Claims Development Result in the Paid-Incurred Chain Reserving Method

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Abstract

We present the one-year claims development result (CDR) in the paid-incurred chain (PIC) reserving model. The PIC reserving model presented in Merz-Wüthrich [6] is a Bayesian stochastic claims reserving model that considers simultaneously claims payments and incurred losses information and allows for deriving the full predictive distribution of the outstanding loss liabilities. In this model we study the conditional mean square error of prediction (MSEP) for the one-year CDR uncertainty, which is the crucial uncertainty view under Solvency II.

Keywords: stochastic claims reserving, PIC method, outstanding loss liabilities, claims payments, incurred losses, prediction uncertainty, conditional mean square error, claims development result, solvency.

1 Introduction

A non-life insurance company needs to hold sufficient reserves (provisions) on its balance sheet in order to meet the outstanding loss liabilities. Therefore, a main task of an actuary in non-life insurance is to predict ultimate loss ratios and outstanding loss liabilities. For these predictions he often has different sources of information and the major difficulty is to combine these information channels appropriately.

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In the present paper we combine claims paid data and claims incurred data (i.e. case estimates for reported claims) to get a unified prediction for the outstanding loss liabilities. A well known method to combine claims paid data and claims incurred data for claims reserving is the Munich chain ladder (MCL) method introduced in Quarg-Mack [10]. However, to the best of our knowledge, there is no way to quantify the prediction uncertainty within the MCL method. Another approach was presented in Dahms [3]. Dahms [3] extended the complementary loss ratio (CLR) method for deriving unified predictions based on claims paid data and claims incurred data simultanously. Unlike the MCL method, the CLR method allows for the derivation of a mean square error of prediction (MSEP) estimate. A recent new approach is the paid-incurred (PIC) reserving method introduced in Merz-Wüthrich [6]. The PIC method has the advantage that it works in a Bayesian framework and therefore it allows for the derivation of the full predictive distribution for the outstanding loss liabilities. This means that within the PIC model one is not only able to calculate the MSEP but one can also calculate any other risk measure, like Value-at-Risk or expected shortfall.

Under the new solvency regulations, such as Solvency II, the so-called one-year claims development result (CDR) is of central interest because it corresponds to a profit & loss statement position that directly influences the financial strength of an insurance company. The one-year CDR is defined as the difference between the prediction of the outstanding loss liabilities today and in one year's time (cf. Merz-Wüthrich [7]). This means that the one-year CDR measures the change in the claims reserves over a one-year time horizon. Due to Solvency II, this one-year view has already attracted a lot of attention in recent research. For references, we refer to Ohlsson-Lauzeningks [8], Merz-Wüthrich [7] and Bühlmann et al. [2]. Dahms et al. [4] analyze the one-year CDR in the framework of the CLR method, which is the first one-year CDR uncertainty analysis for combined claims paid and claims incurred data.

In the present paper we revisit the PIC method within this solvency framework. This means that we consider the one-year CDR for the PIC reserving method. Within our framework we are not only able to calculate the conditional MSEP for the one-year CDR but we can also derive the predictive distribution of the one-year CDR via Monte-Carlo simulations.

Organization of the paper: In Section 2 we recapitulate the assumptions of the PIC model and the definition of the one-year CDR is given in Section 3. We then derive the best estimate of the ultimate claim, based on the paid and incurred data in one year, see Section 4. In Section 5.1 we split this best estimate in an appropiate way and derive the conditional MSEP of the

one-year CDR for single accident years. In Section 5.2 we proceed with the conditional MSEP for aggregated accident years which provides the overall one-year CDR uncertainty. Finally, in Section 6 we present an example and compare it to the results derived in Dahms et al. [4] for the CLR method. Additionally, we provide the full predictive distribution of the one-year CDR via Monte-Carlo simulations.

2 Notation and Model Assumptions

The PIC reserving model combines two channels of information: i) claims payments, which correspond to the payments for reported claims; ii) incurred losses, which refer to the reported claim amounts. Claims payments and incurred losses data are usually aggregated in so-called claims development triangles.

In the following, we denote accident years by $i \in \{0, \ldots, J\}$ and development years by $j \in \{0, \ldots, J\}$. Cumulative payments in accident year i after j development years are denoted by $P_{i,j}$ and the corresponding incurred losses by $I_{i,j}$. We assume that all claims are settled and closed after development year J, i.e. $P_{i,J} = I_{i,J}$ holds with probability 1 for all $i \in \{0, \ldots, J\}$. After accounting year $t = J$ we have observations in the paid and incurred triangles given by (see Figure 1)

$$\mathcal{D}_J = \{P_{i,j}, I_{i,j};\ 0 \leq i \leq J,\ 0 \leq j \leq J,\ 0 \leq i + j \leq J\}$$

