

A Dynamic Model of a Non-Life Insurance Portfolio

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Abstract

We present a simple dynamic model of an insurance portfolio where number of insured policyholders is a stochastic process and the rate of sale of new policies depends on the premium charged. The profit maximizing premium is determined and is compared to the actuarial premium based on the equivalence principle. It turns out that the equivalence principle priced portfolio will be twice as large as the profit maximized portfolio. In addition, expressions are provided for the portfolio's value, value-at-risk, and its tail value-at-risk.

Key words and phrases: *demand function, new sales, birth-death process, profit maximization, value-at-risk, tail value-at-risk*

1 Introduction

Since its introduction in 1909 by Filip Lundberg, collective risk theory has become the dominant model used in actuarial risk theory. The mathematical model underlying collective risk theory is simple and can be described as follows:

- Policyholders (i.e., insured risks) are independent and homogeneous;
- The number of policyholders in the collective remains constant and the aggregate premiums received from all policyholders in the time interval $(0, t)$ is Gt ;
- The aggregate claims paid in $(0, t)$, $S(t)$ is a compound Poisson process:

$$S(t) = \sum_{k=1}^{N(t)} X_k$$

where $N(t)$ is a Poisson process with rate λ and X_k is the size of the k^{th} claim. The X_k are nonnegative and independent and identically distributed (iid) random variables with finite mean p_1 ; and

The risk surplus process, $U(t)$, is given by

$$U(t) = u + Gt - S(t),$$

where $G = (1 + \theta)\lambda p_1$, $\theta > 0$ is the security loading, $u \geq 0$ is the initial reserve. Ruin is said to occur if $U(t) < 0$. The classic problems of collect risk theory are related to determining ruin probabilities and the distribution of surplus levels immediately before and after ruin; see, for example, Gerber and Shiu (1998) and references therein.

Lundberg's collective model has been extended over the years by several authors. The focus of most authors has been in the area of the claims number process, $N(t)$, and the aggregate claims amount process, $S(t)$. For example, one of the first alternative to Lundberg's collective risk model was introduced by Sparre-Andersen (1957) who assumed the inter-occurrence times between successive claims are iid random variables (i.e., an ordinary renewal process) with a general cdf $K(t)$, where $K(0) = 0$. (In the collective risk model, $K(t) = 1 - e^{-\lambda t}$, i.e., an exponential distribution.) A more elaborate risk model was the semi-Markov model introduced in a series of papers by Janssen from 1977 to 1982 (see Janssen (1982) and references therein). Janssen's model assumes there are different types of insured claims possible and the number of insured claims is still a Poisson process, but the inter-occurrence times between the types of insured claims is assumed to follow a semi-Markov process.

One common problem with the classical compound Poisson process model of collective risk theory and the many other alternative models is that they were

static models they implicitly viewed an insurance enterprise as a static system where the number of policyholders remains constant. In reality, however, insurance companies typically experience continuous changes in the number of policyholders. These changes may be due to sales of new policies, deaths, lapses, policy expirations, etc. As a result, the compound Poisson process model is not suitable as a practical model of any type of insurance except, perhaps, a closed block of non-life policies over a short period of time.

As an alternative to the compound Poisson and other static models, Ramsay (1985) introduced a dynamic and more realistic model of an insurance system. Ramsay's model, which is called a compound birth-death model, explicitly focuses on the stochastic evolution of the number of policyholders in-force over time by allowing for increases due to new sales and decreases due to various decrements including death. A model similar to Ramsay's was studied by Zmiev (2000).

A further deficiency in all of the above models is that they assume the number of policyholders in-force is independent of the premium charged, i.e., the demand for insurance is perfectly inelastic. Though some forms of insurance are mandatory (e.g., auto liability and homeowners insurance), most types of insurance are optional and, as such, are ordinary (non-Giffen) goods, i.e., they are subject to a downward sloping demand curve. This fact is ignored in most of traditional actuarial risk theory. For example, it is common in collective risk theory to assume that the premium rate G is exogenous when determining ruin probabilities. This may lead to the false conclusion that ruin probabilities can be made arbitrarily small by increasing G . However, increasing G too much may have the opposite effect: it may result in bankruptcy due to lack of sales.

