\beta,t) \leqslant z \leqslant 0, \\ 0, & z > 0, \end{cases} \tag{12}$$

where

$$\overline{\tau}_{\mu,\sigma}(\alpha,\beta,t) = -\frac{z_{\mu,\sigma}(\alpha,\beta,t)}{\mu t},$$

is called zone-adaptive control strategy.

Table 2.1. Values of $x_{\alpha,\beta}$ calculated numerically using (10).

| | | $\beta = 110\alpha\%$ | $\beta = 120\alpha\%$ | $\beta = 130\alpha\%$ | $\beta = 140\alpha\%$ |
|---|---|---|---|---|---|
| $\alpha = 0.1$ | $c_\alpha = 1.645$ | $x_{\alpha,\beta} = 0.7121$ | $x_{\alpha,\beta} = 0.7522$ | $x_{\alpha,\beta} = 0.7897$ | $x_{\alpha,\beta} = 0.8249$ |
| $\alpha = 0.05$ | $c_\alpha = 1.960$ | $x_{\alpha,\beta} = 0.8360$ | $x_{\alpha,\beta} = 0.8736$ | $x_{\alpha,\beta} = 0.9090$ | $x_{\alpha,\beta} = 0.9422$ |
| $\alpha = 0.01$ | $c_\alpha = 2.576$ | $x_{\alpha,\beta} = 1.0754$ | $x_{\alpha,\beta} = 1.1098$ | $x_{\alpha,\beta} = 1.1423$ | $x_{\alpha,\beta} = 1.1730$ |

Remark 2.3. To keep the risk reserve into a zone[3] associated with a target value, both the initial capital and the premium loading control are applied. This idea may be found in many sources. In particular, in [**6**], Chapter 5, Section 5.5, p. 151, a control is proposed where premium loading "will be increased if the risk reserve ratio passes below a certain alarm barrier. On the other hand, if another barrier is exceeded, the loading will be reduced."

Theorem 2.5. *For $0 < \alpha < \beta < 1$, the zone-adaptive control strategy in the diffusion model is*

$$\widehat{u}_{z,t} = \begin{cases} \sigma\sqrt{t}\,(c_\alpha - x_{\alpha,\beta}), & z < -\sigma\sqrt{t}\,x_{\alpha,\beta}, \\ \sigma\sqrt{t}\,c_\alpha + z, & -\sigma\sqrt{t}\,x_{\alpha,\beta} \leqslant z \leqslant 0, \\ \sigma\sqrt{t}\,c_\alpha, & z > 0 \end{cases} \tag{13}$$

*and*

$$\widehat{\tau}_{z,t} = \begin{cases} \dfrac{\sigma}{\mu\sqrt{t}} x_{\alpha,\beta}, & z < -\sigma\sqrt{t}\,x_{\alpha,\beta}, \\ -\dfrac{z}{\mu t}, & -\sigma\sqrt{t}\,x_{\alpha,\beta} \leqslant z \leqslant 0, \\ 0, & z > 0. \end{cases} \tag{14}$$

Definition 2.5. For the strategy (11)–(12), the random variable

$$S_{z,t} = \begin{cases} 0, & \underline{u}_{\mu,\sigma}(\alpha,\beta,t) \leqslant R_t(\widehat{u}_{z,t},\widehat{\tau}_{z,t}) \leqslant u_{\mu,\sigma}(\alpha,t), \\ R_t(\widehat{u}_{z,t},\widehat{\tau}_{z,t}) - u_{\mu,\sigma}(\alpha,t), & R_t(\widehat{u}_{z,t},\widehat{\tau}_{z,t}) > u_{\mu,\sigma}(\alpha,t), \\ -(\underline{u}_{\mu,\sigma}(\alpha,\beta,t) - R_t(\widehat{u}_{z,t},\widehat{\tau}_{z,t})), & R_t(\widehat{u}_{z,t},\widehat{\tau}_{z,t}) < \underline{u}_{\mu,\sigma}(\alpha,\beta,t) \end{cases}$$

[3]In particular, it is an objective of a rule in Directives [**7**].

is called annual excess (of either sign) of capital.

## 3. Adaptive strategy in multiperiodic model of risk

Address first the multiperiodic diffusion risk model and discuss performance of the zone-adaptive strategy (13)–(14). Recall (see [**14**]) that rigorous definition of a *general* multiperiodic controlled risk model over the elementary state space $(\Omega, \mathcal{F})$ with realizations matching the diagram (1) amounts to a random sequence $(W_k, U_k)$, $k = 0, 1, \dots$, called controlled random sequence, as follows.