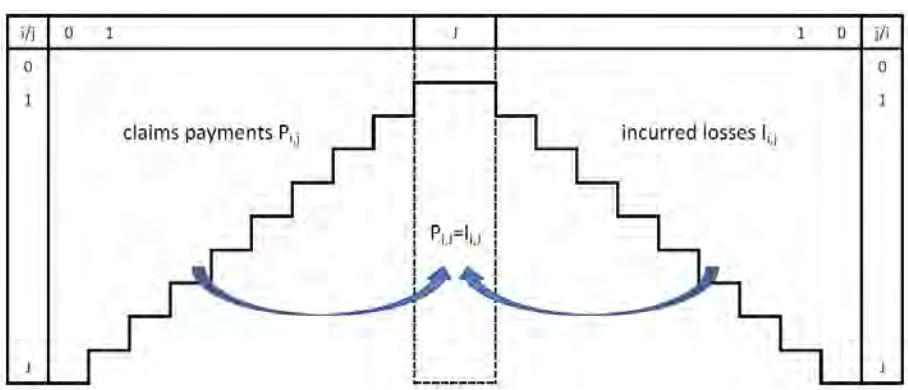

Figure 1: Cumulative claims payments $P_{i,j}$ and incurred losses $I_{i,j}$ observed after accounting year $t = J$ both leading to the same ultimate loss $P_{i,J} = I_{i,J}$.

and after accounting year $t = J + 1$ we have observations in the paid and incurred trapezoids given by (see Figure 2)

$$\mathcal{D}_{J+1} = \{P_{i,j}, I_{i,j};\ 0 \leq i \leq J,\ 0 \leq j \leq J,\ 0 \leq i + j \leq J + 1\}.$$

This means the update $\mathcal{D}_J \mapsto \mathcal{D}_{J+1}$ adds a new diagonal to the observations.

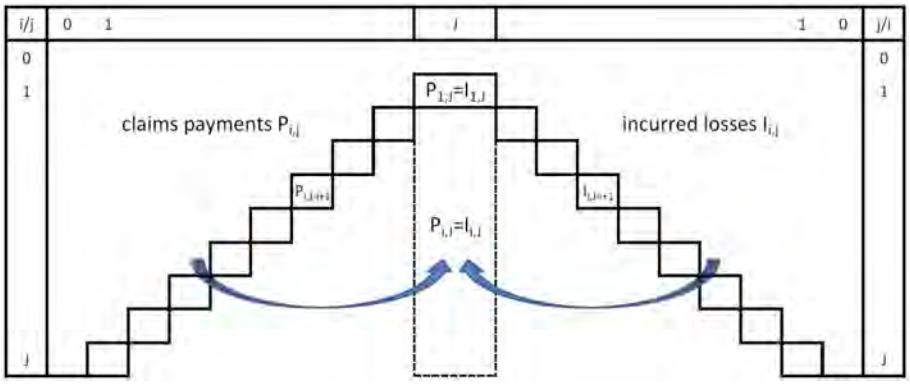

Figure 2: Cumulative claims payments $P_{i,j}$ and incurred losses $I_{i,j}$ observed after accounting year $t = J + 1$ both leading to the same ultimate loss $P_{i,J} = I_{i,J}$.

Next we define the Log-normal PIC model, which combines both cumulative payments and incurred losses information:

Model Assumptions: 2.1 (Log-normal PIC model)

- Conditionally, given $\Theta = (\Phi_0, \Phi_1, \Psi_1, \Phi_2, \Psi_2, \ldots, \Phi_J, \Psi_J)'$, we assume:
  - the random vectors $\Xi_i = (\xi_{i,0}, \xi_{i,1}, \zeta_{i,1}, \xi_{i,2}, \zeta_{i,2}, \ldots, \xi_{i,J}, \zeta_{i,J})$ are i.i.d. with multivariate Gaussian distribution

    $$\Xi_i \sim N(\Theta, V) \qquad \text{for} \quad i \in \{0, 1, \ldots, J\}$$

    with positive definite covariance matrix V and

    $$\xi_{i,j} = \log \frac{P_{i,j}}{P_{i,j-1}} \qquad \text{and} \qquad \zeta_{i,l} = \log \frac{I_{i,l}}{I_{i,l-1}}$$

    for $j \in \{0, 1, \ldots, J\}$ and $l \in \{1, 2, \ldots, J\}$, where we have set $P_{i,-1} = 1$;
  - $P_{i,J} = I_{i,J}$ $\mathbb{P}$-a.s. for all $i = 0, 1, \ldots, J$.
- The components of Θ are independent with prior distributions

  $$\Phi_j \sim N(\phi_j, s_j^2) \quad \text{for } j \in \{0, \ldots, J\} \qquad \text{and} \qquad \Psi_l \sim N(\psi_l, t_l^2) \quad \text{for } l \in \{1, \ldots, J\}$$

  with known parameters $s_j^2 > 0$, $t_l^2 > 0$ and ϕ_j, $\psi_l \in \mathbb{R}$.

3 One-year Claims Development Result

In this paper we consider the short term (one-year) run-off risk described in Merz-Wüthrich [7]. This means, we study the uncertainty in the one-year CDR given by

$$\mathrm{CDR}_i(J+1) = E\left[P_{i,J} \middle| \mathcal{D}_J\right] - E\left[P_{i,J} \middle| \mathcal{D}_{J+1}\right]$$

between the best estimates (w.r.t. the L^2-distance) for the ultimate claim $P_{i,J}$ at times J and $J+1$.

