

# ON THE NON-LIFE SOLVENCY II MODEL

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## **Abstract**

We present a simple actuarial rationale for the non-life economic capital formula proposed for Solvency II in QIS3. On the statistical side we improve on the methodology in QIS3 by defining company specific estimators for all quantities of interest including premium risk and reserve risk volatilities as well as correlation coefficients at the granularity level of lines of business.

We develop the non-life Solvency II economic capital formula applying both the value-at-risk (VaR) and conditional value-at-risk (CVaR) risk measures under a log-normal distribution of the portfolio combined ratio, which is defined as the ratio of incurred claims inclusive “run-off” to the premium and reserve volume. We determine confidence levels under which both methods yield approximately identical practical results. Moreover, we point out that economic capital modeling should neither be restricted to a log-normal distribution assumption nor to the VaR and CVaR risk measures and refer to the actuarial literature for various extensions on this. The portfolio combined ratio is decomposed in a weighted sum of the premium risk ratio and the reserve risk ratio as suggested in QIS3. Based on the basic portfolio risk ratio model summarized in the Appendix, we propose simple weighted estimators for all volatilities and correlation coefficients of interest. A numerical example illustrates the use of the proposed estimators.

## **Key words**

Non-life Solvency II, premium risk, reserve risk, value-at-risk, conditional value-at-risk, weighted estimators

## **1. Introduction**

The present note fulfils a two-fold modelling and statistical purpose. On the modelling side it offers a simple actuarial rationale for the economic capital formula proposed in QIS3 (2007). On the statistical side it improves on the methodology in QIS3 (2007) by defining company specific estimators for all quantities of interest including premium risk and reserve risk volatilities as well as correlation coefficients at the granularity level of lines of business. A more detailed account of the content follows.

Section 2 develops the non-life Solvency II economic capital formula applying both the value-at-risk (VaR) and conditional value-at-risk (CVaR) risk measures under a log-normal distribution of the portfolio combined ratio, which is defined as the ratio of incurred claims inclusive “run-off” to the premium and reserve volume. We determine confidence levels under which both methods yield approximately identical practical results. Moreover, we point out that economic capital modeling should neither be restricted to a log-normal distribution assumption nor to the VaR and CVaR risk measures and refer to the actuarial literature for various extensions on this. In Section 3 the portfolio combined ratio is decomposed in a weighted sum of the premium risk ratio and the reserve risk ratio as suggested in QIS3 (2007). Based on the basic portfolio risk ratio model summarized in the Appendix, we propose simple weighted estimators for all volatilities and correlation coefficients of interest. Finally, Section 4 illustrates the use of the proposed estimators.

## **2. The Non-Life economic capital formula**

Suppose an insurance risk portfolio over a fixed time period, say over a one-year time period  $[0,1]$  between the times  $t=0$  and  $t=1$ , is described by the following quantities:

- $P$  : the (net) *risk premium* of the portfolio for the time period  
 $S$  : the random *aggregate claims* of the portfolio over the time period

While the risk premium is supposed to be known at the beginning of the period, the random aggregate claims are not. The *random loss* of the portfolio at the beginning of the time period is described by the difference between aggregate claims and risk premium and defined by the random variable

$$L = S - P. \quad (2.1)$$

In non-life insurance the aggregate claims over the time period are taken exclusive of the “run-off” and include the claims  $Y$  paid out during the time period and the change in claims reserves  $\Delta R = R_1 - R_0$ , where  $R_t$  denotes the *claims reserves* at time  $t$ , which consists of the total reserves for outstanding claims or RBNS (Reported But Not Settled) claims and the reserves for IBNR (Incurred But Not Reported) claims. Therefore one has the equality  $S = Y + \Delta R$ . At time  $t=0$  the claims reserve  $R_0$  is known while  $R_1$  is unknown. The *volume*  $V = P + R_0$  of the portfolio, which is defined as the sum of the risk premium and the claims reserves at the beginning of the period, is known at time  $t=0$ . Consider the ratio of the random loss to the volume, which can be written as

$$\frac{L}{V} = \frac{Y + R_1 - (P + R_0)}{P + R_0} = X - 1, \quad X = \frac{Y + R_1}{P + R_0} \quad (2.2)$$

where  $X$  represents a *combined ratio* of the portfolio (ratio of incurred claims inclusive “run-off” to the premium and reserve volume). By the actuarial equivalence principle or fair value principle, the random loss vanishes in the average, that is  $E[L] = 0$ . This implies that the expected target of the combined ratio is one or  $E[X] = 1$ . The Solvency II model assumes that the random combined ratio is log-normally distributed, say with parameters  $\mu_x$  and  $\sigma_x$  (see QIS3 (2007), I.3.236, p.81). The *portfolio volatility* parameter  $\sigma = \sqrt{\text{Var}[X]}$  is defined to be the standard deviation of the combined ratio of the portfolio. By (2.2) it identifies with the standard deviation of the ratio of the random loss to the volume, that is one has alternatively

$$\sigma^2 = \text{Var}\left[\frac{L}{V}\right]. \quad (2.3)$$

Since  $X$  is log-normally distributed with mean one, one has the equalities

$$E[X] = e^{\mu_x + \frac{1}{2}\sigma_x^2} = 1, \quad \sigma^2 = \text{Var}[X] = e^{\sigma_x^2} - 1, \quad (2.4)$$

which imply the relationships

$$\mu_x = -\frac{1}{2}\sigma_x^2, \quad \sigma_x^2 = \ln(1 + \sigma^2). \quad (2.5)$$

The *economic capital* of the insurance risk portfolio to the confidence level  $\alpha$  is supposed to depend only on the random loss and is denoted by  $EC_\alpha[L]$ . In the standard Solvency II approach, the economic capital is defined to be the value-at-risk (VaR) of the random loss taken at the confidence level  $\alpha = 99.5\%$ , that is  $EC_\alpha[L] = \text{VaR}_\alpha[L]$ . Using (2.2), the log-normal assumption on  $X$  and (2.5) one obtains the non-life economic capital formula (QIS3 (2007), I.3.235, p.81) as follows:

$$\begin{aligned} EC_\alpha[L] &= \text{VaR}_\alpha[L] = \text{VaR}_\alpha\left[\frac{L}{V}\right] \cdot V = (\text{VaR}_\alpha[X] - 1) \cdot V \\ &= \left(\exp\left\{\sigma_x \Phi^{-1}(\alpha) - \frac{1}{2}\sigma_x^2\right\} - 1\right) \cdot V = \rho_\alpha(\sigma) \cdot V \end{aligned} \quad (2.6)$$