Let the random variables $W_k$, $k = 0, 1, \dots$, with realizations $\mathfrak{w}_k$ assume values in a state space $\mathrm{W}$ endowed with a $\sigma$-algebra $\mathcal{W}$, called $k$-th year output state space. The random variables $U_k$, $k = 0, 1, \dots$, with realization $\mathfrak{u}_k$ assume values in a state space $\mathrm{U}$ endowed with a $\sigma$-algebra $\mathcal{U}$, called $k$-th year input or control state space. Particular choice of the spaces $(\mathrm{W}, \mathcal{W})$ and $(\mathrm{U}, \mathcal{U})$ will depend on the context.

For brevity, set $\mathcal{Y}_0 = \{\mathfrak{w}_0\}$ and for $k = 2, 3, \dots$ put

$$\mathcal{Y}_{k-1} = \{\mathfrak{w}_0, \dots, \mathfrak{w}_{k-1}; \mathfrak{u}_0, \dots, \mathfrak{u}_{k-2}\}$$

for the "history" up to the $(k-1)$-th year inclusively. Introduce

$$\pi_k(\mathcal{Y}_{k-1}, \mathfrak{u}_{k-1}; d\mathfrak{w}_k) = \pi_k(\mathfrak{w}_0, \dots, \mathfrak{w}_{k-1}, \mathfrak{u}_0, \dots, \mathfrak{u}_{k-1}; d\mathfrak{w}_k),$$
$$\gamma_{k-1}(\mathcal{Y}_{k-1}; d\mathfrak{u}_{k-1}) = \gamma_{k-1}(\mathfrak{w}_0, \dots, \mathfrak{w}_{k-1}, \mathfrak{u}_0, \dots, \mathfrak{u}_{k-2}; d\mathfrak{u}_{k-1})$$

called transition function of the probability mechanism (t.f.p.m.) of insurance and transition functions of control mechanism (t.f.c.m.) respectively.

The $k$-th year t.f.p.m. is any kernel on $(\mathrm{W} \times [\mathrm{W}^{\times k} \times \mathrm{U}^{\times k}], \mathcal{W} \otimes [\mathcal{W}^{\otimes k} \otimes \mathcal{U}^{\otimes k}])$, i.e., a measure on $(\mathrm{W}, \mathcal{W})$ with respect to its last argument, and a measurable function on $\mathcal{W}^{\otimes k} \otimes \mathcal{U}^{\otimes k}$ with respect to $\mathfrak{w}_0, \dots, \mathfrak{w}_{k-1}, \mathfrak{u}_0, \dots, \mathfrak{u}_{k-1}$. The $k$-th year t.f.c.m. is a measure on $(\mathrm{U}, \mathcal{U})$ with respect to its last argument, and a measurable function on $\mathcal{W}^{\otimes k} \otimes \mathcal{U}^{\otimes k-1}$ with respect to $\mathfrak{w}_0, \dots, \mathfrak{w}_{k-1}$, $\mathfrak{u}_0, \dots, \mathfrak{u}_{k-2}$. Sequences of t.f.c.m.

$$\boldsymbol{\gamma} = \{\gamma_k(\cdot\,;\cdot),\ k = 0, 1, \dots\} \quad \text{or} \quad \boldsymbol{\gamma}_n = \{\gamma_k(\cdot\,;\cdot),\ k = 0, 1, \dots, n-1\}$$

are called infinite- or finite-time ($n$-years) planning horizon control strategies. It is noteworthy that practical insurers are interested in control strategies with quite a limited number of years, rather than in large- or even infinite-time ones.

It is known (see, e.g., § 1 of Chapter 1 in [**9**]) that under certain mild regularity conditions on the measurable spaces $(\mathrm{W}, \mathcal{W})$ and $(\mathrm{U}, \mathcal{U})$ the initial distribution $\pi_0(\cdot)$ and the families $\pi_k(\cdot\,;\cdot)$, $k = 1, 2, \dots$, and $\gamma_k(\cdot\,;\cdot)$, $k = 0, 1, \dots$, define over the elementary state space $(\Omega, \mathcal{F})$ a random sequence $(W_k, U_k)$, $k = 0, 1, \dots$, having finite-dimensional distributions

$$\mathsf{P}^{\boldsymbol{\pi},\boldsymbol{\gamma}}\{W_0 \in A_0, U_0 \in B_0, \dots, W_n \in A_n, U_n \in B_n\} = \int_{A_0} \pi_0(d\mathfrak{w}_0)$$
$$\times \int_{B_0} \gamma_0(\mathfrak{w}_0\,; d\mathfrak{u}_0) \dots \int_{A_n} \pi_n(\mathcal{Y}_{n-1}, \mathfrak{u}_{n-1}; d\mathfrak{w}_n) \int_{B_n} \gamma_n(\mathcal{Y}_n\,; d\mathfrak{u}_n), \quad (15)$$

where $n \in \mathsf{N}$ and $A_k \in \mathcal{W}$, $B_k \in \mathcal{U}$, $k = 0, 1, \dots, n$. In other words, these families specify a unique measure $\mathsf{P}^{\boldsymbol{\pi},\boldsymbol{\gamma}}$ on $(\Omega, \mathcal{F})$ which value on any rectangle $A_0 \times \dots \times A_n \times B_0 \times \dots \times B_n$ is given by the right hand side of (15).