The one-year CDR measures the change in the prediction by updating the information from $\mathcal{D}_J$ to $\mathcal{D}_{J+1}$. With the tower property of the conditional expectation we obtain for the expected one-year CDR for accident year i viewed from time J

$$E\left[\mathrm{CDR}_i(J+1) \middle| \mathcal{D}_J\right] = 0,$$

which is the martingale property of successive predictions. This justifies the fact that, in the budget statement, the one-year CDR is usually predicted by 0 at time J. In the following we study the uncertainty in this prediction by means of the conditional MSEP, given the observations $\mathcal{D}_J$. In other words we calculate

$$\begin{aligned} \mathrm{msep}_{\mathrm{CDR}_i(J+1)|\mathcal{D}_J}(0) &= E\left[\left(\mathrm{CDR}_i(J+1) - 0\right)^2 \middle| \mathcal{D}_J\right] \\ &= \mathrm{Var}\left(\mathrm{CDR}_i(J+1) \middle| \mathcal{D}_J\right) = \mathrm{Var}\left(E\left[P_{i,J} | \mathcal{D}_{J+1}\right] \middle| \mathcal{D}_J\right) \end{aligned} \tag{1}$$

for single and aggregated accident years. The conditional MSEP is probably the most popular uncertainty measure in claims reserving practice and has the advantage that it can be derived analytically in the PIC model.

4 Expected ultimate claim at time $J+1$

In this section we derive the conditional expected ultimate claim $E[P_{i,J}|\mathcal{D}_{J+1}]$ in two steps. In the first step we derive $E[P_{i,J}|\Theta, \mathcal{D}_{J+1}]$, and in the second step we calculate $E[P_{i,J}|\mathcal{D}_{J+1}]$, see Corollary 4.3.

In the following we can either work with the random vector $\Xi_i \in \mathbb{R}^{2J+1}$ (see Model Assumptions 2.1) or with the logarithmized observations of accident year i, namely

$$X_i = (\log P_{i,0}, \log I_{i,0}, \log P_{i,1}, \ldots, \log P_{i,J-1}, \log I_{i,J-1}, \log P_{i,J})' \in \mathbb{R}^{2J+1}.$$

This is possible, since there exist an invertible matrix $B \in \mathbb{R}^{(2J+1)\times(2J+1)}$ such that $X_i = B\ \Xi_i$ and

$$X_i|_{\{\Theta\}} = B\ \Xi_i|_{\{\Theta\}} \sim N(\mu = B\Theta, \Sigma = BVB')$$

holds. At time $J+1$ we observe

$$X_i^{(1)}(J+1) = \begin{cases} (\log P_{i,0}, \log I_{i,0}, \log P_{i,1}, \ldots, \log P_{i,J-i+1}, \log I_{i,J-i+1})' \in \mathbb{R}^{2(J-i+2)} & \text{for } i \geq 2 \\ (\log P_{i,0}, \log I_{i,0}, \ldots, \log P_{i,J-1}, \log I_{i,J-1}, \log P_{i,J})' \in \mathbb{R}^{2J+1} & \text{for } i = 0, 1 \end{cases}$$

and want to predict the lower triangle given by

$$X_i^{(2)}(J+1) = \begin{cases} (\log P_{i,J-i+2}, \log I_{i,J-i+2}, \ldots, \log P_{i,J-1}, \log I_{i,J-1}, \log P_{i,J})' \in \mathbb{R}^{2i-3} & \text{for } i \geq 2 \\ (\log P_{i,J})' & \text{for } i = 1 \end{cases}$$

for $i = 2, \ldots, J$. The mean vector

$$\mu = B\Theta = \left(\mu_i^{(1)}(J+1), \mu_i^{(2)}(J+1)\right)' \in \mathbb{R}^{2J+1}$$

is decomposed into

$$\mu_i^{(1)}(J+1) = E[X_i^{(1)}(J+1)|\Theta] = B_{i-1}^{(1)}\Theta \quad \text{and} \quad \mu_i^{(2)}(J+1) = E[X_i^{(2)}(J+1)|\Theta] = B_{i-1}^{(2)}\Theta$$

for $i = 2, \ldots, J$, where the matrices $B_i^{(1)} \in \mathbb{R}^{2(J-i+1)\times(2J+1)}$ and $B_i^{(2)} \in \mathbb{R}^{(2i-1)\times(2J+1)}$ are, for $i = 1, \ldots, J$, defined by

$$B = \begin{pmatrix} B_i^{(1)} \\ B_i^{(2)} \end{pmatrix}.$$

For $i = 0$ we set $B_0^{(1)} = B$ and $B_0^{(2)} = B_1^{(2)}$ and the mean vectors are given by

$$\mu_1^{(1)}(J+1) = E[X_1^{(1)}(J+1)|\Theta] = B\Theta \quad \text{and} \quad \mu_1^{(2)}(J+1) = E[X_1^{(2)}(J+1)|\Theta] = B_0^{(2)}\Theta.$$

