On the distortion of a copula and its margins
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19-22 June 2011, Madrid, Spain  ASTIN Colloquium
Introduction and motivation

In insurance and financial risk modeling, practitioners may be required to compute aggregate risk distribution for a portfolio of correlated risks:

- pricing or premium calculation of contingent payoffs on these multiple risks
- capital allocation among several lines of business
- analyzing diversification benefits within an enterprise
- reporting of risks to external parties, e.g. regulators

Models used to describe the correlation structure:

- multivariate distributions with correlation
- “copulas” - separates the peculiar characteristics of marginals
The concept of distortion

Apply a probability distortion to multivariate distributions:

- to adjust for risk and uncertainty in aggregating a portfolio of correlated risks

- to change probability measure to price contingent claims involving multiple risks

- a direct extension of the distortion concept in the univariate case

Be careful in the extension because you want to preserve properties of a copula:

- three kinds of multivariate distortion - will or will not affect the dependence structure

In the paper, we show much more: numerous examples, multivariate ordering of risks, integral transform with distortion
Copulas - recipe for disaster?

Article on *Wired Magazine*, 23 Feb 2009, by F. Salmon titled “Recipe for Disaster: The Formula that Killed Wall Street”\(^1\).

\[
\Pr[T_A < 1, T_B < 1] = \\
\phi_2(\phi^{-1}(F_A(1)), \phi^{-1}(F_B(1), \gamma))
\]

- Collapse of the market on defaultable loans, collaterized debt obligations, other credit derivatives (huge $$$’s involved!!!)


- Pricing basis: Gaussian or normal copula.

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\(^1\) Source: P. Embrechts slides, “Did a Mathematical Formula Really Blow up Wall Street?”
Sklar\'s representation theorem

Sklar (1959): There exists a copula function $C$ such that

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

where $F_i$ is the marginal for $X_i$, $i = 1, \ldots, n$.

Equivalently, we write

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) = C(P(X_1 \leq x_1), \ldots, P(X_n \leq x_n)).$$

$C$ need not be unique, but it is unique for continuous marginals. Else, $C$ is uniquely determined on $\text{Ran}F_1 \times \ldots \times \text{Ran}F_n$.

In the continuous case, this unique copula can be expressed as

$$C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)),$$

where $F_i^{-1}$ are the respective quantile functions.
Examples of (implicit) copulas

Normal copula:

\[ C_R^\mathbf{u}(u) = \Phi_R(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)), \]

where \( \Phi \) is the cdf of standard univariate normal, \( \Phi_R \) is the joint cdf of \( X \sim \mathcal{N}_n(0, R) \) with \( R \), the correlation matrix.

The case where \( R = I_n \) results in independence, and \( R = J_n \) gives comonotonicity.

\( \mathfrak{t} \) copula:

\[ C_{\nu, R}^\mathbf{u}(u) = t_{\nu, R}(t_{\nu}^{-1}(u_1), \ldots, t_{\nu}^{-1}(u_n)), \]

where \( t_{\nu} \) is the cdf of standard univariate \( \mathfrak{t} \), \( t_{\nu, R} \) is the joint cdf of \( X \sim \mathcal{t}_n(\nu, 0, R) \) with \( R \), the correlation matrix.

The case where \( R = J_n \) gives comonotonicity, but \( R = I_n \) does not result in independence.
Simulation - normal vs t copula

Normal copula (rho=-0.90)

Normal copula (rho=0.20)

t3 copula (rho=-0.90)

t3 copula (rho=0.20)
Some problems with multivariate normal

Some believe that there are deficiencies of the normal for multivariate modeling in finance/insurance:

- The tails of the margins may be too thin, and hence fail to generate some extreme values.

- As a consequence, in the multivariate sense, it fails to capture phenomenon of joint extreme movements. Simultaneous large values may be relatively infrequent - generally believed to lack tail dependence.

- Too much symmetry - lack of presence of skewness. Some financial/insurance data exhibits long tails.
Special class: Archimedean copulas

$C$ is an *Archimedean* if it has the form

$$C(u_1, \ldots, u_n) = \psi^{-1}(\psi(u_1) + \cdots + \psi(u_n)),$$

for some function $\psi$ (called the generator) satisfying:

- $\psi(1) = 0$;
- $\psi$ is decreasing; and
- $\psi$ is convex.

To ensure you get a legitimate copula for higher dimensions, $\psi^{-1}$ must be completely *monotonic*, i.e. its derivatives alternate in signs.