In such a way, controlled random sequence is the random sequence $(W_k, U_k)$, $k = 0, 1, \dots$, on $(\Omega, \mathcal{F}, \mathsf{P}^{\boldsymbol{\pi},\boldsymbol{\gamma}})$ assuming values on the product space $(\mathrm{W} \times \mathrm{U}, \mathcal{W} \otimes \mathcal{U})$. It defines the multiperiodic controlled insurance process.

In the control theory the $k$-th annual probability mechanism is called *Markov* if t.f.p.m. is of the form

$$\pi_k(\mathcal{Y}_{k-1}, \mathfrak{u}_{k-1}; d\mathfrak{w}_k) = \pi_k(\mathfrak{w}_{k-1}, \mathfrak{u}_{k-1}; d\mathfrak{w}_k),$$

being a function of the previous state and action only. The $k$-th year control is called non-randomized or *pure* if t.f.c.m. is degenerate,

$$\gamma_{k-1}(\mathfrak{Y}_{k-1}\,;d\mathfrak{u}_{k-1}) = \delta_{\gamma_{k-1}(\mathfrak{Y}_{k-1})}(d\mathfrak{u}_{k-1}).$$

In other words, one has

$$\mathfrak{u}_{k-1} = \gamma_{k-1}(\mathfrak{Y}_{k-1}), \tag{16}$$

where $\gamma_{k-1}(\cdot)$ is a mapping from $\mathrm{W}^{\times k} \times \mathrm{U}^{\times(k-1)}$ to U. The strategies $\boldsymbol{\gamma}$ or $\boldsymbol{\gamma}_n$ are called non-randomized or pure if each t.f.c.m. $\gamma_k$ is non-randomized.

In stochastic control theory the non-randomized strategies are emphasized because of their mathematical convenience and "sufficiency" (see, e.g., Theorem 1.2 in [**9**]). In this paper they are accentuated because of their pragmatic advantage: supervisors will hardly approve application of a randomized control strategy.

The $k$-th year non-randomized control is called Markov if

$$\mathfrak{u}_{k-1} = \gamma_{k-1}(\mathfrak{w}_{k-1}). \tag{17}$$

Pure strategies $\boldsymbol{\gamma}$ or $\boldsymbol{\gamma}_n$ are called *Markov* if each t.f.c.m. $\gamma_k$ is Markov. The decision maker who applies a Markov strategy deals with the immediate history ignoring the more ancient one.

Remark 3.1. If we deal with Markov t.f.p.m. and restrict ourselves to pure Markov strategies, then the controlled random sequence $(W_k, U_k)$, $k = 0, 1, \ldots$, is reduced to a Markov chain with transition probability

$$P(\mathfrak{w}_{k-1}\,;d\mathfrak{w}_k) = \pi_k(\mathfrak{w}_{k-1}, \gamma_{k-1}(\mathfrak{w}_{k-1})\,;d\mathfrak{w}_k).$$

We continue to write $\mathsf{P}^{\pi,\gamma}$ for the Markov chain with transition probability $P$ and denote by $\mathsf{E}^{\pi,\gamma}$ the mean with respect to that measure.

Come back to diffusion multiperiodic controlled risk model and to the adaptive strategies synthesized in Section 2. For the mutually independent standard Brownian motions $\mathsf{W}_t^{[k]}$, $k = 1, 2, \ldots$, select

$$\mathrm{W} = \mathsf{R} \times \{0, 1\} \quad \text{and} \quad \mathrm{U} = \mathsf{R}^+ \times \mathsf{R}^+$$

and set $\mathfrak{w}_0^{\langle 1\rangle} = u > 0$, $\mathfrak{w}_k = (\mathfrak{w}_k^{\langle 1\rangle}, \mathfrak{w}_k^{\langle 2\rangle}) \in \mathrm{W}$, $\mathfrak{u}_{k-1} = (\mathfrak{u}_{k-1}^{\langle 1\rangle}, \mathfrak{u}_{k-1}^{\langle 2\rangle}) \in \mathrm{U}$ and $d\mathfrak{w}_k = (d\mathfrak{w}_k^{\langle 1\rangle} \times d\mathfrak{w}_k^{\langle 2\rangle}) \in \mathcal{W}$. For simplicity's sake consider in this section the *stationary* case, which means that $\mu$, $\sigma$, $t$, $\alpha$ and $\beta$ are year-by-year equal.