Actuaries such as Karl Borch and economists working in the areas of economics of insurance, risk, and uncertainty have made extensive use of utility theory to study insurance risks.¹ However, very little of this research has entered the core of collective risk theory. An example of an application of traditional economic theory to collective risk theory is Kliger and Levikson (1998) where they consider a group of N independent and identically distributed potential property/casualty risks, and use their aggregate demand function for insurance to determine the insurer's optimal premium per insured to maximize the insurer's expected profits subject to a (solvency) constraint.

Though Ramsay's (1985) model was developed within the context of a life insurance enterprise, in the sequel we will adapt Ramsay's model to a non-life environment. In addition, we will use the Kliger-Levikson approach whereby potential policyholders are price sensitive and the insurer is a profit maximizing monopolist facing a downward sloping demand for its insurance policies. The insurer's objective is to determine G^* , the premium that maximizes its expected discounted profits. We will determine G^* , the insurer's expected discounted profits, the vari-

¹Boyle (1990) gives a summary of Borch's contributions to actuarial science and insurance. Dionne and Harrington (1992) provide a collection of some of the most important contributions to insurance economics.

ance of the insurer's discounted profits, the insurer's value-at-risk, the insurer's expected size, and explore ruin probabilities.

2 The Model

Consider a monopolistic, risk neutral, profit maximizing insurer who, at time 0, is about to launch a new product: a one-period (e.g., a year, six months, a quarter, etc., which represents one unit of time) indemnity policy that reimburses policyholder's claims (subject to possible ordinary deductibles and coinsurance payments) immediately upon their occurrence. The insurer plans to create a portfolio or block of such policies by selling them in a market consisting of very large number of M independent risk averse individuals (potential policyholders). These individuals are homogeneous except for their degree of risk aversion and hence their reservation price, i.e., the amount he or she is willing to pay, for a policy. The distribution of the population's reservation price is captured by a known function called the demand function.

Each policy covers a single policyholder who produces claims in a Poisson process with known constant rate ω over the duration of the policy. All claims are assumed to be iid random variables and independent of claims produced by other policyholders. In addition, there is no claims inflation, adverse selection, moral hazard or morale hazard problems. A person is allowed to purchase at most one policy at any time.

Each policy is purchased by a single premium of G paid at the start of the policy's term and the premium is the same for each policyholder. As there is assumed to be no adverse selection, moral hazard or morale hazard problems, the insurer is willing and able to sell any number of policies that is demanded. Once sold, a policy cannot be terminated before its term (one period) expires.

To procure policies, the insurer incurs the following expenses: (i) an expense of f (including taxes and commissions) per unit of premium received, (ii) a continuous expense at the rate of e per policyholder, (iii) a continuous expense at the rate of c regardless of the number of policyholders present (c can be considered as fixed overhead expenses), and (iii) b per unit of claim incurred and is paid upon the occurrence of each claim, with $0 \leq f, b < 1$ and $e, c \geq 0$. The quantities f, b, e , and c are not "conservative" estimates, rather they are assumed to reflect the actuary's best estimate of the insurer's expenses.

For a policy sold at time t , consider the present value at time t of the future profits produced by this policy over the period $(t, t + 1)$. Here profit is defined simply as premiums plus net interest less expenses and claims paid. As policies are time homogeneous, this present value must be independent of t . Let \mathcal{P} denote this policy's discounted future profit at the time of its sale, i.e.,

$$\mathcal{P} = (1 - f)G - e\bar{a}_{\overline{1}|\delta} - (1 + b)S \quad (1)$$

where $\bar{a}_{\overline{1}|\delta}$ denotes the traditional continuous annuity certain evaluated at the constant risk-free force of interest δ , and S is a discounted compound Poisson process given by

$$S = \sum_{r=1}^N e^{-\delta U_{(r)}} X_r, \quad (2)$$

where N is a Poisson process with rate ω , X_r is the size of the insurer's payment on the r^{th} claim, $t + U_{(r)}$ is the time of occurrence of the r^{th} claim in the interval $(t, t + 1)$. Note that the $\{U_{(r)}\}$ and $\{X_r\}$ sequences are mutually independent.