An important source of Archimedean generators is the inverses of the Laplace transforms of distribution functions.

**Feller (1971):** A function $\varphi$ on $[0, \infty]$ is the Laplace transform of a cdf $F$ if and only if $\varphi$ is completely monotonic with $\varphi(0) = 1$. 
# Archimedean copulas and their generators

<table>
<thead>
<tr>
<th>Family</th>
<th>Generator $\psi(t)$</th>
<th>Range of $\alpha$</th>
<th>$C(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Independence</td>
<td>$-\log(t)$</td>
<td>na</td>
<td>$\prod_{i=1}^{n} u_i$</td>
</tr>
<tr>
<td>Clayton</td>
<td>$t^{-\alpha} - 1$</td>
<td>$\alpha &gt; 0$</td>
<td>$\left[ \sum_{i=1}^{n} u_i^{-\alpha} - n + 1 \right]^{-1/\alpha}$</td>
</tr>
<tr>
<td>Gumbel-Hougaard</td>
<td>$(-\log t)^\alpha$</td>
<td>$\alpha \geq 1$</td>
<td>$\exp \left{ - \left[ \sum_{i=1}^{n} (-\log u_i)^\alpha \right]^{1/\alpha} \right}$</td>
</tr>
<tr>
<td>Frank</td>
<td>$-\log \left( \frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1} \right)$</td>
<td>$\alpha &gt; 0$</td>
<td>$-\frac{1}{\alpha} \log \left[ 1 + \frac{\prod_{i=1}^{n} (e^{-\alpha u_i} - 1)}{(e^{-\alpha} - 1)^{n-1}} \right]$</td>
</tr>
</tbody>
</table>
Normal, t, and Clayton copulas

Marginals: Gamma(5,1), $\rho = 0.75$, and $\nu = 3$
Review of univariate distortion

We say \( g : [0, 1] \rightarrow [0, 1] \) is a \textit{distortion function} if it satisfies the following properties:

\begin{itemize}
  \item \( g(0) = 0 \) and \( g(1) = 1 \); and
  \item \( g \) is continuous and non-decreasing.
\end{itemize}

The transformation of the distribution function \( F_X \)

\[ F_{X^*}(x) = g[F_X(x)] = g \circ F_X(x) \]

is the df of \( X^* \) that leads to a \textit{probability distortion} of \( X \) to \( X^* \).

\textbf{Wang Transform:} Here \( g(t) = \Phi[\Phi^{-1}(t) + \gamma] \) preserves Normal and Lognormal distributions:

\begin{itemize}
  \item \( X \sim \text{Normal}(\mu, \sigma^2) \) implies \( X^* \sim \text{Normal}(\mu - \gamma\sigma, \sigma^2) \)
  \item \( X \sim \text{Lognormal}(\mu, \sigma^2) \) implies \( X^* \sim \text{Lognormal}(\mu - \gamma\sigma, \sigma^2) \)
\end{itemize}
## Some well-known distortion functions

<table>
<thead>
<tr>
<th>Distortion</th>
<th>Functional form $g(t)$</th>
<th>Inverse form $g^{-1}(s)$</th>
<th>Convex constraints</th>
<th>Concave constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proportional hazard</td>
<td>$t^{1/\gamma}$</td>
<td>$s^\gamma$</td>
<td>$0 &lt; \gamma \leq 1$</td>
<td>$\gamma \geq 1$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$\frac{1 - e^{-\gamma t}}{1 - e^{-\gamma}}$</td>
<td>$\log[1 - s(1 - e^{-\gamma})]$</td>
<td>$\gamma &lt; 0$</td>
<td>$\gamma &gt; 0$</td>
</tr>
<tr>
<td>Logarithmic</td>
<td>$\frac{1}{\gamma} \log[1 - t(1 - e^{\gamma})]$</td>
<td>$\frac{e^{\gamma t - 1}}{e^{\gamma - 1}}$</td>
<td>$\gamma &lt; 0$</td>
<td>$\gamma &gt; 0$</td>
</tr>
<tr>
<td>Wang transform</td>
<td>$\Phi[\Phi^{-1}(t) + \gamma]$</td>
<td>$\Phi[\Phi^{-1}(s) - \gamma]$</td>
<td>$\gamma \leq 0$</td>
<td>$\gamma \geq 0$</td>
</tr>
<tr>
<td>Dual-power</td>
<td>$1 - (1 - t)^{\gamma}$</td>
<td>$1 - (1 - s)^{1/\gamma}$</td>
<td>$\gamma \leq 1$</td>
<td>$\gamma \geq 1$</td>
</tr>
</tbody>
</table>