with the volatility dependent function

$$\rho_\alpha(\sigma) = \frac{\exp\left\{\Phi^{-1}(\alpha) \cdot \sqrt{\ln(1 + \sigma^2)}\right\}}{\sqrt{1 + \sigma^2}} - 1, \quad (2.7)$$

where  $\Phi^{-1}(\alpha)$  denotes the  $\alpha$ -quantile of the standard normal distribution  $\Phi(x)$ . Alternatively, and as first suggested in the CEIOPS consultation paper CP20 (2006), 5.309, p.137, one can instead define the economic capital to be the tail value-at-risk (TailVaR) or conditional value-at-risk (CVaR) of the random loss taken at the confidence level  $\alpha = 99\%$ . With this choice of risk measure, one obtains the following economic capital formula:

$$\begin{aligned}
 EC_\alpha[L] &= CVaR_\alpha[L] = CVaR_\alpha\left[\frac{L}{V}\right] \cdot V = (CVaR_\alpha[X] - 1) \cdot V \\
 &= \left( VaR_\alpha[X] + \frac{1}{1-\alpha} E[(X - VaR_\alpha[X])_+] - 1 \right) \cdot V = \rho_\alpha(\sigma) \cdot V
 \end{aligned}
 \tag{2.8}$$

with the volatility dependent function

$$\rho_\alpha(\sigma) = \frac{\alpha - \Phi\left(\Phi^{-1}(\alpha) - \sqrt{\ln(1 + \sigma^2)}\right)}{1 - \alpha},
 \tag{2.9}$$

which is obtained by noting that for a log-normal distribution with parameters (2.5) one has

$$\begin{aligned}
 E[(X - VaR_\alpha[X])_+] &= e^{\mu_x + \frac{1}{2}\sigma_x^2} \cdot \Phi\left(\frac{\mu_x - \ln(VaR_\alpha[X])}{\sigma_x} + \sigma_x\right) - VaR_\alpha[X] \cdot \Phi\left(\frac{\mu_x - \ln(VaR_\alpha[X])}{\sigma_x}\right) \\
 &= \Phi(\sigma_x - \Phi^{-1}(\alpha)) - \Phi(-\Phi^{-1}(\alpha)) \cdot VaR_\alpha[X] = 1 - \Phi\left(\Phi^{-1}(\alpha) - \sqrt{\ln(1 + \sigma^2)}\right) - (1 - \alpha) \cdot VaR_\alpha[X].
 \end{aligned}$$

It is interesting to compare numerically the formulas (2.7) and (2.9). The QIS3 VaR proposal (2.7) with  $\alpha = 99.5\%$  is an implementation of the rule of thumb  $\rho_\alpha(\sigma) \approx 3 \cdot \sigma$  (for another explanation see Hürlimann(2004), Example 7.1, p.95). For  $\alpha = 99\%$  the CVaR method requires only slightly more economic capital. Table 2.1 provides a numerical comparison of the quotients  $\rho_\alpha(\sigma)/\sigma$  and determines confidence levels under which both methods coincide approximately up to the third decimal place (green and yellow columns).

**Table 2.1:** Comparison of the standard non-life Solvency II VaR and CVaR formulas

VaR Method				CVaR Method			
confidence level	0.99	<b>0.995</b>	0.99624	confidence level	<b>0.98675</b>	0.99	0.995
percentile	2.326	<b>2.576</b>	2.673	percentile	<b>2.219</b>	2.326	2.576
volatility				volatility			
12.0%	2.594	<b>2.925</b>	3.056	12.0%	<b>2.923</b>	3.054	3.366
12.5%	2.605	<b>2.940</b>	3.073	12.5%	<b>2.939</b>	3.071	3.387
13.0%	2.617	<b>2.955</b>	3.090	13.0%	<b>2.954</b>	3.088	3.408
13.5%	2.628	<b>2.970</b>	3.106	13.5%	<b>2.969</b>	3.105	3.429
14.0%	2.639	<b>2.985</b>	3.123	14.0%	<b>2.985</b>	3.122	3.450
<b>14.5%</b>	2.650	<b>3.000</b>	3.139	<b>14.5%</b>	<b>3.000</b>	3.139	3.471
15.0%	2.661	<b>3.015</b>	3.156	15.0%	<b>3.015</b>	3.156	3.492
15.5%	2.672	<b>3.030</b>	3.173	15.5%	<b>3.031</b>	3.173	3.514
16.0%	2.684	<b>3.045</b>	3.190	16.0%	<b>3.046</b>	3.190	3.535
16.5%	2.695	<b>3.060</b>	3.206	16.5%	<b>3.062</b>	3.207	3.556
17.0%	2.706	<b>3.075</b>	3.223	17.0%	<b>3.077</b>	3.224	3.578

At this stage, it must be pointed out that from an actuarial viewpoint, economic capital modeling should neither be restricted to a log-normal distribution assumption nor to the VaR and CVaR risk measures. In general, it is possible to consider Gamma and elliptical type distributions (e.g. Hürlimann(2001), Landsman and Valdez(2003), Valdez(2005), Furman and Landsman(2005/07)), compound Poisson distributions (e.g. Hürlimann(2003)) or even distribution-free methods (e.g. Hürlimann(2002)). On the other side, different economic capital models can be designed and other risk measures can be considered (e.g. Dhaene, Goovaerts and Kaas(2003), Hürlimann(2004)). This opens the way to a wide variety of flexible internal models for Solvency II.

### 3. Estimation of the volatility parameter

To estimate the volatility parameter, historical data on the risk portfolio and insurance market information can be used. In contrast to Section 2, the time horizon for estimation is not restricted to the current one-year time horizon for economic capital evaluation, but it may include past observation periods or even simulated future periods. Let  $V_p$  denote the risk premium volume,  $V_r$  the claims reserve volume, and  $V = V_p + V_r$  the portfolio volume of the current risk portfolio at time  $t = 0$ . The combined ratio (2.2) can be rewritten as

$$X = \left( \frac{V_p}{V} \right) \cdot X^p + \left( \frac{V_r}{V} \right) \cdot X^r, \quad (3.1)$$

where  $X^p = \frac{Y}{P}$  represent the random ratio of paid claims to risk premiums and  $X^r = \frac{R_1}{R_0}$

the random ratio of end of year claims reserves to beginning of year claims reserves, which in the Solvency II terminology are called *premium risk* ratio respectively *reserve risk* ratio (QIS3 (2007), I.3.226-I.3.229, p.79). The correlation coefficient between the premium risk and the reserve risk, denoted by  $\rho_{pr}$ , is defined by the covariance relationship

$$\text{Cov}[X^p, X^r] = \rho_{pr} \sigma_p \sigma_r, \quad \sigma_p = \sqrt{\text{Var}[X^p]}, \quad \sigma_r = \sqrt{\text{Var}[X^r]}. \quad (3.2)$$