For $k = 1, 2, \ldots$, let the $k$-th annual t.f.p.m. be

$$\pi_k(\mathfrak{w}_{k-1}, \mathfrak{u}_{k-1}\,;d\mathfrak{w}_k) = \mathsf{P}\{R_t(\mathfrak{u}_{k-1}) \in d\mathfrak{w}_k^{\langle 1\rangle}, \mathbf{1}_{\{M_k(\mathfrak{u}_{k-1})<0\}} \in d\mathfrak{w}_k^{\langle 2\rangle}\}, \tag{18}$$

where, in accordance with (2),

$$R_t(\mathfrak{u}_{k-1}) = \mathfrak{u}_{k-1}^{\langle 1\rangle} + (1 + \mathfrak{u}_{k-1}^{\langle 2\rangle})\mu t - V_k(t), \quad V_k(t) = \mu t + \sigma \mathsf{W}_t^{[k]}, \quad t \geqslant 0.$$

For $k = 1, 2, \ldots$, let the $k$-th annual t.f.c.m. be $\mathfrak{u}_{k-1} = \gamma_{k-1}(\mathfrak{w}_{k-1})$, where

$$\gamma_{k-1}(\mathfrak{w}_{k-1}) = (\gamma_{k-1}^{\langle 1\rangle}(\mathfrak{w}_{k-1}), \gamma_{k-1}^{\langle 2\rangle}(\mathfrak{w}_{k-1}))$$

with $z(\mathfrak{w}_{k-1}^{\langle 1\rangle}) = \mathfrak{w}_{k-1}^{\langle 1\rangle} - \sigma\sqrt{t}c_\alpha$ and (see (13) and (14))

$$\gamma_{k-1}^{\langle 1\rangle}(\mathfrak{w}_{k-1}) = \widehat{u}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t} \quad \text{and} \quad \gamma_{k-1}^{\langle 2\rangle}(\mathfrak{w}_{k-1}) = \widehat{\tau}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}. \tag{19}$$

Evidently, t.f.p.m. (18) is Markov and the $n$-year strategy $\boldsymbol{\gamma}_n = \{\gamma_k(\cdot\,;\cdot),\ k = 0, 1, \ldots, n-1\}$ generated by (19) is pure and Markov. The diffusion multiperiodic controlled risk model is

reduced (see Remark 3.1) to a Markov chain on the state space $(\mathrm{W}, \mathcal{W})$, $\mathrm{W} = \mathsf{R} \times \{0,1\}$, with the transition probability[4]

$$P(\mathfrak{w}_{k-1}\,; d\mathfrak{w}_k) = \mathsf{P}\big\{R_t\big(\widehat{u}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}, \widehat{\tau}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}\big) \in d\mathfrak{w}_k^{\langle 1\rangle}, \mathbf{1}_{\big\{M_t\big(\widehat{u}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}, \widehat{\tau}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}\big)<0\big\}} \in d\mathfrak{w}_k^{\langle 2\rangle}\big\}.$$

Bearing in mind Theorem 5.3, this transition probability may be further elaborated as follows: the "ruin" kernel (recall that $z(\mathfrak{w}_{k-1}^{\langle 1\rangle}) = \mathfrak{w}_{k-1}^{\langle 1\rangle} - \sigma\sqrt{t}c_\alpha$)

$$P(\mathfrak{w}_{k-1}\,; d\mathfrak{w}_k^{\langle 1\rangle} \times \{1\}) = \mathsf{P}\big\{R_t\big(\widehat{u}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}, \widehat{\tau}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}\big) \in d\mathfrak{w}_k^{\langle 1\rangle}, M_t\big(\widehat{u}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}, \widehat{\tau}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}\big) < 0\big\}$$

is equal[5] to

$$\mathsf{P}\{\sigma\sqrt{t}c_\alpha - \sigma\mathsf{W}_t \in d\mathfrak{w}_k^{\langle 1\rangle}, \sup_{0<s\leqslant t}\Big(\sigma\mathsf{W}_s + \frac{z(\mathfrak{w}_{k-1}^{\langle 1\rangle})}{t}\,s\Big) > \sigma\sqrt{t}\,c_\alpha + z(\mathfrak{w}_{k-1}^{\langle 1\rangle})\} \tag{20}$$

if $-\sigma\sqrt{t}\,x_{\alpha,\beta} \leqslant z(\mathfrak{w}_{k-1}^{\langle 1\rangle}) \leqslant 0$, is equal to

$$\mathsf{P}\{\sigma\sqrt{t}c_\alpha - \sigma\mathsf{W}_t \in d\mathfrak{w}_k^{\langle 1\rangle}, \sup_{0<s\leqslant t}\Big(\sigma\mathsf{W}_s - \frac{\sigma x_{\alpha,\beta}}{\sqrt{t}}s\Big) > \sigma\sqrt{t}\,(c_\alpha - x_{\alpha,\beta})\} \tag{21}$$