For $i \geq 1$ the covariance matrix is decomposed in a similar way such that

$$\Sigma = BVB' = \begin{pmatrix} \Sigma_{11}^{(i)} & \Sigma_{12}^{(i)} \\ \Sigma_{21}^{(i)} & \Sigma_{22}^{(i)} \end{pmatrix},$$

with $\Sigma_{11}^{(i)} \in \mathbb{R}^{2(J-i+1)\times 2(J-i+1)}$. For $i = 0$ we set

$$\Sigma_{k,j}^{(0)} = \text{Cov}\left(X_1^{(k)}(J+1), X_1^{(j)}(J+1)\middle|\, \Theta\right)$$

for $k, j \in \{1, 2\}$. Note that with this notation for all $i \in \{1, 2, \ldots, J\}$ and all $k, j \in \{1, 2\}$ holds

$$\Sigma_{k,j}^{(i-1)} = \text{Cov}\left(X_i^{(k)}(J+1), X_i^{(j)}(J+1)\middle|\, \Theta\right)$$

Lemma 4.1 *Under Model Assumptions 2.1 we obtain for the conditional distribution of* $X_i^{(2)}(J+1)$*, given* $\{\Theta, \mathcal{D}_J\}$*,*

$$X_i^{(2)}(J+1)|_{\{\Theta,\mathcal{D}_J\}} = X_i^{(2)}(J+1)|_{\{\Theta,X_i^{(1)}(J+1)\}} \sim N\left(\tilde{\mu}_i^{(2)}(J+1), \tilde{\Sigma}_{22}^{(i-1)}\right)$$

where

$$\tilde{\mu}_i^{(2)}(J+1) = \mu_i^{(2)}(J+1) + \Sigma_{21}^{i-1}(\Sigma_{11}^{i-1})^{-1}\left(X_i^{(1)}(J+1) - \mu_i^{(1)}(J+1)\right)$$
$$\tilde{\Sigma}_{22}^{(i-1)} = \Sigma_{22}^{(i-1)} - \Sigma_{21}^{(i-1)}(\Sigma_{11}^{(i-1)})^{-1}\Sigma_{12}^{(i-1)} \in \mathbb{R}^{(2i-3)\vee 1\times(2i-3)\vee 1}$$

for $i = 1, \ldots, J$*.*

As a direct consequence we get for the ultimate claim

$$\log P_{i,J}|_{\{\Theta,\mathcal{D}_{J+1}\}} \sim N\left(e_i'\tilde{\mu}_i^{(2)}(J+1), e_i'\tilde{\Sigma}_{22}^{(i-1)}e_i\right)$$

for $i \in \{1, \ldots, J\}$, where $e_i = (0, \ldots, 0, 1)' \in \mathbb{R}^{(2i-3)}$ and $e_1 = e_2$. This immediately implies for the predictor of the ultimate claim, given $\{\Theta, \mathcal{D}_J\}$,

$$\mathrm{E}[P_{i,J}|\Theta, \mathcal{D}_{J+1}] = \exp\left\{e_i'\tilde{\mu}_i^{(2)}(J+1) + e_i'\tilde{\Sigma}_{22}^{(i-1)}e_i/2\right\} \qquad \text{for} \quad i \in \{1, \ldots, J\}. \tag{2}$$

We see that the ultimate claim predictor in (2) still depends on Θ, namely

$$\begin{aligned} e_i'\tilde{\mu}_i^{(2)}(J+1) &= e_i'\left(\mu_i^{(2)}(J+1) + \Sigma_{21}^{i-1}(\Sigma_{11}^{(i-1)})^{-1}\left(X_i^{(1)}(J+1) - \mu_i^{(1)}(J+1)\right)\right) \\ &= e_i'\left(B_{i-1}^{(2)}\Theta + \Sigma_{21}^{i-1}(\Sigma_{11}^{(i-1)})^{-1}\left(X_i^{(1)}(J+1) - B_{i-1}^{(1)}\Theta\right)\right) \\ &= \Gamma_{i-1}\Theta + e_i'\Sigma_{21}^{i-1}(\Sigma_{11}^{(i-1)})^{-1}X_i^{(1)}(J+1) \end{aligned} \tag{3}$$

with

$$\Gamma_{i-1} = e_i'\left(B_{i-1}^{(2)} - \Sigma_{21}^{i-1}(\Sigma_{11}^{(i-1)})^{-1}B_{i-1}^{(1)}\right).$$

Our aim is to calculate the posterior distribution of Θ, conditionally given observations $\mathcal{D}_{J+1}$.