Using (3.1) and (3.2) one obtains the relationship

$$\sigma = \frac{1}{V} \sqrt{(\sigma_p V_p)^2 + (\sigma_r V_r)^2 + 2\rho_{pr} \sigma_p \sigma_r V_p V_r}. \quad (3.3)$$

According to (3.3) the portfolio volatility depends on the *premium volatility*  $\sigma_p$ , the *reserve volatility*  $\sigma_r$ , the correlation coefficient  $\rho_{pr}$  and appropriate volumes. To estimate volatilities of random ratios, we apply the general method presented in the Appendix. For this, suppose the following portfolio data over  $m > 1$  (past) years is available:

$P_k$  : (net) risk premiums over  $m$  years,  $k = 1, \dots, m$

$Y_k$  : paid claims during the time periods  $[k-1, k]$ ,  $k = 1, \dots, m$

$R_k$  : claims reserve at the beginning and end of the  $m$  years,  $k = 0, \dots, m$

$\Delta R_k = R_k - R_{k-1}$  : change in claims reserves during the time periods  $[k-1, k]$ ,  $k = 1, \dots, m$

A portfolio based estimator of the premium volatility is

$$\hat{\sigma}_p = \sqrt{\sum_{k=1}^m \frac{P_k}{P^\bullet} \cdot \left( \frac{Y_k}{P_k} - \hat{\mu}_p \right)^2}, \quad \hat{\mu}_p = \frac{\sum_{k=1}^m Y_k}{P^\bullet}, \quad P^\bullet = \sum_{k=1}^m P_k. \quad (3.4)$$

A similar portfolio based estimator of the reserve volatility is

$$\hat{\sigma}_r = \sqrt{\sum_{k=1}^m \frac{R_{k-1}}{R^\bullet} \cdot \left( \frac{\Delta R_k}{R_{k-1}} - \hat{\mu}_r \right)^2}, \quad \hat{\mu}_r = \frac{\sum_{k=1}^m \Delta R_k}{R^\bullet} = \frac{R_m - R_0}{R^\bullet}, \quad R^\bullet = \sum_{k=1}^m R_{k-1}, \quad (3.5)$$

which is obtained by noting that  $\frac{\Delta R_k}{R_{k-1}} - \frac{R_m - R_0}{R^\bullet} = \frac{R_k}{R_{k-1}} - \frac{\sum_{j=1}^m R_j}{R^\bullet}$ . Applying the same technique, the following estimator of the portfolio volatility is obtained:

$$\hat{\sigma} = \sqrt{\sum_{k=1}^m \frac{P_k + R_{k-1}}{P^\bullet + R^\bullet} \cdot \left( \frac{Y_k + R_k}{P_k + R_{k-1}} - \hat{\mu} \right)^2}, \quad (3.6)$$

$$\hat{\mu} = \frac{\sum_{k=1}^m Y_k + \sum_{k=1}^m R_k}{P^\bullet + R^\bullet}, \quad P^\bullet + R^\bullet = \sum_{k=1}^m P_k + \sum_{k=1}^m R_{k-1}.$$

Using these estimators, the equation (3.3) yields the following compatible estimator of the correlation coefficient between premium risk and reserve risk

$$\hat{\rho}_{pr} = \frac{1}{2} \cdot \frac{(\hat{\sigma}V)^2 - (\hat{\sigma}_p V_p)^2 - (\hat{\sigma}_r V_r)^2}{(\hat{\sigma}_p V_p) \cdot (\hat{\sigma}_r V_r)}. \quad (3.7)$$

In the current framework of the standard approach to Solvency II, the estimation of the portfolio volatility should be done under the more refined granularity level of lines of business. Suppose that the portfolio consists of  $n$  lines of business (LoB), for which the following historical data over the  $m > 1$  (past) years is available:

$P_i^j$  : (net) risk premiums of LoB  $i \in \{1, \dots, n\}$  in year  $j \in \{1, \dots, m\}$

$Y_i^j$  : paid claims of LoB  $i \in \{1, \dots, n\}$  during year  $[j-1, j]$ ,  $j \in \{1, \dots, m\}$

$R_i^j$  : claims reserve of LoB  $i \in \{1, \dots, n\}$  at beginning and end of year  $j \in \{0, \dots, m\}$

$\Delta R_i^j = R_i^j - R_i^{j-1}$  : change in claims reserves of LoB  $i \in \{1, \dots, n\}$  during year  $[j-1, j]$

Again, let us apply the general method of the Appendix. For each line of business  $i \in \{1, \dots, n\}$ , let  $V_{p,i}$  denote its line of business risk premium volume,  $V_{r,i}$  its line of business claims reserve volume, and  $V_i = V_{p,i} + V_{r,i}$  its line of business volume at time  $t = 0$ . Then

$V_p = \sum_{i=1}^n V_{p,i}$  represents the risk premium volume,  $V_r = \sum_{i=1}^n V_{r,i}$  the claims reserve volume

and  $V = \sum_{i=1}^n V_i$  the portfolio volume of the current risk portfolio at time  $t = 0$ . Consider the

premium risk ratio  $X_i^p$  and the reserve risk ratio  $X_i^r$  of the lines of business  $i \in \{1, \dots, n\}$ .

Let further  $w_i^p = \frac{V_{p,i}}{V_p}$  be the risk premium weight and  $w_i^r = \frac{V_{r,i}}{V_r}$  be the claims reserve

weight associated to the line of business  $i \in \{1, \dots, n\}$ , and let  $w^p = (w_1^p, \dots, w_n^p)$  and

$w^r = (w_1^r, \dots, w_n^r)$  be the corresponding weight vectors. The overall premium risk ratio and reserve risk ratio of the portfolio are defined by the linear combinations

$$X^p = \sum_{i=1}^n w_i^p X_i^p, \quad X^r = \sum_{i=1}^n w_i^r X_i^r. \quad (3.8)$$

The parameters of the premium risk ratio are described by the mean vector  $v^p = (\mu_1^p, \dots, \mu_n^p)$ , where  $\mu_i^p = E[X_i^p]$  is the mean premium risk ratio of the  $i$ -th line of

business, and the covariance matrix  $\Sigma^p = (\rho_{ij}^p \sigma_i^p \sigma_j^p)$ , where  $\sigma_i^p = \sqrt{\text{Var}[X_i^p]}$  is the standard deviation of the premium risk ratio of the  $i$ -th line of business and  $\rho_{ij}^p$  is the correlation coefficient between the premium risk ratios  $X_i^p$  and  $X_j^p$ ,  $i, j = 1, \dots, n$ .