if $z(\mathfrak{w}_{k-1}^{\langle 1\rangle}) < -\sigma\sqrt{t}\,x_{\alpha,\beta}$, and is equal to

$$\mathsf{P}\{\sigma\sqrt{t}c_\alpha - \sigma\mathsf{W}_t \in d\mathfrak{w}_k^{\langle 1\rangle}, \sup_{0<s\leqslant t}\sigma\mathsf{W}_s > \sigma\sqrt{t}\,c_\alpha\} \tag{22}$$

if $z(\mathfrak{w}_{k-1}^{\langle 1\rangle}) > 0$. The analysis of "non-ruin" kernel

$$\begin{aligned} &P(\mathfrak{w}_{k-1}\,; d\mathfrak{w}_k^{\langle 1\rangle} \times \{0\}) \\ &\qquad = \mathsf{P}\big\{R_t\big(\widehat{u}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}, \widehat{\tau}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}\big) \in d\mathfrak{w}_k^{\langle 1\rangle}, M_t\big(\widehat{u}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}, \widehat{\tau}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}\big) > 0\big\} \end{aligned} \tag{23}$$

is analogous.

Remark 3.2. Emphasize again our reason to work with the diffusion annual probability mechanisms of insurance (2). Seeking for a telling illustration of the adaptive control approach, we came from a general multiperiodic model to a particular one which allows, due to the well-known auxiliary Theorem 5.3, the explicit expressions for (20), (21), (22) and for (23). If the explicit formulae are not of paramount interest, e.g., for the computer-oriented analysts, the extensions to more general annual probability mechanisms of insurance may be straightforward.

We have defined diffusion multiperiodic controlled risk model with $n$-years zone-adaptive strategy (19). Let us analyze its performance.

**3.1. Targeting.** Targeting, or ability to keep the risk reserve inside a strip zone associated with the target value $u_{\mu,\sigma}(\alpha, t)$, is the central property of zone-adaptive strategies.

Theorem 3.1. *In the above multiperiodic diffusion stationary risk model, for the strategy* (19) *and for each* $k = 1, 2, \ldots,$

$$\mathsf{E}^{\boldsymbol{\pi},\gamma}\{\text{capital at the end of year } k\} = \sigma\sqrt{t}\,c_\alpha.$$

[4]It is noteworthy that $P(\mathfrak{w}_{k-1}\,; d\mathfrak{w}_k)$ depends on $\mathfrak{w}_{k-1}$ through $\mathfrak{w}_{k-1}^{\langle 1\rangle}$ only.

[5]It is noteworthy that $P(\mathfrak{w}_{k-1}; \mathsf{R} \times \{1\}) = \boldsymbol{\psi}_t(\widehat{u}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}, \widehat{\tau}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t})$.

Proof. Bearing in mind Theorem 2.1, one has (see Remark 2.2, recall that $V_k(t) = \mu t + \sigma \mathsf{W}_t^{[k]}$ and see (19), (11) and (12) for definition of $U_{k-1}^{\langle 1\rangle}$ and $U_{k-1}^{\langle 2\rangle}$)

$$W_k^{\langle 1\rangle} = R_t(U_{k-1}) = U_{k-1}^{\langle 1\rangle} + (1 + U_{k-1}^{\langle 2\rangle})\mu t - V_k(t) = \sigma\sqrt{t}\, c_\alpha - \sigma \mathsf{W}_t^{[k]}.$$

The proof is complete by taking expectation from both sides. □

**3.2. Solvency.** The following result is fundamental.

Theorem 3.2. *In the above multiperiodic diffusion stationary risk model, for the strategy* (19) *and for each* $k = 1, 2, \dots,$

$$\mathsf{P}^{\pi,\gamma}\{\text{first ruin in year } k\} \leqslant \beta.$$

Proof. One has[6]

$$\begin{aligned}\mathsf{P}^{\pi,\gamma}\{\text{first ruin in year } k\} &= \mathsf{P}^{\pi,\gamma}\{W_1^{\langle 2\rangle} = 0, \dots, W_{k-1}^{\langle 2\rangle} = 0, W_k^{\langle 2\rangle} = 1\} \\ &= \int_{\mathsf{R}^+\times\{0\}} P(\mathfrak{w}_0, d\mathfrak{w}_1) \dots \int_{\mathsf{R}^+\times\{0\}} P(\mathfrak{w}_{k-2}, d\mathfrak{w}_{k-1}) P(\mathfrak{w}_{k-1}, \mathsf{R}\times\{1\}),\end{aligned}$$

where

$$P(\mathfrak{w}_{k-1}, \mathsf{R}\times\{1\}) = \boldsymbol{\psi}_t(\widehat{u}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}, \widehat{\tau}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}) \leqslant \beta$$

by Section 2. It yields the result immediately. □

Corollary 3.1. For each integer $n$,

$$\mathsf{P}^{\pi,\gamma}\{\text{ruin within } n \text{ years}\} = \sum_{k=1}^{n} \mathsf{P}^{\pi,\gamma}\{\text{first ruin in year } k\} \leqslant n\beta.$$