The likelihood of the logarithmized observations at time $J+1$, given Θ, is given by

$$L_{\mathcal{D}_{J+1}}(\Theta) \propto \prod_{i=0}^{J} \exp\left\{-\frac{1}{2}\left(X_i^{(1)}(J+1) - B_{i-1\vee 0}^{(1)}\Theta\right)'(\Sigma_{11}^{(i-1)\vee 0})^{-1}\left(X_i^{(1)}(J+1) - B_{i-1\vee 0}^{(1)}\Theta\right)\right\} \tag{4}$$

and with Model Assumptions 2.1 and Bayes' theorem follows that the posterior distribution $u(\Theta|\mathcal{D}_{J+1})$ has the form

$$u(\Theta|\mathcal{D}_{J+1}) \propto L_{\mathcal{D}_{J+1}}(\Theta) \exp\left\{-\frac{1}{2}(\Theta-\nu)'T^{-1}(\Theta-\nu)\right\}, \tag{5}$$

with prior mean

$$\nu = (\phi_0, \phi_1, \psi_1, \phi_2, \psi_2, \ldots, \phi_J, \psi_J)'$$

and prior covariance matrix

$$T = \mathrm{diag}(s_0^2, s_1^2, t_1^2, s_2^2, t_2^2, \ldots, s_J^2, t_J^2).$$

An immediate consequence is Theorem 4.2, whose proof is provided in the appendix:

Theorem 4.2 (Posterior distribution of Θ)

Under Model Assumptions 2.1 the posterior distribution $u(\Theta|\mathcal{D}_{J+1})$ is a multivariate Gaussian distribution with posterior mean $\nu(\mathcal{D}_{J+1}) \in \mathbb{R}^{(2J+1)}$ and posterior covariance matrix $T(\mathcal{D}_{J+1})$ with

$$T(\mathcal{D}_{J+1}) = \left(T^{-1} + \sum_{i=0}^{J} (B^{(1)}_{(i-1)\vee 0})'(\Sigma_{11}^{(i-1)\vee 0})^{-1} B^{(1)}_{(i-1)\vee 0}\right)^{-1}$$

and mean

$$\nu(\mathcal{D}_{J+1}) = T(\mathcal{D}_{J+1})\left[T^{-1}\nu + \sum_{i=0}^{J} (B^{(1)}_{(i-1\vee 0)})'(\Sigma_{11}^{((i-1)\vee 0)})^{-1} X_i^{(1)}(J+1)\right].$$

From (3) we obtain that the exponent of predictor (2) is a affin-linear function of the components of Θ. Using Theorem 4.2 this implies the following corollary:

Corollary 4.3 (Expected ultimate claim given $\mathcal{D}_{J+1}$)

The expected ultimate claim for accident year $i \in \{1, \ldots, J\}$, given $\mathcal{D}_{J+1}$, is given by

$$\begin{aligned} &E[P_{i,J}|\mathcal{D}_{J+1}] \\ &= \exp\{\Gamma_{i-1}\nu(\mathcal{D}_{J+1}) + \Gamma_{i-1}T(\mathcal{D}_{J+1})(\Gamma_{i-1})'/2 + e_i'\Sigma_{21}^{i-1}(\Sigma_{11}^{(i-1)})^{-1}X_i^{(1)}(J+1) + e_i'\tilde{\Sigma}_{22}^{(i-1)}e_i/2\}. \end{aligned} \tag{6}$$

5 Mean Square Error of Prediction of the CDR

5.1 Single Accident Years

In the last section we have calculated the expected ultimate claim in the PIC reserving model, given the observations $\mathcal{D}_{J+1}$. Our aim now is to calculate the prediction uncertainty in terms of the conditional MSEP. From (1) we see that the problem to derive the conditional MSEP for the one-year CDR is solved by calculating $\mathrm{Var}\left(E\left[P_{i,J}|\mathcal{D}_{J+1}\right]|\,\mathcal{D}_J\right)$. Since $E\left[P_{i,J}|\mathcal{D}_J\right]^2$ is given by (6) with $i-1$ and $J+1$ replaced by i and J (see Happ-Wüthrich [5]), this conditional variance can be derived by calculating $\mathrm{E}\left(E\left[P_{i,J}|\mathcal{D}_{J+1}\right]^2\Big|\mathcal{D}_J\right)$.

We see that the exponential term in (6), namely

$$\Gamma_{i-1}\nu(\mathcal{D}_{J+1}) + \Gamma_{i-1}T(\mathcal{D}_{J+1})(\Gamma_{i-1})'/2 + e_i'\Sigma_{21}^{i-1}\left(\Sigma_{11}^{(i-1)}\right)^{-1}X_i^{(1)}(J+1) + e_i'\tilde{\Sigma}_{22}^{(i-1)}e_i/2,$$

is affin-linear in

$$X = (\log P_{1,J}, \log P_{2,J-1}, \log I_{2,J-1}, \ldots, \log P_{J,1}, \log I_{J,1})'.$$