To be viable, the insurer must attract new customers by offering attractive premiums. For a given premium of G , we assume for simplicity that the sale of new policies occurs in a time homogeneous Poisson process with rate $\lambda(G)$, which is a decreasing function of G called the demand function for policies.² Let $Q(t)$ be the number of policyholders in-force at time t and let $A(t)$ and $W(t)$ denote the number of new policies and expired policies, respectively, in $(0, t)$ so that

$$Q(t) = Q(0) + A(t) - W(t). \quad (3)$$

As policies last for exactly one period, $Q(t)$ depends only on those arrivals during the preceding period, i.e., during the interval $(t - 1, t)$. Thus $Q(t)$ has a Poisson distribution with mean $\lambda(G)t$ if $t \leq 1$ and mean $\lambda(G)$ if $t \geq 1$.

Let $\Pi_i(t, s, G)$ denote the insurer's total discounted profits at time t for profits earned during the period $(t, t + s)$ given $Q(t) = i$ and the insurer charges every policyholder a premium of G , and let $\Pi_i(t, G) = \lim_{s \rightarrow \infty} \Pi_i(t, s, G)$. Assuming $Q(0) = 0$, the insurer's objective is to find the optimal premium G^* that maximizes its expected discounted profits, i.e.,

$$G^* = \arg \max_G \{E[\Pi_0(0, G)]\}. \quad (4)$$

We will provide an expression for G^* and determine the insurer's expected discounted profits and size.

²As M is finite, the assumption of Poisson sales is valid only if M is so large that the rate of new sales is unaffected by the number of policyholders already insured. In addition, demand is measured in terms the expected rate of sale of new policies rather than actual number of policies sold.

3 The Main Results

3.1 Optimum Premium, G^*

For convenience we introduce \mathcal{P}_k , $k = 1, 2, \dots$, which denotes a sequence of iid random variables with the same distribution as \mathcal{P} . The present value at time 0 of the profits from all policies sold up to time t is given by

$$\Pi_0(0, t, G) = \sum_{r=1}^{A(t)} e^{-\delta T_{(r)}(t)} \mathcal{P}_r - c \bar{a}_{\overline{t}| \delta} \quad (5)$$

where $T_{(r)}(t)$ is the time of arrival of the r^{th} new policyholder in $(0, t)$, and $c \bar{a}_{\overline{t}| \delta}$ represents the present value of the insurer's fixed costs (whether or not policies are sold).

To determine the moments of $\Pi_0(0, t, G)$ we need the moments of \mathcal{P}_r and hence the moments of S from equation (2). However, from Karlin and Taylor (1975, Chapter 4, Theorem 2.3) it is known that given $N = n$, the $U_{(r)}$ s in equation (2) are the order statistics from an iid sample (U_1, U_2, \dots, U_n) where the U_r s are iid uniform random variables on $(0, 1)$. Thus S can be written as

$$S = \sum_{r=1}^N e^{-\delta U_r} X_r, \quad (6)$$

which implies S is a simple compound Poisson sum. Assuming the first three moments of X_r exist and are given by $\beta_j = \mathbb{E}[X_r^j]$, then using a well known result on the cumulants of compound Poisson sums (e.g., Daykin et al. 1994, Chapter 3.2), gives

$$\mathbb{E}[\mathcal{P}] = (1 - f)G - (e + (1 + b)\omega\beta_1) \bar{a}_{\overline{t}| \delta} \quad (7)$$

$$\text{Var}[\mathcal{P}] = (1 + b)^2 \omega \beta_2 \bar{a}_{\overline{t}| \delta} \quad (8)$$

$$\mathbb{E}[(\mathcal{P} - \mathbb{E}[\mathcal{P}])^3] = -(1 + b)^3 \omega \beta_3 \bar{a}_{\overline{t}| \delta} \quad (9)$$

and the coefficient of skewness of \mathcal{P} , $\gamma(\mathcal{P})$, is

$$\gamma(\mathcal{P}) = \frac{-\beta_3 \bar{a}_{\overline{t}| \delta}}{\sqrt{\omega} (\beta_2 \bar{a}_{\overline{t}| \delta})^{3/2}}. \quad (10)$$