**3.3. Dynamic solvency provisions.** To redirect, as in (19), the risk reserve into the strip zone when deficit at the end of the insurance year occurs, provisions have to be established. These provisions should be backed by appropriate assets, which are formed by surplus values reserved in those years when the year-end risk reserve exceeds the upper level of the strip zone. Commonly, these provisions are invested, but we consciously ignore the investment aspects in this paper. The interested reader may introduce them at the price of more cumbersome transition probabilities.

Bearing in mind Definition 2.5, consider

$$\mathsf{E}^{\pi,\gamma} S_n = \sum_{k=1}^{n} \mathsf{E}^{\pi,\gamma} S_{(z(W_{k-1}^{\langle 1\rangle}),t)},$$

where

$$\begin{aligned}\mathsf{E}^{\pi,\gamma} S_{(z(W_{k-1}^{\langle 1\rangle}),t)} &= \int_{\mathrm{W}} P(\mathfrak{w}_0, d\mathfrak{w}_1) \dots \int_{\mathrm{W}} P(\mathfrak{w}_{k-2}, d\mathfrak{w}_{k-1}) \\ &\times \Big( \int_{\{\mathfrak{w}_k^{\langle 1\rangle} > \sigma\sqrt{t}\, c_\alpha\}} (\mathfrak{w}_k^{\langle 1\rangle} - \sigma\sqrt{t}\, c_\alpha) P(\mathfrak{w}_{k-1}, d\mathfrak{w}_k^{\langle 1\rangle} \times \{0,1\}) \\ &- \int_{\{\mathfrak{w}_k^{\langle 1\rangle} < \sigma\sqrt{t}\,(c_\alpha - x_{\alpha,\beta})\}} (\sigma\sqrt{t}\,(c_\alpha - x_{\alpha,\beta}) - \mathfrak{w}_k^{\langle 1\rangle}) P(\mathfrak{w}_{k-1}, d\mathfrak{w}_k^{\langle 1\rangle} \times \{0,1\}) \Big).\end{aligned}$$

The following theorem shows that application of the strategy (19) implicates the increase (in terms of mean values) of dynamic solvency provisions.

[6]Note that $\mathfrak{w}_k^{\langle 2\rangle} = 0$ implies $\mathfrak{w}_k^{\langle 1\rangle} > 0$, $k = 1, 2, \dots$.

THEOREM 3.3. *In the above multiperiodic diffusion stationary risk model, for the strategy* (19) *and for each* $k = 1, 2, \dots,$

$$\mathsf{E}^{\pi,\gamma} S_{(z(W_{k-1}^{\langle 1\rangle}),t)} > 0.$$

PROOF. Note that

$$P(\mathfrak{w}_{k-1}\,; d\mathfrak{w}_k^{\langle 1\rangle} \times \{0,1\}) = \mathsf{P}\big\{R_t\big(\widehat{u}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}, \widehat{\tau}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}\big) \in d\mathfrak{w}_k^{\langle 1\rangle}\big\},$$

which equals $\mathsf{P}\{\sigma\sqrt{t}c_\alpha - \sigma \mathsf{W}_t \in d\mathfrak{w}_k^{\langle 1\rangle}\} = \Phi_{\{\sigma\sqrt{t}c_\alpha,\sigma^2 t\}}(d\mathfrak{w}_k^{\langle 1\rangle})$ for each $\mathfrak{w}_{k-1}^{\langle 1\rangle}$ (see (20), (21), (22) and their analogues in case on "non-ruin" kernel). Here $\Phi_{\{\sigma\sqrt{t}c_\alpha,\sigma^2 t\}}(\cdot)$ stands for the normal distribution function with expectation $\sigma\sqrt{t}c_\alpha$ and variance $\sigma^2 t$. It is easy to verify that

$$\int_{\{x>\sigma\sqrt{t}\,c_\alpha\}} (x - \sigma\sqrt{t}\,c_\alpha)\Phi_{\{\sigma\sqrt{t}c_\alpha,\sigma^2 t\}}(dx) \\ - \int_{\{x<\sigma\sqrt{t}\,(c_\alpha - x_{\alpha,\beta})\}} (\sigma\sqrt{t}\,(c_\alpha - x_{\alpha,\beta}) - x)\Phi_{\{\sigma\sqrt{t}c_\alpha,\sigma^2 t\}}(dx) > 0,$$

which completes the proof. □

## 4. Some generalizations and discussion

In Section 3 we dealt with Markov t.f.p.m. and pure Markov control strategy, which yields relatively compact formulae. Moreover, we dealt with the stationary case, when duration $t$, the parameters of the annual risks $\mu$, $\sigma$ and the parameters of the strategy $\alpha$, $\beta$ are year-by-year equal. Generalizations of this framework are numerous.