Note: The convex/concave constraints are for the function $g(t)$. 
Adjustment for risk


- For a (non-negative) risk $X$, the premium principle associated with the distortion function:

  $$\pi_g(X) = E(X^*) = \int_0^\infty [1 - g[F_X(x)]] \, dx.$$

- The difference $\pi_g(X) - E(X)$ is risk premium (or adjustment for risk), and is positive if $g$ is convex. (Jensen’s inequality)

- Distortion can also be used to price contingent payoffs, say $h(X)$, associated with an underlying asset with value $X$. In case of no-arbitrage, these risk-neutral (distorted) probabilities can be derived from observable prices in the market.
The effect of distortion

Wang transform

Proportional hazard

Exponential

Logarithmic

Gamma parameter = -0.25

Gamma parameter = 0.25

Gamma parameter = -1

Gamma parameter = -1
Parameter uncertainty

In practice, we estimate probability distributions usually based on limited data so that parameter uncertainty is always present.

To illustrate, consider the case where $X$, conditional on the risk parameter $\gamma$, is Exponential with: $F_X(x|\gamma) = 1 - \exp(-\gamma x)$.

If $\gamma$ has a Gamma distribution with a scale and shape parameters $\lambda$ and $\alpha$, respectively, the unconditional distribution of $X$ is a Pareto distribution expressed as

$$F_X(x) = 1 - (1 + \lambda x)^{-\alpha}.$$

Indeed, one can easily derive the corresponding distortion function in this case:

$$g(t) = 1 - (1 + \log(1 - t)^{-\lambda/\gamma})^{-\alpha}.$$

Note that this distortion function is neither strictly convex nor concave.
Effect of distortion for parameter uncertainty

gamma=1, lambda=0.75, alpha=2

distorted density
Distortion of the first kind

Let \( g_1, \ldots, g_n \) be \( n \) distortion functions. Then the transformation of the copula associated with \( X \) defined by

\[
C_X(u_1^*, \ldots, u_n^*) = C_X(g_1(u_1), \ldots, g_n(u_n))
\]

induces a multivariate probability distortion of \( X \) to \( X^* \).

This type of a distortion leads to a simple distortion of the margins while preserving the copula structure.

An example of this type is the multivariate extension of the Wang transform constructed by Kijima (2006).
Example - Multivariate Burr I

Consider the Weibull margins

\[ F_i(x_i) = 1 - \exp(-x_i^k), \quad x_i \geq 0, k > 0, \]

for \( i = 1, \ldots, n \), linked with a legitimate copula, for example, a Clayton copula defined by

\[ C_X(u_1, \ldots, u_n) = \left[ \sum_{i=1}^{n} u_i^{-\alpha} - n + 1 \right]^{-1/\alpha}. \]

With the distortion function \( g(t) = 1 - (1 - \log(1 - t))^{-\gamma} \), this leads to Burr margins

\[ F_i^*(x_i) = 1 - [(1 + x_i^k)]^{-\gamma}, \quad x_i \geq 0, k > 0, \gamma > 0. \]
Distortion of the second kind

Let \( g_1, \ldots, g_n \) be \( n \) distortion functions. Then the transformation of the copula associated with \( X \) defined by

\[
\hat{C}(u_1^*, \ldots, u_n^*) = \hat{C}(g_1(u_1), \ldots, g_n(u_n)),
\]

where \( \hat{C} \) is a copula function, induces a multivariate probability distortion of \( X \) to \( \hat{X} \).

This leads to a simultaneous distortion of the margins and the copula structure.

**Multivariate Burr II:** Similarly distort margins from Weibull to Burr, but transform the copula structure to Gumbel-Hougaard

\[
\hat{C}(u_1, \ldots, u_n) = \exp\left\{-\left[\sum_{i=1}^{n} (-\log u_i)^\alpha\right]^{1/\alpha}\right\}.
\]

Result is yet another multivariate Burr distribution.
Distortion of the third kind

Let \( g \) be a distortion function with inverse \( g^{-1} \) that is absolutely monotonic of order \( n \) on \([0, 1] \). Then the transformation of the copula associated with \( X \) defined by

\[
C_g(u_1, \ldots, u_n) = g^{-1}(C_X(g(u_1), \ldots, g(u_n)))
\]

induces a distortion of \( X \) to \( \tilde{X} \).