Similarly, the parameters of the reserve risk ratio are described by the mean vector  $v^r = (\mu_1^r, \dots, \mu_n^r)$ , where  $\mu_i^r = E[X_i^r]$  is the mean reserve risk ratio of the  $i$ -th line of

business, and the covariance matrix  $\Sigma^r = (\rho_{ij}^r \sigma_i^r \sigma_j^r)$ , where  $\sigma_i^r = \sqrt{\text{Var}[X_i^r]}$  is the standard deviation of the  $i$ -th line of business reserve risk ratio and  $\rho_{ij}^r$  is the correlation coefficient

between the reserve risk ratios  $X_i^r$  and  $X_j^r$ ,  $i, j = 1, \dots, n$ . According to (3.8), estimators of the mean and variance of the portfolio premium risk ratio and reserve risk ratio necessarily satisfy relationships

$$\begin{aligned} \hat{\mu}_p &= \hat{v}^p \cdot w^{p,T}, & \hat{\mu}_r &= \hat{v}^r \cdot w^{r,T}, \\ \hat{\sigma}_p^2 &= w^r \cdot \hat{\Sigma}^p \cdot w^{r,T}, & \hat{\sigma}_r^2 &= w^r \cdot \hat{\Sigma}^r \cdot w^{r,T}, \end{aligned} \quad (3.9)$$

where  $\hat{v}^p, \hat{v}^r, \hat{\Sigma}^p, \hat{\Sigma}^r$  are estimators of the corresponding mean vectors and covariance matrices, which are obtained as follows. Let us begin with the premium risk ratio. Consider

the premium risk ratios  $X_i^{p,j} = \frac{Y_i^j}{P_i^j}$  with the weights  $w_i^{p,j} = \frac{P_i^j}{P_i^\bullet}$ ,  $P_i^\bullet = \sum_{j=1}^m P_i^j$ , and the

reserve risk ratios  $X_i^{r,j} = \frac{R_i^j}{R_i^{j-1}}$  with the weights  $w_i^{r,j} = \frac{R_i^{j-1}}{R_i^\bullet}$ ,  $R_i^\bullet = \sum_{j=1}^m R_i^{j-1}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ . Then one has the following mean and standard deviation estimators

$$\begin{aligned}\hat{\mu}_i^p &= \sum_{j=1}^m w_i^{p,j} X_i^{p,j}, & \hat{\sigma}_i^p &= \sqrt{\sum_{j=1}^m w_i^{p,j} \cdot (X_i^{p,j} - \hat{\mu}_i^p)^2}, \\ \hat{\mu}_i^r &= \sum_{j=1}^m w_i^{r,j} X_i^{r,j}, & \hat{\sigma}_i^r &= \sqrt{\sum_{j=1}^m w_i^{r,j} \cdot (X_i^{r,j} - \hat{\mu}_i^r)^2}.\end{aligned}\tag{3.10}$$

To estimate the correlation coefficients  $\rho_{ij}^p$ ,  $i \neq j$  between line of business premium risk ratios, consider the sub-portfolios with premium risk ratios  $X_i^p$  and  $X_j^p$ , whose overall premium risk ratios are determined by the weighted mean

$$X_{ij}^p = \frac{P_i^\bullet}{P_i^\bullet + P_j^\bullet} X_i^p + \frac{P_j^\bullet}{P_i^\bullet + P_j^\bullet} X_j^p, \quad i \neq j, i, j = 1, \dots, n.\tag{3.11}$$

The historical data consists of premium risk ratios  $X_{ij}^{p,k}$  and weights  $w_{ij}^{p,k}$  defined by

$$\begin{aligned}X_{ij}^{p,k} &= \frac{P_i^k}{P_i^k + P_j^k} X_i^{p,k} + \frac{P_j^k}{P_i^k + P_j^k} X_j^{p,k}, \\ w_{ij}^{p,k} &= \frac{P_i^k + P_j^k}{P_i^\bullet + P_j^\bullet}, \quad i \neq j, i, j = 1, \dots, n, k = 1, \dots, m.\end{aligned}\tag{3.12}$$

Interpreting the  $X_{ij}^{p,k}$ 's as outcomes of  $X_{ij}^p$  with probability function  $\Pr(X_{ij}^p = X_{ij}^{p,k}) = w_{ij}^{p,k}$ , one obtains the following mean and standard deviation estimators of  $X_{ij}^p$ :

$$\begin{aligned}\hat{\mu}_{ij}^p &= \sum_{k=1}^m w_{ij}^{p,k} X_{ij}^{p,k} = \frac{P_i^\bullet}{P_i^\bullet + P_j^\bullet} \hat{\mu}_i^p + \frac{P_j^\bullet}{P_i^\bullet + P_j^\bullet} \hat{\mu}_j^p, \\ \hat{\sigma}_{ij}^p &= \sqrt{\sum_{k=1}^m w_{ij}^{p,k} (X_{ij}^{p,k} - \hat{\mu}_{ij}^p)^2}.\end{aligned}\tag{3.13}$$

A compatible estimator of the correlation coefficient  $\rho_{ij}^p$  is defined through the relationship

$$\hat{\rho}_{ij}^p = \frac{1}{2} \cdot \frac{[(P_i^\bullet + P_j^\bullet) \hat{\sigma}_{ij}^p]^2 - [P_i^\bullet \hat{\sigma}_i^p]^2 - [P_j^\bullet \hat{\sigma}_j^p]^2}{(P_i^\bullet \hat{\sigma}_i^p)(P_j^\bullet \hat{\sigma}_j^p)}.\tag{3.14}$$

Similarly, to estimate the correlation coefficients  $\rho_{ij}^r$ ,  $i \neq j$  between line of business reserve risk ratios, consider the sub-portfolios with reserve risk ratios  $X_i^r$  and  $X_j^r$ , whose overall reserve risk ratios are determined by the weighted mean

$$X_{ij}^r = \frac{R_i^*}{R_i^* + R_j^*} X_i^r + \frac{R_j^*}{R_i^* + R_j^*} X_j^r, \quad i \neq j, i, j = 1, \dots, n. \quad (3.15)$$