**4.1. Non-stationary case.** Generalization to the series of annual risks which remain known but different throughout successive insurance years is straightforward: endow $t$, $\mu$, $\sigma$, $\alpha$, $\beta$ in Section 3 with subscripts indicating the number of year, i.e., $t_k$, $\mu_k$, $\sigma_k$, $\alpha_k$, $\beta_k$. The arguments in the proofs remain essentially the same.

**4.2. Refined control.** There could be proposed many refined control strategies backed by insurance practice. A straightforward example of refined control is adaptive selection of the control parameters $\alpha$, $\beta$ feed-backed on the past history. One has

$$\mathrm{W} = \mathsf{R} \times \{0,1\} \quad \text{and} \quad \mathrm{U} = \mathsf{R}^+ \times \mathsf{R}^+ \times (0,1) \times (0,1),$$

with t.f.p.m. (18) and the control strategy defined (compare to (19)) by the quadruple

$$\gamma_{k-1}^{\langle 1\rangle}(\mathfrak{w}_{k-1}) = \widehat{u}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t}, \quad \gamma_{k-1}^{\langle 2\rangle}(\mathfrak{w}_{k-1}) = \widehat{\tau}_{z(\mathfrak{w}_{k-1}^{\langle 1\rangle}),t},$$
$$\gamma_{k-1}^{\langle 3\rangle}(\mathfrak{w}_{k-1}) = \alpha(\mathfrak{w}_{k-1}), \quad \gamma_{k-1}^{\langle 4\rangle}(\mathfrak{w}_{k-1}) = \beta(\mathfrak{w}_{k-1}),$$

with $\alpha = \alpha(\mathfrak{w}_{k-1})$, $\beta = \beta(\mathfrak{w}_{k-1})$ in $\gamma_{k-1}^{\langle 1\rangle}(\mathfrak{w}_{k-1})$ and $\gamma_{k-1}^{\langle 2\rangle}(\mathfrak{w}_{k-1})$. A sensible choice of the functions $\alpha : \mathrm{W} \to (0,1)$ and $\beta : \mathrm{W} \to (0,1)$ must, e.g., return "tougher" control values, as $\mathfrak{w}_{k-1}^{\langle 2\rangle} = 1$ (i.e., when ruin occurs in the previous year).

Another example addresses solvency control levels which are early warning levels with regard to the ruin (see [**20**] where solvency control levels are discussed from the positions of regulation). Introduce $m$ control levels $0 < \varrho_1 < \dots < \varrho_m$. Set t.f.p.m. (compare to (18))

$$\pi_k(\mathfrak{w}_{k-1}, \mathfrak{u}_{k-1}\,; d\mathfrak{w}_k) = \mathsf{P}\{R_t(\mathfrak{u}_{k-1}) \in d\mathfrak{w}_k^{\langle 1\rangle}, \mathbf{1}_{\{M_k(\mathfrak{u}_{k-1})<0\}} \in d\mathfrak{w}_k^{\langle 2\rangle}, \\ \mathbf{1}_{\{M_k(\mathfrak{u}_{k-1})<\varrho_1\}} \in d\mathfrak{w}_k^{\langle 3\rangle}, \dots, \mathbf{1}_{\{M_k(\mathfrak{u}_{k-1})<\varrho_m\}} \in d\mathfrak{w}_k^{\langle m+2\rangle}\} \quad (24)$$

and put $\mathrm{W} = \mathsf{R} \times \{0,1\} \times \dots \times \{0,1\}$ with $\mathfrak{w}_k = (\mathfrak{w}_k^{\langle 1\rangle}, \mathfrak{w}_k^{\langle 2\rangle}, \dots, \mathfrak{w}_k^{\langle m+2\rangle}) \in \mathrm{W}$. A sensible control should impose gradually tougher restrictions after downward crossings of the control levels.