That means that for all $i \in \{1,\ldots,J\}$ there exist a matrix L_i and a $\mathcal{D}_J$-measurable random variable $g_i(\mathcal{D}_J)$ such that

$$\begin{aligned}
&L_iX + g_i(\mathcal{D}_J)\\
&\quad = \Gamma_{i-1}\nu(\mathcal{D}_{J+1}) + \Gamma_{i-1}T(\mathcal{D}_{J+1})(\Gamma_{i-1})'/2 + e_i'\Sigma_{21}^{i-1}\left(\Sigma_{11}^{(i-1)}\right)^{-1}X_i^{(1)}(J+1) + e_i'\tilde{\Sigma}_{22}^{(i-1)}e_i/2.
\end{aligned}$$

This means it holds for the ultimate claim

$$E[P_{i,J}|\mathcal{D}_{J+1}] = \exp\{L_iX + g_i(\mathcal{D}_J)\}.$$

For Lemma 5.1 we recall

$$\begin{aligned}
\tilde{\mu}_i^{(2)}(J) &= \mu_i^{(2)}(J) + \Sigma_{21}^i\left(\Sigma_{11}^i\right)^{-1}\left(X_i^{(1)}(J) - \mu_i^{(1)}(J)\right) = B_i^{(2)}\Theta + \Sigma_{21}^i\left(\Sigma_{11}^i\right)^{-1}\left(X_i^{(1)}(J) - \mu_i^{(1)}(J)\right)\\
\tilde{\Sigma}_{22}^{(i)} &= \Sigma_{22}^{(i)} - \Sigma_{21}^{(i)}\left(\Sigma_{11}^{(i)}\right)^{-1}\Sigma_{12}^{(i)}
\end{aligned}$$

Lemma 5.1 *Under Model Assumptions 2.1 we obtain*

$$(\log P_{i,J-i+1}, \log I_{i,J-i+1})|_{\{\Theta,\mathcal{D}_J\}} \sim N(\mu_i, \Sigma_i)$$

for $i \in \{2,\ldots,J\}$*, where*

$$(\mu_i)' = (\tilde{\mu}_i^{(1)}, \tilde{\mu}_i^{(2)}) = (e_i^{(1)}\tilde{\mu}_i^{(2)}(J), e_i^{(2)}\tilde{\mu}_i^{(2)}(J)) \qquad \textit{and} \qquad \Sigma_i = E_i\tilde{\Sigma}_{22}^{(i)}E_i'$$

with $e_i^{(1)} = (1, 0, \ldots, 0), e_i^{(2)} = (0, 1, 0, \ldots, 0) \in \mathbb{R}^{(2(i-1)+1)}$ *and the projection matrix* E_i *on the first two coordinates, i.e. the first two rows of* E_i *are given by* $e_i^{(1)}$ *and* $e_i^{(2)}$*. Moreover, for* $i = 1$ *we have*

$$\log P_{1,J}|_{\{\Theta,\mathcal{D}_J\}} \sim N\left(\tilde{\mu}_1^{(1)} = \tilde{\mu}_1^{(2)}(J), \Sigma_1 = \tilde{\Sigma}_{22}^{(1)}\right).$$

Using the independence of different accident years, given Θ, Lemma 5.1 leads to the joint distribution of X, given $\{\mathcal{D}_J, \Theta\}$:

Corollary 5.2 *Under Model Assumptions 2.1 we have*

$$X|_{\{\mathcal{D}_J,\Theta\}} = (\log P_{1,J}, \log P_{2,J-1}, \log I_{2,J-1}, \ldots, \log P_{J,1}, \log I_{J,1})'|_{\{\mathcal{D}_J,\Theta\}} \sim N(\mu, \Sigma)$$

where $\mu = \left(\tilde{\mu}_1^{(1)}, \tilde{\mu}_2^{(1)}, \tilde{\mu}_2^{(2)} \ldots, \tilde{\mu}_J^{(1)}, \tilde{\mu}_J^{(2)}\right)' \in \mathbb{R}^{2J-1}$ *and*

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \Sigma_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Sigma_J \end{pmatrix} \in \mathbb{R}^{(2J-1)\times(2J-1)}.$$

Corollary 5.2 implies

$$E\left[E[P_{i,J}|\mathcal{D}_{J+1}]^2 \middle| \Theta, \mathcal{D}_J\right] = \exp\{2L_i\mu + 2L_i\Sigma L_i' + 2g_i(\mathcal{D}_J)\}$$

for all $i \in \{1, 2, \ldots, J\}$. We see that $E\left[E[P_{i,J}|\mathcal{D}_{J+1}]^2 \middle| \Theta, \mathcal{D}_J\right]$ depends on Θ via

$$\mu = \left(\tilde{\mu}_1^{(1)}, \tilde{\mu}_2^{(1)}, \tilde{\mu}_2^{(2)}, \tilde{\mu}_3^{(1)} \ldots, \tilde{\mu}_J^{(1)}, \tilde{\mu}_J^{(2)}\right)'$$

and recalling the definition of $\tilde{\mu}_i^{(k)}$ for $k \in \{1, 2\}$ we obtain

$$\begin{aligned} \tilde{\mu}_i^{(k)} &= (e_i^{(k)})'(\mu_i^{(2)}(J) + \Sigma_{21}^i(\Sigma_{11}^{(i)})^{-1}(X_i^{(1)} - \mu_i^{(1)}(J))) \\ &= (e_i^{(k)})'(B_i^{(2)}\Theta + \Sigma_{21}^i(\Sigma_{11}^{(i)})^{-1}(X_i^{(1)} - B_i^{(1)}\Theta)) \\ &= \Gamma_i^{(k)}\Theta + (e_i^{(k)})'\Sigma_{21}^i(\Sigma_{11}^{(i)})^{-1}X_i^{(1)}, \end{aligned} \tag{7}$$

where

$$\Gamma_i^{(k)} = (e_i^{(k)})'\left(B_i^{(2)} - \Sigma_{21}^i(\Sigma_{11}^{(i)})^{-1}B_i^{(1)}\right).$$