As $A(t)$ is a Poisson process with rate $\lambda(G)$, we can once again appeal to the uniform distribution argument used to derive equation (6) to prove that

$$\Pi_0(0, t, G) = \sum_{r=1}^{A(t)} e^{-\delta T_r(t)} \mathcal{P}_r - c \bar{a}_{\overline{t}| \delta} \quad (11)$$

where the $T_r(t)$ s are iid from a uniform distribution on $(0, t)$. For notational convenience, let $V_i(t, s, G) = \mathbb{E} [\Pi_i(t, s, G)]$ and $V_i(t, G) = \mathbb{E} [\Pi_i(t, G)]$ be the discounted expected value of the portfolio. It follows that

$$V_0(0, t, G) = [(1 - f)\lambda(G)(G - G^{\text{equiv}}) - c] \bar{a}_{\overline{t}| \delta} \quad (12)$$

$$V_0(0, G) = \frac{1}{\delta} [(1 - f)\lambda(G)(G - G^{\text{equiv}}) - c] \quad (13)$$

where

$$G^{\text{equiv}} = \frac{(e + (1 + b)\omega\beta_1) \bar{a}_{\overline{t}| \delta}}{1 - f} \quad (14)$$

is the gross premium per policy based on the equivalence principle, i.e., G^{equiv} ensures a zero expected net revenue from each policy sold. In addition,

$$\text{Var} [\Pi_0(0, t, G)] = \lambda(G) \mathbb{E} [\mathcal{P}^2] \bar{a}_{\overline{t}| 2\delta} \quad (15)$$

$$\mathbb{E} [(\Pi_0(0, t, G) - V_0(0, t, G))^3] = \lambda(G) \mathbb{E} [\mathcal{P}^3] \bar{a}_{\overline{t}| 3\delta} \quad (16)$$

and the coefficient of skewness of $\Pi_0(0, t, G)$, $\gamma(\Pi_0(0, t, G))$, is

$$\gamma(\Pi_0(0, t, G)) = \frac{\mathbb{E} [\mathcal{P}^3] \bar{a}_{\overline{t}| 3\delta}}{\sqrt{\lambda(G)} (\mathbb{E} [\mathcal{P}^2] \bar{a}_{\overline{t}| 2\delta})^{3/2}} \quad (17)$$

To proceed further we need an explicit expression for the demand function. For simplicity we assume a linear demand function, which is one of the most commonly used demand function in economics (La France 1985):

$$\lambda(G) = \alpha(G^{\text{max}} - G) \quad (18)$$

where $\alpha \gg 0$ and $G^{\text{max}} > 0$ are known constants, $0 \leq G \leq G^{\text{max}}$. As all potential policyholders are risk averse, each would buy a policy if $G = \omega\beta_1\bar{a}_{\overline{t}| \delta}$, the pure premium. However, as $G^{\text{equiv}} > \omega\beta_1\bar{a}_{\overline{t}| \delta}$ some individuals would not pay G^{equiv} for a policy. To calibrate $\lambda(G)$, we assume the insurer knows that a premium of G^{equiv} would result in a rate of sale of new policies of M^{equiv} , i.e.,

$$\lambda(G^{\text{equiv}}) = M^{\text{equiv}} < M. \quad (19)$$

From equation (18)

$$V_0(0, G) = \frac{1}{\delta} [(1 - f)\alpha(G^{\text{max}} - G)(G - G^{\text{equiv}}) - c], \quad (20)$$

which is a quadratic in G that goes to $-\infty$ as $G \rightarrow \infty$, then the insurer can be profitable if and only if the equation $V_0(0, G) = 0$ has distinct real roots and the premium lies between these roots. Let $G^{(1)} \leq G^{(2)}$ denote the roots of the equation $V_0(0, G) = 0$, i.e.,

$$G^{(1)}, G^{(2)} = G^* \pm \sqrt{(G^*)^2 - G^{\max} \left(G^{\text{equiv}} + \frac{c}{\alpha(1-f)G^{\max}} \right)} \quad (21)$$

where G^* and G^{equiv} are defined in equations (23) and (14), respectively, and the following profitability constraint holds:

$$G^* > \sqrt{G^{\max} \left(G^{\text{equiv}} + \frac{c}{\alpha(1-f)G^{\max}} \right)}, \quad (22)$$

which ensures the insurer the possibility of a positive profit by the appropriate choice of premium.