The historical data consists of reserve risk ratios  $X_{ij}^{r,k}$  and weights  $w_{ij}^{r,k}$  defined by

$$\begin{aligned} X_{ij}^{r,k} &= \frac{R_i^{k-1}}{R_i^{k-1} + R_j^{k-1}} X_i^{r,k} + \frac{R_j^{k-1}}{R_i^{k-1} + R_j^{k-1}} X_j^{r,k}, \\ w_{ij}^{r,k} &= \frac{R_i^{k-1} + R_j^{k-1}}{R_i^* + R_j^*}, \quad i \neq j, i, j = 1, \dots, n, k = 1, \dots, m. \end{aligned} \quad (3.16)$$

Interpreting the  $X_{ij}^{r,k}$ 's as outcomes of  $X_{ij}^r$  with probability function  $\Pr(X_{ij}^r = X_{ij}^{r,k}) = w_{ij}^{r,k}$ , one obtains the following mean and standard deviation estimators of  $X_{ij}^r$ :

$$\begin{aligned} \hat{\mu}_{ij}^r &= \sum_{k=1}^m w_{ij}^{r,k} X_{ij}^{r,k} = \frac{R_i^*}{R_i^* + R_j^*} \hat{\mu}_i^r + \frac{R_j^*}{R_i^* + R_j^*} \hat{\mu}_j^r, \\ \hat{\sigma}_{ij}^r &= \sqrt{\sum_{k=1}^m w_{ij}^{r,k} (X_{ij}^{r,k} - \hat{\mu}_{ij}^r)^2}. \end{aligned} \quad (3.17)$$

A compatible estimator of the correlation coefficient  $\rho_{ij}^r$  is defined through the relationship

$$\hat{\rho}_{ij}^r = \frac{1}{2} \cdot \frac{[(R_i^* + R_j^*) \hat{\sigma}_{ij}^r]^2 - [R_i^* \hat{\sigma}_i^r]^2 - [R_j^* \hat{\sigma}_j^r]^2}{(R_i^* \hat{\sigma}_i^r)(R_j^* \hat{\sigma}_j^r)}. \quad (3.18)$$

On the other side, to estimate the portfolio volatility, a similar technique is applied. Let  $X_i$  be the combined ratio of the line of business  $i \in \{1, \dots, n\}$  with weight  $w_i = \frac{V_i}{V}$ . Then the combined ratio of the portfolio is defined by the linear combination of combined ratios

$$X = \sum_{i=1}^n w_i X_i. \quad (3.19)$$

The parameters of the combined ratio are described by the mean vector  $v = (\mu_1, \dots, \mu_n)$ , where  $\mu_i = E[X_i]$  is the mean combined ratio of the  $i$ -th line of business, and the covariance matrix  $\Sigma = (\rho_{ij} \sigma_i \sigma_j)$ , where  $\sigma_i = \sqrt{\text{Var}[X_i]}$  is the standard deviation of the combined ratio of the  $i$ -th line of business and  $\rho_{ij}$  is the correlation coefficient between the

combined ratios  $X_i$  and  $X_j$ ,  $i, j = 1, \dots, n$ . According to (3.19), estimators of the mean and variance of the portfolio combined ratio necessarily satisfy the relationships

$$\hat{\mu} = \hat{\nu} \cdot w^T, \quad \hat{\sigma}^2 = w \cdot \hat{\Sigma} \cdot w^T, \quad (3.20)$$

where  $\hat{\nu}, \hat{\Sigma}$  are estimators of the corresponding mean vectors and covariance matrices,

which are obtained as follows. Consider the combined ratios  $X_i^j = \frac{Y_i^j + R_i^j}{P_i^j + R_i^{j-1}}$  with the

weights  $w_i^j = \frac{P_i^j + R_i^{j-1}}{P_i^* + R_i^*}$ ,  $i = 1, \dots, n, j = 1, \dots, m$ . Then one has the following mean and

standard deviation estimators

$$\hat{\mu}_i = \sum_{j=1}^m w_i^j X_i^j, \quad \hat{\sigma}_i = \sqrt{\sum_{j=1}^m w_i^j \cdot (X_i^j - \hat{\mu}_i)^2}. \quad (3.21)$$

To estimate the correlation coefficients  $\rho_{ij}$ ,  $i \neq j$  between the line of business combined ratios, consider the sub-portfolios with combined ratios  $X_i$  and  $X_j$ , whose overall combined ratios are determined by the weighted mean

$$X_{ij} = \frac{P_i^* + R_i^*}{P_i^* + R_i^* + P_j^* + R_j^*} X_i + \frac{P_j^* + R_j^*}{P_i^* + R_i^* + P_j^* + R_j^*} X_j, \quad i \neq j, i, j = 1, \dots, n. \quad (3.22)$$

The historical data consists of combined ratios  $X_{ij}^k$  and weights  $w_{ij}^k$  defined by

$$X_{ij}^k = \frac{P_i^k + R_i^{k-1}}{P_i^k + R_i^{k-1} + P_j^k + R_j^{k-1}} X_i^k + \frac{P_j^k + R_j^{k-1}}{P_i^k + R_i^{k-1} + P_j^k + R_j^{k-1}} X_j^k, \quad (3.23)$$

$$w_{ij}^k = \frac{P_i^k + R_i^{k-1} + P_j^k + R_j^{k-1}}{P_i^* + R_i^* + P_j^* + R_j^*}, \quad i \neq j, i, j = 1, \dots, n, k = 1, \dots, m.$$

Interpreting the  $X_{ij}^k$ 's as outcomes of  $X_{ij}$  with probability function  $\Pr(X_{ij} = X_{ij}^k) = w_{ij}^k$ , one obtains the mean and standard deviation estimators of  $X_{ij}$

$$\hat{\mu}_{ij} = \sum_{k=1}^m w_{ij}^k X_{ij}^k = \frac{P_i^* + R_i^*}{P_i^* + R_i^* + P_j^* + R_j^*} \hat{\mu}_i + \frac{P_j^* + R_j^*}{P_i^* + R_i^* + P_j^* + R_j^*} \hat{\mu}_j, \quad (3.24)$$

$$\hat{\sigma}_{ij} = \sqrt{\sum_{k=1}^m w_{ij}^k (X_{ij}^k - \hat{\mu}_{ij})^2}.$$

A compatible estimator of the correlation coefficient  $\rho_{ij}$  is defined through the relationship

$$\hat{\rho}_{ij} = \frac{1}{2} \cdot \frac{[(P_i^* + R_i^* + P_j^* + R_j^*)\hat{\sigma}_{ij}]^2 - [(P_i^* + R_i^*)\hat{\sigma}_i]^2 - [(P_j^* + R_j^*)\hat{\sigma}_j]^2}{[(P_i^* + R_i^*)\hat{\sigma}_i][(P_j^* + R_j^*)\hat{\sigma}_j]}. \quad (3.25)$$

Up to the correlation coefficient  $\rho_{pr}$  between the premium risk ratio and the reserve risk ratio, all parameters have been now estimated. A compatible estimator of  $\rho_{pr}$  is of the form (3.7) with volatilities defined in (3.9) and (3.20).