\( C_g \) induced by this distortion satisfies the necessary properties of a copula and is then the copula associated with the distorted \( \tilde{X} \) and therefore can be written as

\[
C_g(u_1, \ldots, u_n) = C_{\tilde{X}}(u_1, \ldots, u_n).
\]

For proof, see Morillas (2005). This leads to a synchronized distortion of the margins and the copula structure, and a new method of constructing new copulas from a given one.

Interesting to note that this preserves the margins; it simply distorts the dependence structure.
An actuarial application

Consider an insurance portfolio of fire insurance policies where the loss amounts vary according to:

- buildings $X_1$
- contents $X_2$
- loss of profits $X_3$

To accommodate the possible large number of zeroes in each type of loss, we use a mixture model of the form:

$$f_k(x) = \begin{cases} p_k, & \text{for } x = 0 \\ (1 - p_k)f_{LN,k}(x), & \text{for } x > 0 \end{cases}$$

LN refers to the log-normal distribution with parameters $\mu$ and $\sigma$.

It is also easy to prove that the marginal CDF for the mixture is:

$$F_k(x) = p_k + (1 - p_k)F_{LN,k}(x), \text{ for } k = 1, 2, 3.$$
Marginal parameter and choice of copula

We assume the following parameter values for the margins:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Building ((X_1))</th>
<th>Contents ((X_2))</th>
<th>Profits ((X_3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>0.05</td>
<td>0.10</td>
<td>0.20</td>
</tr>
<tr>
<td>(\mu)</td>
<td>0.01</td>
<td>-0.50</td>
<td>-1.25</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>0.20</td>
<td>1.30</td>
<td>1.40</td>
</tr>
</tbody>
</table>

For purposes of making the illustration simple, we use a Clayton copula with

\[
C(u_1, u_2, u_3) = \left( u_1^{-\alpha} + u_2^{-\alpha} + u_3^{-\alpha} - 2 \right)^{-1/\alpha},
\]

where the \(\alpha\) parameter value is assumed to be 5. This translates to a Kendall’s tau correlation of approximately 70%.
Valuing excess of loss reinsurance

- We apply distortion to the case where we value excess of loss reinsurance with retention \( d \) so that our variable of interest is:

\[
(S - d)_+ = (X_1 + X_2 + X_3 - d)_+,
\]

where \( S \) denotes the aggregate loss.

- To accommodate parameter uncertainty, we apply *distortion of the third kind* based on \( g(t) = t^{1/\gamma} \) with \( \gamma = 10 \), leading to a re-parameterized Clayton copula

\[
C(u_1, u_2, u_3) = \left( u_1^{-\alpha \gamma} + u_2^{-\alpha \gamma} + u_3^{-\alpha \gamma} - 2 \right)^{-1/\alpha \gamma}.
\]

- We then simulated values of the excess of loss and examined the resulting distribution, with and without the distortion.
Kernel density of the logarithm of sum

Excess Coverage = 2

Excess Coverage = 5

Excess Coverage = 10

Excess Coverage = 20
## Summary of risk adjustments

<table>
<thead>
<tr>
<th>Expectation</th>
<th>Excess of Loss Amount ((d))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>without distortion</td>
<td></td>
</tr>
<tr>
<td>(E(S - d)_+)</td>
<td>1.2764</td>
</tr>
<tr>
<td>with distortion</td>
<td></td>
</tr>
<tr>
<td>(E(S^* - d)_+)</td>
<td>1.3159</td>
</tr>
<tr>
<td>risk adjustment</td>
<td>0.0395</td>
</tr>
<tr>
<td>loading percentage</td>
<td>3.1%</td>
</tr>
</tbody>
</table>
Additional materials in the paper

You can find additional discussion of materials in the paper:

- Multivariate ordering of risks with distortion
  - supermodular ordering
- Multivariate probability integral transform with distortion
  - extended Genest and Rivest (2001) results
Concluding remarks

- Increasingly important to assess the aggregate risk distribution of a portfolio of often correlated risks.

- Some limitations as to specifying just the correlation structures to model the dependencies of risks - users are warned of use of copulas.

- Copulas provide flexibility to allow modeling various dependence structures, allowing to separate the effects of peculiar characteristics of the margins such as thickness of tails.

- We advocate applying distortion to multivariate distributions, and hence to copulas, as a means to adjust for risk and uncertainty in the aggregation of portfolios of correlated risks.

- We caution practical users to understand the implications of distortion.
Some useful references