The first order equation becomes

$$\frac{\partial}{\partial G} V_0(0, G) = (1-f)(G^{\max} - G) - (1-f)(G - G^{\text{equiv}}) = 0 \quad (23)$$

which yields the optimum G^* as

$$G^* = \frac{1}{2} (G^{\max} + G^{\text{equiv}}) = \frac{1}{2} (G^{(1)} + G^{(2)}) \quad (24)$$

$$V_0(0, t, G^*) = \left[\frac{(1-f)}{4} (G^{\max} - G^{\text{equiv}})^2 - c \right] \bar{a}_{\overline{t}|} \delta \quad (25)$$

and

$$V_0(0, G^*) = \frac{(1-f)}{4\delta} (G^{\max} - G^{\text{equiv}})^2 - \frac{c}{\delta}. \quad (26)$$

Note that G^* is independent of the company's fixed overhead expenses c and the slope of the its demand curve, α . However, the inequality (22) ensures the insurer's profitability depends on these two parameters. The optimal arrival rate for new business is

$$\lambda^* = \lambda(G^*) = \frac{\alpha}{2} (G^{\max} - G^{\text{equiv}}) = \frac{M^{\text{equiv}}}{2} \quad (27)$$

and the expected number of policyholders at the end of the first period is λ^* .

As mentioned earlier, under traditional actuarial pricing based on the equivalence principle (e.g., McClenahan 2001, Chapter 3, equation (5) with interest ignored, or Bowers et al. 1997, Chapter 15), the gross premium rate for each policyholder is G^{equiv} , which yields an expected profit of zero per policy sold. In practice,

however, insurers traditionally add a loading factor of $(1 + \theta)$ to G^{equiv} for contingencies, fixed costs, and profits. Thus the traditional gross premium becomes

$$G^{\text{trad}} = (1 + \theta)G^{\text{equiv}} \quad (28)$$

with $\theta > 0$. Of course, the problem here is: how does one choose θ ?³ The profit maximizing loading contained in G^* is θ^* where

$$G^* = (1 + \theta^*)G^{\text{equiv}}. \quad (29)$$

3.2 Value-At-Risk and Valuations

Let $C(t)$ denote the insurer's capital at time t , $L_i(t, s, G) = -\Pi_i(t, s, G)$ be the present value of the insurer's net losses during the period $(t, t + s)$ with $L_i(t, G) = \lim_{s \rightarrow \infty} \Pi_i(t, s, G)$. Given $Q(t) = i$, it follows that

$$e^{-s\delta}C(t + s) = C(t) - L_i(t, s, G). \quad (30)$$

For a confidence level $1 - \epsilon$ with $0 < \epsilon < 1$, we define the insurer's value-at-risk (VaR_ϵ) and tail value-at-risk (TailVaR_ϵ) over the period $(t, t + s)$ given $Q(t) = i$ as

$$\text{VaR}_\epsilon(t, t + s | i, G) = \inf\{\ell \in \mathbb{R} : \Pr[L_i(t, s, G) > \ell] \leq \epsilon\} \quad (31)$$

$$\text{TailVaR}_\epsilon(t, s | i, G) = \mathbb{E}[L_i(t, s, G) | L_i(t, s, G) > \text{VaR}_\epsilon(t, s | i)] \quad (32)$$

where ϵ is small, e.g., $\epsilon = 0.05, 0.01, 0.001$; see, for example, McNeil et al. (2005, Chapter 2) and Sandström (2007).

Following Sandström (2007) we assume the percentiles of $L_i(t, s, G)$ can be determined using the normal power approximation. For a positively skewed random variable X with mean μ_X , variance σ_X^2 and skewness γ_X the normal power approximation (Daykin et al. 1994, Chapter 4.2.4) is as follows: for $x > \mu_X$ and $0 < \gamma_X \leq 1$

$$\Pr[X \leq x] \approx \Phi \left[-\frac{3}{\gamma_X} + \sqrt{\left(\frac{3}{\gamma_X}\right)^2 + 1 + 2\left(\frac{3}{\gamma_X}\right)\left(\frac{x - \mu_X}{\sigma_X}\right)} \right] \quad (33)$$

where $\Phi(x)$ is the cdf of the standard normal distribution. The percentile function of the standard normal distribution is z_ϵ where $\Phi(z_\epsilon) = \epsilon$.