In fact, to estimate the economic capital formula (2.7) or (2.9), it suffices to determine (3.6) provided overall portfolio information is used, or (3.20), (3.21) and (3.25) in case detailed information on the lines of business is available. However, bear in mind that a desirable goal of any solvency model consists to measure also all possible kinds of diversification effects, for example diversification between premium risk and reserve risk and diversification across the lines of business (e.g. CRO Forum (2005)). To determine these diversification effects precisely, it is clear that all of the remaining estimators must be used.

Finally, it is important to note that the standard approach to Solvency II proposes quite different estimators for the above quantities. For the premium risk volatilities it uses a credibility mix between company specific estimators and market wide estimators, for the reserve risk volatilities it prescribes only line of business dependent insurance market based fixed numerical values, and all required correlation coefficients are fixed numerical values (see QIS3 (2007), I.3.242, p.82, I.3.246, p.83, I.3.249-250, p.84). Our method proposes company specific estimators for both the premium risk and reserve risk as well as for the correlation coefficient between these risk factors at the granularity level of lines of business.

#### **4. Numerical illustration**

We illustrate the use of the estimation method presented in Section 3 to calculate the economical capital according to Solvency II including various diversification effects. Table 4.1 lists the available information for a portfolio with  $m = 5$  lines of business over  $n = 5$  years. A short look at the statistics shows that that LoB 1 and LoB 3 are relatively small and stable business segments, LoB 2 loses constantly risk premium volume but has still important claims reserves, LoB 4 and LoB 5 contribute strongly to the overall slight growing of the portfolio volume.

The economic capital is calculated according to the VaR method (2.7) at the confidence level  $\alpha = 99.5\%$  and reported in Table 4.2. At the portfolio level the formulas (3.4)-(3.6) are applied and at the lines of business level (3.10) is used. The correlation coefficients between the premium risk and reserve risk are evaluated with a formula of type (3.7).

The economic capital of the portfolio combined risk is 33'731 and compares with the sum of the stand-alone lines of business economic capitals for premium and reserve risk of total amount 65'350. The total diversification effect of amount 31'619 is due to a diversification effect across the lines of business of amount 25'824 and a portfolio diversification effect between the premium risk and the reserve risk of amount 5'795. The correlation between premium risk and reserve risk is positive at the portfolio level and almost negative at the lines of business level.

On the other side it is also possible to evaluate the economic capital at the portfolio level using the formulas (3.9) and (3.20). In this situation it is necessary to calculate the correlation

coefficients between the lines of business for the premium risk, the reserve risk and the combined risk using the formulas (3.14), (3.18) and (3.25). The Table 4.3 lists the obtained correlation matrices and Table 4.4 summarizes the economic capital evaluation using this second estimation method. In Table 4.3 we observe positive correlation coefficients between the lines of business for the combined risk and mixed positive and negative correlation coefficients for the premium risk and the reserve risk.

The economic capitals of Table 4.4 and 4.2 differ slightly: 9'330 compared to 10'021 for the premium risk, 29'553 compared to 29'505 for the reserve risk, and 31'371 compared to 33'731 for the combined risk. In this situation, the increased total diversification effect of amount 33'979 (compared to 31'619) is due to a diversification effect across the lines of business of amount 26'467 (compared to 25'824) and a portfolio diversification effect between the premium risk and the reserve risk of amount 7'512 (compared to 5'795). The correlation between premium risk and reserve risk is again positive but smaller, which explains the increased diversification.

**Table 4.1:** risk premiums, paid claims and claims reserves for a non-life portfolio

Development year	0	1	2	3	4	5	Total years
<b>Risk premiums / LoB</b>							
1		1000	800	1200	1000	1000	5000
2		8000	7000	6000	5000	4000	30000
3		2100	1700	2700	2100	2400	11000
4		6000	6000	7500	8000	8500	36000
5		1000	2000	3500	5000	6500	18000
<b>Total LoB</b>		18100	17500	20900	21100	22400	100000
<b>Paid claims / LoB</b>							
1		350	200	400	300	400	1650
2		3500	3500	3000	2500	2000	14500
3		800	650	900	600	800	3750
4		2250	1800	2200	2000	2500	10750
5		350	500	500	750	2250	4350
<b>Total LoB</b>		7250	6650	7000	6150	7950	35000
<b>Claims reserve / LoB</b>							
1	900	1100	1300	1300	1200	1000	5800
2	8200	8000	10000	8000	6000	7000	40200
3	1500	1900	3200	2600	2600	2500	11800
4	4200	4800	5500	5500	6700	6900	26700
5	600	1100	1300	2500	3500	4500	9000
<b>Total LoB</b>	15400	16900	21300	19900	20000	21900	93500
<b>Δ Claims reserve / LoB</b>							
1		200	200	0	-100	-200	100
2		-200	2000	-2000	-2000	1000	-1200
3		400	1300	-600	0	-100	1000
4		600	700	0	1200	200	2700
5		500	200	1200	1000	1000	3900
<b>Total LoB</b>		1500	4400	-1400	100	1900	6500

**Table 4.2:** economic capital and diversification effects using (3.6) at the portfolio level

risk	portfolio	economic capital					diversification
		LoB 1	LoB 2	LoB 3	LoB 4	LoB 5	LoB
premium risk	10021	646	2201	1010	3756	4818	2410
reserve risk	29505	2501	24766	12024	6213	7414	23414
combined risk	33731	1758	21033	10868	5901	6160	11990
diversification risks	5795	1389	5934	2166	4069	6072	13834
premium risk vs.		correlation coefficient					
reserve risk	0.415	-0.864	-0.691	0.207	-0.318	-0.405	