Next, we define the matrix Γ with rows $\Gamma_i^{(k)}$, i.e.

$$\Gamma = \left(\Gamma_1^{(1)\prime} \quad \Gamma_2^{(1)\prime} \quad \Gamma_2^{(2)\prime} \quad \ldots \quad \Gamma_J^{(1)\prime} \quad \Gamma_J^{(2)\prime}\right)'$$

and γ by

$$\gamma = \left((e_1^{(1)})'\Sigma_{21}^1(\Sigma_{11}^{(1)})^{-1}X_1^{(1)}, (e_2^{(1)})'\Sigma_{21}^2(\Sigma_{11}^{(2)})^{-1}X_2^{(1)}, (e_2^{(2)})'\Sigma_{21}^2(\Sigma_{11}^{(2)})^{-1}X_2^{(1)}, \ldots, (e_J^{(2)})'\Sigma_{21}^J(\Sigma_{11}^{(J)})^{-1}X_J^{(1)}\right)'.$$

We see that $\mu = \Gamma\Theta' + \gamma$ is a affin-linear function of Θ. With $E[P_{i,J}|\mathcal{D}_J] = E[E[P_{i,J}|\mathcal{D}_{J+1}]|\mathcal{D}_J]$ and Theorem 4.2 applied to $u(\Theta|\mathcal{D}_J)$ (see Happ-Wüthrich [5]) we obtain the following result:

Theorem 5.3 *Under Model Assumptions 2.1 we obtain for $i \in \{1, \ldots, J\}$*

$$\begin{aligned} E\left[E[P_{i,J}|\mathcal{D}_{J+1}]^2\middle|\mathcal{D}_J\right] &= \exp\{2L_i(\Gamma\nu(\mathcal{D}_J) + \gamma) + 2L_i\Gamma T(\mathcal{D}_J)\Gamma' L_i' + 2L_i\Sigma L_i' + 2g_i(\mathcal{D}_J)\} \\ &= E[P_{i,J}|\mathcal{D}_J]^2 \cdot \exp\{L_i\Gamma T(\mathcal{D}_J)\Gamma' L_i' + L_i\Sigma L_i'\}. \end{aligned}$$

By means of this relationship between $E\left[E[P_{i,J}|\mathcal{D}_{J+1}]^2\middle|\mathcal{D}_J\right]$ and $E[P_{i,J}|\mathcal{D}_J]^2$ it is straightforward to derive the MSEP for the one-year CDR of a single accident year, which is given in the next theorem:

Theorem 5.4 (Conditional MSEP for single accident years)
Under Model Assumptions 2.1 the conditional MSEP, given $\mathcal{D}_J$, for the one-year CDR of accident year $i \in \{1, \ldots, J\}$ is given by

$$\mathrm{msep}_{\mathrm{CDR}_i(I+1)|\mathcal{D}_J}(0) = (E[P_{i,J}|\mathcal{D}_J])^2\left(\exp\{L_i\Gamma T(\mathcal{D}_J)\Gamma' L_i' + L_i\Sigma L_i'\} - 1\right).$$

In the following section we consider the conditional MSEP for aggregated accident years.

5.2 Aggregated Accident Years

We study now the conditional MSEP of the one-year CDR for aggregated accident years:

$$\begin{aligned} \mathrm{msep}_{\sum_{i=1}^J \mathrm{CDR}_i(I+1)|\mathcal{D}_J}(0) &= E\left[\left(\sum_{i=1}^J \mathrm{CDR}_i(I+1) - 0\right)^2\middle|\mathcal{D}_J\right] \\ &= \mathrm{Var}\left(\sum_{i=1}^J \mathrm{CDR}_i(I+1)\middle|\mathcal{D}_J\right) = \mathrm{Var}\left(\sum_{i=1}^J E\left[P_{i,J}|\mathcal{D}_{J+1}\right]\middle|\mathcal{D}_J\right). \end{aligned} \tag{8}$$

For the calculation of this conditional MSEP we need the following result, whose proof is provided in the appendix.

Proposition 5.5 *Under Model Assumptions 2.1 it holds*

$$E\left[\exp\{L_iX + L_jX\}|\mathcal{D}_J\right] = E\left[\exp\{L_iX\}|\mathcal{D}_J\right]\ E\left[\exp\{L_jX\}|\mathcal{D}_J\right] \\ \times \exp\left\{L_i\Sigma L_j'\right\} \times \exp\left\{L_i\Gamma T(\mathcal{D}_J)\Gamma' L_j'\right\}$$

for $i, j \in \{1, \ldots, J\}$.