As we are interested in $\text{VaR}_\epsilon(0, 1 | 0, G^*)$ and $\text{TailVaR}_\epsilon(0, 1 | 0, G^*)$ certain moments of $L_0(0, 1, G^*)$ are needed. From equations (12), (15), and (16) we note

³It must be pointed out that G^{trad} is based on cost-plus pricing or full-cost pricing and is viewed very unfavorably by experts; see, for example, Phillips (2005, Chapter 2.2.1) and Atkinson and Dallas (2000, Chapter 3.5.3).

$$\mu_0^* = \mathbb{E}[L_0(0, 1, G^*)] = c\bar{a}_{\overline{1}|\delta} - \frac{(1-f)}{4} (\theta^* G^{\text{equiv}})^2 \bar{a}_{\overline{1}|\delta} \quad (34)$$

$$(\sigma_0^*)^2 = \text{Var}[L_0(0, 1, G^*)] = \frac{M^{\text{equiv}}}{2} \mathbb{E}[\mathcal{P}^2] \bar{a}_{\overline{1}|\delta} \quad (35)$$

$$\mu_{03}^* = \mathbb{E}[(L_0(0, 1, G^*) - \mu_0^*)^3] = -\lambda(G^*) \mathbb{E}[\mathcal{P}^3] \bar{a}_{\overline{1}|\delta} \quad (36)$$

and the coefficient of skewness of $L_0(0, 1, G^*)$ is

$$\gamma_0^* = \frac{-\mathbb{E}[\mathcal{P}^3] \bar{a}_{\overline{1}|\delta}}{\sqrt{\lambda(G^*)} \left(\mathbb{E}[\mathcal{P}^2] \bar{a}_{\overline{1}|\delta} \right)^{3/2}}. \quad (37)$$

Note that for sufficiently large M^{equiv} , $\gamma_0^* \approx 0$.

The normal power approximation can now be used to obtain $\text{VaR}_\epsilon(0, 1|0, G^*)$ by solving the equation

$$z_{1-\epsilon} = -\frac{3}{\gamma_0^*} + \sqrt{\left(\frac{3}{\gamma_0^*}\right)^2 + 1 + 2\left(\frac{3}{\gamma_0^*}\right) \left(\frac{\text{VaR}_\epsilon(0, 1|0, G^*) - \mu_0^*}{\sigma_0^*}\right)}. \quad (38)$$

Following Sandström (2007), this gives

$$\text{VaR}_\epsilon(0, 1|0, G^*) = \mu_0^* + \sigma_0^* K_V(\gamma_0^*, \epsilon) \quad (39)$$

where

$$K_V(\gamma_0^*, \epsilon) = z_{1-\epsilon} + \frac{1}{6} \gamma_0^* (z_{1-\epsilon}^2 - 1). \quad (40)$$

Similarly, following Sandström (2007) we get the following expression for the tail value-at-risk:

$$\text{TailVaR}_\epsilon(0, 1|0, G^*) = \mu_0^* + \sigma_0^* K_{TV}(\gamma_0^*, \epsilon) \quad (41)$$

where

$$K_{TV}(\gamma_0^*, \epsilon) = \frac{1}{\epsilon} \phi(z_{1-\epsilon}) \left[1 + \frac{1}{6} \gamma_0^* z_{1-\epsilon}^3 \right] \quad (42)$$

and $\phi(x)$ is the pdf of the standard normal random variable.