**4.3. Non-Markov modelling.** A kind of non-Markov control is mentioned, e.g., in Directives [**7**], where three- and seven-years feedback is applied. For such control, of which the simplest case is the two-year feedback

$$\mathfrak{u}_{k-1} = \gamma_{k-1}(\mathfrak{w}_{k-2}, \mathfrak{w}_{k-1}),$$

non-Markov generalizations of t.f.p.m. (18) may be applied, e.g.,

$$\pi_k(\mathfrak{w}_{k-2}, \mathfrak{w}_{k-1}, \mathfrak{u}_{k-1}\,; d\mathfrak{w}_k) = \begin{cases} \pi_k^{[1]}(\mathfrak{w}_{k-1}, \mathfrak{u}_{k-1}\,; d\mathfrak{w}_k), & \mathfrak{w}_{k-2}^{\langle 2\rangle} = 0, \\ \pi_k^{[2]}(\mathfrak{w}_{k-1}, \mathfrak{u}_{k-1}\,; d\mathfrak{w}_k), & \mathfrak{w}_{k-2}^{\langle 2\rangle} = 1, \end{cases}$$

where $\pi_k^{[1]}$ is a "standard" t.f.p.m. (i.e., t.f.p.m. (18)) applied when in year $k-2$ there is no ruin, and $\pi_k^{[2]}$ is a t.f.p.m. (for example, (24)) designed for a more cautious control after a ruin in year $k-2$.

**4.4. Diffusion approximations.** The Brownian model is of particular attention because of its role in diffusion approximation, when the random walk is replaced by an appropriate Brownian motion process for which the probabilities of interest are explicit. In our context the standard conditions (see [**12**], [**10**], [**11**], [**19**], [**18**], [**8**]) may be applied to approximate the explicit transition probabilities found in Section 3 and the lower and upper bounds of the zone-adaptive strategy found in Section 2. This analysis, though strategically clear, requires many technicalities and will be done elsewhere.

**4.5. Stress testing and scenario analysis.** Recall that scenario analysis typically refers to varying a range of parameters in the model of multiperiodic insurance process according to some scenario of nature, and stress testing refers to such shifting the values of individual parameters that affects critically the insurer's financial position. Both are made to evaluate the impact of these effects on the insurer's business.

Important is to examine the multi-periodic diffusion risk models with incompletely known underlying risk. The risk structure may be specified by a set of particular scenarios of nature. For example, a volatile nature scenario is formed by assuming the successive annual claims out-pay rates i.i.d. outcomes of a random variable, always unknown at the moment of decision making. The adaptive control strategies introduced in the paper under complete information may be developed to compensate the inevitable errors of decision making under incomplete information.

## 5. Auxiliary results for Brownian motion

For the reader's convenience we collect in this section some formulae for real-valued Brownian motion with linear drift, $\theta t + \sigma \mathsf{W}_t$, $t \geqslant 0$, where $\theta \in \mathsf{R}$, $\sigma > 0$.

THEOREM 5.1. *For* $x \geqslant 0$

$$\mathsf{P}\{\sup_{0\leqslant s\leqslant t} \mathsf{W}_s > x\} = 2\mathsf{P}\{\mathsf{W}_t > x\}.$$

This result is well known. Refer to formula 1.1.4 in Part II, Chapter 1 of [**4**].

THEOREM 5.2. *For* $x \geqslant 0$ *and* $\theta \in \mathsf{R}$, $\sigma > 0$

$$\mathsf{P}\{\sup_{0\leqslant s\leqslant t} (\theta s + \sigma \mathsf{W}_s) \leqslant x\} = \Phi\Big(\frac{x-\theta t}{\sigma\sqrt{t}}\Big) - \exp\{2\theta x/\sigma^2\}\Phi\Big(\frac{-x-\theta t}{\sigma\sqrt{t}}\Big).$$

This result is well known (see, e.g., [**13**], Example 1 on p. 27). Refer to formula 1.1.4 in Part II, Chapter 2 of [**4**].

THEOREM 5.3. *For $x \geqslant 0$ and $\theta \in \mathsf{R}$, $\sigma > 0$*

$$\mathsf{P}\{\theta t + \sigma \mathsf{W}_t \in dy, \sup_{0 \leqslant s \leqslant t} (\theta s + \sigma \mathsf{W}_s) \leqslant x\} = \mathsf{P}\{\theta t + \sigma \mathsf{W}_t \in dy\} - \mathsf{P}\{\theta t + \sigma \mathsf{W}_t \in dy, \sup_{0 \leqslant s \leqslant t} (\theta s + \sigma \mathsf{W}_s) \geqslant x\},$$

*where*

$$\mathsf{P}\{\theta t + \sigma \mathsf{W}_t \in dy\} = \frac{1}{\sigma\sqrt{2\pi t}} \exp\{-(y - \theta t)^2/2\sigma^2 t\}\, dy$$

*and*

$$\mathsf{P}\{\theta t + \sigma \mathsf{W}_t \in dy, \sup_{0 \leqslant s \leqslant t} (\theta s + \sigma \mathsf{W}_s) \geqslant x\} = \frac{1}{\sigma\sqrt{2\pi t}} \exp\{(2\theta y t - \theta^2 t^2 - (|y - x| + x)^2)/2\sigma^2 t\}\, dy.$$

This result is well known. Refer to formulae 1.0.6 and 1.1.8 in Part II, Chapter 2 of [**4**].

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