**Table 4.3:** correlation matrices for the premium risk, reserve risk and combined risk

premium risk					
	1	2	3	4	5
1	1.000	-0.624	-0.107	0.289	0.579
2	-0.624	1.000	-0.434	0.084	0.243
3	-0.107	-0.434	1.000	0.912	0.818
4	0.289	0.084	0.912	1.000	0.730
5	0.579	0.243	0.818	0.730	1.000
reserve risk					
	1	2	3	4	5
1	1.000	0.836	0.632	0.248	-0.650
2	0.836	1.000	0.633	-0.054	-0.545
3	0.632	0.633	1.000	0.490	-0.579
4	0.248	-0.054	0.490	1.000	-0.541
5	-0.650	-0.545	-0.579	-0.541	1.000
combined risk					
	1	2	3	4	5
1	1.000	0.242	0.723	0.769	0.218
2	0.242	1.000	0.555	0.122	0.043
3	0.723	0.555	1.000	0.632	0.043
4	0.769	0.122	0.632	1.000	0.090
5	0.218	0.043	0.043	0.090	1.000

**Table 4.4:** economic capital and diversification effects using (3.9) and (3.20)

risk	portfolio	economic capital					diversification
		LoB 1	LoB 2	LoB 3	LoB 4	LoB 5	LoB
premium risk	9330	646	2201	1010	3756	4818	3101
reserve risk	29553	2501	24766	12024	6213	7414	23366
combined risk	31371	1758	21033	10868	5901	6160	14349
diversification risks	7512	1389	5934	2166	4069	6072	12118
premium risk vs.		correlation coefficient					
reserve risk	0.198	-0.864	-0.691	0.207	-0.318	-0.405	

### APPENDIX: A basic portfolio risk factor ratio model

To evaluate the economic capital of portfolios of risks in practice, it is often judicious to express risk factors using random ratios. For example, insurance risk loss ratios are quotients from claims to risk premiums while financial risk loss ratios are just negative returns. In the general context of risk factors, it is possible to define the risk factor ratio of a portfolio as a weighted linear combination of the risk factor ratios of its components.

Consider a portfolio with random risk factors  $S = (S_1, \dots, S_n)$  and deterministic vector of volumes  $V = (V_1, \dots, V_n)$  associated to these risk factors. The vector of random risk factor ratios  $X = (X_1, \dots, X_n)$  is defined by  $X_i = \frac{S_i}{V_i}$ ,  $i = 1, \dots, n$ . Let  $V = V_1 + \dots + V_n$  be the total volume of the portfolio, and let  $w = (w_1, \dots, w_n)$  be the vector of portfolio weights with  $w_i = \frac{V_i}{V}$  the weight associated to risk factor  $i$ ,  $i = 1, \dots, n$ . Then the overall risk factor ratio of the portfolio is defined by the linear combination

$$X = \sum_{i=1}^n w_i X_i. \quad (\text{A.1})$$

We assume the existence of the mean vector  $v = (\mu_1, \dots, \mu_n)$ , where  $\mu_i = E[X_i]$  is the mean, and the covariance matrix  $\Sigma = (\rho_{ij} \sigma_i \sigma_j)$  such that  $\sigma_i^2 = \text{Var}[X_i]$  is the variance and  $\rho_{ij}$  is the correlation between the risk factor ratios  $X_i$  and  $X_j$ ,  $i, j = 1, \dots, n$ . Moreover, a joint multivariate distribution function  $F_X(x_1, \dots, x_n)$  with marginal distributions  $F_i(x)$ ,  $i = 1, \dots, n$ , has to be specified, which is compatible with the parameters  $v, \Sigma$ . In terms of the defined risk factor ratios, one is interested in the stand-alone (standard) risk factors  $L_i = V_i \cdot (X_i - \mu_i)$ ,  $i = 1, \dots, n$ , and in the portfolio (standard) risk factor  $L = V \cdot (X - \mu)$ . Using (A.1) the mean and variance of the portfolio risk factor ratio are

$$\mu = v \cdot w^T, \quad \sigma^2 = w \cdot \Sigma \cdot w^T. \quad (\text{A.2})$$

For practical evaluation, there remains the modeling choice for the joint multivariate distribution function  $F_X(x_1, \dots, x_n)$ , and the statistical estimation of the parameters  $v, \Sigma$ . Provided historical data is available, the latter task is solved as follows.

Let  $S^j = (S_1^j, \dots, S_n^j)$  and  $P^j = (P_1^j, \dots, P_n^j)$ ,  $j = 1, \dots, m$ , represent the risk factors and volumes of the portfolio in  $m$  past periods. Consider the risk factor ratios  $X_i^j = \frac{S_i^j}{P_i^j}$ , and the weights  $w_i^j = \frac{P_i^j}{P_i^\bullet}$ ,  $P_i^\bullet = \sum_{j=1}^m P_i^j$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ . Interpreting the historical risk factor ratios  $X_i^j$  as outcomes of the ratio random variables  $X_i$  with probability function  $\Pr(X_i = X_i^j) = w_i^j$ , one obtains the mean and variance parameters

$$\begin{aligned}\mu_i &= E[X_i] = \sum_{j=1}^m w_i^j X_i^j, \\ \sigma_i^2 &= \text{Var}[X_i] = \sum_{j=1}^m w_i^j \cdot (X_i^j - \mu_i)^2, \quad i = 1, \dots, n.\end{aligned}\tag{A.3}$$

To estimate the correlation coefficients  $\rho_{ij}$ ,  $i \neq j$  between risk factors, consider the sub-portfolios with risk factor ratios  $X_i$  and  $X_j$ , whose overall risk factor ratios are determined by the weighted mean

$$X_{ij} = \frac{P_i^\bullet}{P_i^\bullet + P_j^\bullet} X_i + \frac{P_j^\bullet}{P_i^\bullet + P_j^\bullet} X_j, \quad i \neq j, i, j = 1, \dots, n.\tag{A.4}$$

The associated historical data consists of risk factor ratios  $X_{ij}^k$  and weights  $w_{ij}^k$  defined by

$$\begin{aligned}X_{ij}^k &= \frac{P_i^k}{P_i^k + P_j^k} X_i^k + \frac{P_j^k}{P_i^k + P_j^k} X_j^k, \\ w_{ij}^k &= \frac{P_i^k + P_j^k}{P_i^\bullet + P_j^\bullet}, \quad i \neq j, i, j = 1, \dots, n, k = 1, \dots, m.\end{aligned}\tag{A.5}$$

Interpreting the  $X_{ij}^k$ 's as outcomes of  $X_{ij}$  with probability function  $\Pr(X_{ij} = X_{ij}^k) = w_{ij}^k$ , one obtains the mean and variance parameters

$$\mu_{ij} = E[X_{ij}] = \sum_{k=1}^m w_{ij}^k X_{ij}^k = \frac{P_i^\bullet}{P_i^\bullet + P_j^\bullet} \mu_i + \frac{P_j^\bullet}{P_i^\bullet + P_j^\bullet} \mu_j,\tag{A.6}$$

$$\sigma_{ij}^2 = \text{Var}[X_{ij}] = \sum_{k=1}^m w_{ij}^k (X_{ij}^k - \mu_{ij})^2.\tag{A.7}$$