Suppose the insurer performs valuations at the end of each period, i.e., at $t = k$ for $k = 0, 1, \dots$. Let \mathcal{F}_k denote the entire history of the portfolio up to and including time k . We assume \mathcal{F}_k is known to the insurer. For example, the information contained in \mathcal{F}_k includes the following:

- the number of policyholders remaining in force at k , $Q(k) = j$;
- the remaining time, measured from time k , until each policy

- of the $Q(k)$ policies expires, τ_{ki} , for $i = 1, 2, \dots, Q(k)$; and
- the capital at k , $C(k)$;

In this situation we are interested in $\text{VaR}_\epsilon(k, k+1|j, G^*)$, $\text{TailVaR}_\epsilon(k, k+1|j, G^*)$ and the value of the portfolio, $V_j(k, G^*)$. Because of the unexpired policies, we note that

$$L_j(k, k+1, G^*) = \sum_{i=1}^j e \bar{a}_{\tau_{ki}|} \delta + (1+b)S_i + L_0(k, k+1, G^*) \quad (43)$$

where S_i is a discounted compound Poisson process given by

$$S_i = \sum_{r=1}^{N_i} e^{-\delta U_{i(r)}} X_{ir}, \quad (44)$$

where N_i is a Poisson process with rate ω over the interval $(k, k + \tau_{ki})$, X_{ir} is the size of the insurer's payment on the r^{th} claim produced by the i^{th} unexpired policy, $k + U_{i(r)}$ is the time of occurrence of the r^{th} claim in the interval $(k, k + \tau_{ki})$. We note that the $L_0(k, k+1, G^*)$ are iid for $k = 0, 1, \dots$, and they are also independent of the expenses and losses produced by the unexpired policies. Thus the first three cumulants of $L_j(k, k+1, G^*)$ are:

$$\mu_j^* = \mathbb{E} [L_j(k, k+1, G^*)] = \mu_0^* + \sum_{i=1}^j (e + (1+b)\omega\tau_{ki}\beta_1) \bar{a}_{\tau_{ki}|} \delta \quad (45)$$

$$(\sigma_j^*)^2 = \text{Var} [L_j(k, k+1, G^*)] = (\sigma_0^*)^2 + \sum_{i=1}^j (1+b)^2 \omega \tau_{ki} \beta_2 \bar{a}_{\tau_{ki}|}^2 \delta \quad (46)$$

$$\mu_{j3}^* = \mathbb{E} \left[(L_j(k, k+1, G^*) - \mu_j^*)^3 \right] = \mu_{03}^* + \sum_{i=1}^j (1+b)^3 \omega \tau_{ki} \beta_3 \bar{a}_{\tau_{ki}|}^3 \delta \quad (47)$$

and the coefficient of skewness of $L_j(k, k+1, G^*)$ is

$$\gamma_j^* = \frac{\mu_{j3}^*}{(\sigma_j^*)^3}. \quad (48)$$

This gives the following expressions:

$$\text{VaR}_\epsilon(k, k+1|\mathcal{F}_k, G^*) = \mu_j^* + \sigma_j^* K_V(\gamma_j^*, \epsilon) \quad (49)$$

and

$$\text{TailVaR}_\epsilon(k, k+1|\mathcal{F}_k, G^*) = \mu_j^* + \sigma_j^* K_{TV}(\gamma_j^*, \epsilon). \quad (50)$$

We note that at time k there is an unearned premium of:

$$\text{Unearned Premium}_k = G^* \sum_{i=1}^j \tau_{ki} \quad (51)$$

and a reserve (including allocated expenses) for current policies in force of

$$\text{Reserve}_k = \sum_{i=1}^j (e + (1 + b)\omega\tau_{ki}\beta_1) \bar{a}_{\tau_{ki}} \delta. \quad (52)$$

Finally, to determine the value of the portfolio at k we note that $V_0(k, G^*)$ is independent of k . Thus

$$V_j(k, G^* | \mathcal{F}_k) = C(k) - \text{Reserve}_k + \frac{(1-f)}{4\delta} (G^{\max} - G^{\text{equiv}})^2 - \frac{c}{\delta}. \quad (53)$$

In closing, given G^* we may be interested in determining the probability of ultimate ruin of the portfolio. Of course this is a much more challenging problem than the equivalent problem under the traditional collective risk model. Results may have to be obtained via simulations. Another question that has no counterpart under the traditional collective risk model is the following: does the profit maximized portfolio have a smaller probability of ruin than any other portfolio given the same initial reserve?

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