Note that the formula (A.6) is compatible with the definitions (A.3) and (A.4). In order that (A.7) is also compatible with (A.3) and (A.4), the correlation coefficient  $\rho_{ij}$  must satisfy the relationship

$$\rho_{ij} = \frac{1}{2} \cdot \frac{[(P_i^\bullet + P_j^\bullet)\sigma_{ij}]^2 - [P_i^\bullet\sigma_i]^2 - [P_j^\bullet\sigma_j]^2}{(P_i^\bullet\sigma_i)(P_j^\bullet\sigma_j)}.\tag{A.8}$$

and the probabilities  $\Pr(X_{ij} = X_{ij}^k) = w_{ij}^k$  must be compatible with the probabilities  $\Pr(X_i = X_i^k) = w_i^k$  and  $\Pr(X_j = X_j^k) = w_j^k$ . Using (A.3) and (A.8) one sees that the mean and variance of the portfolio risk factor ratio are given by (A.2).

Exhaustive conditions under which the estimator (A.8) defines a “true” correlation coefficient  $\rho_{ij} \in [-1, 1]$  are not known to the author. Simulation examples show that  $\rho_{ij} \notin [-1, 1]$  may occur, however with a very small probability.

In the following let us specify a simple practical method under which (A.8) will define a “true” correlation coefficient  $\rho_{ij} \in [-1,1]$ . Using regression analysis it is possible to fit for each  $i \in \{1, \dots, n\}$  the premium volumes to the following function

$$P_i^k = r^{k-1} \alpha_i, \quad k = 1, \dots, m, \quad (\text{A.9})$$

where  $r$  is interpreted as a price index adjustment factor. Furthermore set

$$P_i^* = \sum_{k=1}^m P_i^k = \alpha_i \cdot S_r, \quad S_r = \sum_{k=1}^m r^{k-1} = \frac{r^m - 1}{r - 1} \quad i = 1, \dots, n. \quad (\text{A.10})$$

Under (A.9) and (A.10) the weights are simply given by

$$w_i^k = w_j^k = w_{ij}^k = \frac{r^{k-1}}{S_r}, \quad i \neq j, i, j = 1, \dots, n, k = 1, \dots, m. \quad (\text{A.11})$$

One sees that (A.3), (A.6) and (A.7) can be rewritten as

$$\mu_i = S_r^{-1} \cdot \sum_{k=1}^m r^{k-1} \cdot X_i^k, \quad (\text{A.3}')$$

$$\sigma_i^2 = S_r^{-1} \cdot \sum_{k=1}^m r^{k-1} \cdot (X_i^k - \mu_i)^2, \quad i = 1, \dots, n.$$

$$\mu_{ij} = \frac{\alpha_i}{\alpha_i + \alpha_j} \mu_i + \frac{\alpha_j}{\alpha_i + \alpha_j} \mu_j, \quad (\text{A.6}')$$

$$\sigma_{ij}^2 = S_r^{-1} \cdot \sum_{k=1}^m r^{k-1} \cdot \left[ \frac{\alpha_i}{\alpha_i + \alpha_j} (X_i^k - \mu_i) + \frac{\alpha_j}{\alpha_i + \alpha_j} (X_j^k - \mu_j) \right]^2. \quad (\text{A.7}')$$

Inserting into (A.8) the correlation coefficient simplifies to the analytical expression

$$\rho_{ij} = \frac{\sum_{k=1}^m r^{k-1} \cdot (X_i^k - \mu_i) \cdot (X_j^k - \mu_j)}{\sqrt{\sum_{k=1}^m r^{k-1} \cdot (X_i^k - \mu_i)^2} \cdot \sqrt{\sum_{k=1}^m r^{k-1} \cdot (X_j^k - \mu_j)^2}}, \quad (\text{A.12})$$

which reminds one in the special case  $r = 1$  of the classical product moment estimator of the correlation coefficient, for which  $\rho_{ij} \in [-1,1]$  clearly holds.

## References

- Dhaene, J., Goovaerts, M. and R. Kaas* (2003). Economic Capital Allocation derived from Risk Measures. *North American Actuarial Journal* 7(2), 44-59. Available as <http://www.soa.org/library/journals/north-american-actuarial-journal/2003/april/naaj0304-3.pdf>
- Furman, E. and Z. Landsman* (2005). Risk capital decomposition for a multivariate dependent gamma portfolio. *Insurance: Mathematics and Economics* 37, 635-649. <http://www.sciencedirect.com/science/journal/01676687/>
- Furman, E. and Z. Landsman* (2007). Economic capital allocations for non-negative portfolios of dependent risks. Proceedings of the 37<sup>th</sup> International ASTIN Colloquium, Orlando. <http://www.actuaries.org/ASTIN/Colloquia/Orlando/Papers/Furman.pdf>
- Hürlimann, W.* (2001). Analytical evaluation of economic risk capital for portfolios of Gamma risks. *ASTIN Bulletin* 31, 107-122.
- Hürlimann, W.* (2002). Analytical bounds for two value-at-risk functionals. *ASTIN Bulletin* 32(2), 235-265.
- Hürlimann, W.* (2003). Conditional value-at-risk bounds for compound Poisson risks and a normal approximation. *Journal of Applied Mathematics* 3(3), 141-154.
- Hürlimann, W.* (2004). Distortion risk measures and economic capital. *North American Actuarial Journal* 8(1), 86-95.
- Landsman, Z. and E.A. Valdez* (2003). Tail conditional expectations for elliptical distributions. *North American Actuarial Journal* 7, 55-71. <http://www.soa.org/library/journals/north-american-actuarial-journal/2003/october/naaj0310-5.pdf>
- QIS3 (2007). QIS3 Technical Specifications – Part I: Instructions. CEIOPS Quantitative Impact Study 3, April 2007. Available at <http://www.ceiops.org>.
- CRO Forum* (2005). A Framework for Incorporating Diversification in the Solvency Assessment of Insurers. Available at [www.actuaries.org.uk](http://www.actuaries.org.uk).
- Valdez, E.A.* (2005). Tail conditional variance for elliptically contoured distributions. *Belgian Actuarial Bulletin* 5(1), 26-36. [http://www.actuaweb.be/BAB/volume5/05-Vol5\\_3-Valdez.pdf](http://www.actuaweb.be/BAB/volume5/05-Vol5_3-Valdez.pdf)