Using the tower property of conditional expectations we obtain for (8)

$$\begin{aligned}
&\mathrm{msep}_{\sum_{i=1}^J \mathrm{CDR}_i(I+1)|\mathcal{D}_J}(0) \\
&= \sum_{i=1}^J \mathrm{Var}\left(E\left[P_{i,J}|\mathcal{D}_{J+1}\right]|\,\mathcal{D}_J\right) + \sum_{\substack{k,i=1\\k\neq i}}^J E\left[E[P_{i,J}|\mathcal{D}_{J+1}]E[P_{k,J}|\mathcal{D}_{J+1}]\big|\,\mathcal{D}_J\right] \\
&\qquad - \sum_{\substack{k,i=1\\k\neq i}}^J E[P_{i,J}|\mathcal{D}_J]\,E[P_{k,J}|\mathcal{D}_J] \\
&= \sum_{i=1}^J \mathrm{msep}_{\mathrm{CDR}_i(I+1)|\mathcal{D}_J}(0) + \sum_{\substack{k,i=1\\k\neq i}}^J \exp\{g_i(\mathcal{D}_J)\}\exp\{g_k(\mathcal{D}_J)\}E\left[\exp\{L_iX + L_kX|\mathcal{D}_J\right] \\
&\qquad - \sum_{\substack{k,i=1\\k\neq i}}^J E[P_{i,J}|\mathcal{D}_J]E[P_{k,J}|\mathcal{D}_J].
\end{aligned}$$

Together with Proposition 5.5 this leads to the following result:

Theorem 5.6 (Conditional MSEP for aggregated accident years)
Under Model Assumptions 2.1 the conditional MSEP, given $\mathcal{D}_J$*, for the one-year CDR of aggregated accident years is given by*

$$\begin{aligned}
&\mathrm{msep}_{\sum_{i=1}^J \mathrm{CDR}_i(I+1)|\mathcal{D}_J}(0) \\
&= \sum_{i=1}^J \mathrm{msep}_{\mathrm{CDR}_i(I+1)|\mathcal{D}_J}(0) \\
&\qquad + \sum_{\substack{k,i=1\\k\neq i}}^J E[P_{i,J}|\mathcal{D}_J]E[P_{k,J}|\mathcal{D}_J]\left(\exp\left\{L_i\Sigma L_k' + L_i\Gamma T(\mathcal{D}_J)\Gamma' L_k'\right\} - 1\right).
\end{aligned}$$

6 Example

We revisit the data given in Dahms [3]. Our analysis is based on Model Assumptions 2.1, where we assume that σ_j and τ_j are deterministic parameters. These parameters are estimated by

the plug-in estimates provided in Table 2 in Merz-Wüthrich [6]. In Table 1 we compare the prediction uncertainty measured by the square root of the conditional MSEP for the one-year CDR calculated by the PIC method and the CLR method (cf. Dahms et al. [4]). Under Model Assumptions 2.1, these values are calculated analytically with Theorem 5.4 for single accident years and with Theorem 5.6 for aggregated accident years. We observe that for most accident years and aggregated accident years the prediction uncertainty for the one-year CDR in the PIC model is smaller than the two values for the prediction uncertainty in the CLR model. This is caused by the fact that in contrast to the CLR model in the PIC model the prediction is based on both information channels, i.e. claims paid data and incurred loss data.

Table 2 provides the ratios between the square root of the conditional MSEP for the one-year CDR and the square root of the conditional MSEP for the total run-off of the ultimate claim. We observe that for later accident years (i.e. $i \geq 7$) and aggregated accident years the values for the CLR method (based on paid data or incurred loss data) and for the PIC method only slightly differ. Moreover, we see that for aggregated accident years the one-year uncertainty is about 75% of the total run-off uncertainty. This result is in-line with the field study conducted by AISAM-ACME [1].

	$\text{msep}_{\text{CDR}}^{1/2}$		$\text{msep}_{\text{CDR}}^{1/2}$
accident	CLR method	CLR method	PIC method
year i	Paid	Incurred	Paid & Incurred
1	194	14.639	2.637
2	4.557	4.678	4.597
3	5.597	6.628	7.656
4	33.675	34.258	6.606
5	30.574	30.997	31.594
6	42.598	43.074	43.168
7	166.154	166.255	139.352
8	138.685	138.740	127.053
9	210.899	210.979	173.721
Total	346.576	350.534	292.879

Table 1: Prediction uncertainty for the one-year CDR from the CLR method for claims payments and incurred losses (cf. Dahms et al. [4]) and from the PIC method.

As mentioned in the introduction we can not only calculate the conditional MSEP for the one-

accident year	$\text{msep}^{1/2}_{\text{CDR}}/\text{msep}^{1/2}_{\text{Ultimate}}$ CLR method Incurred	CLR method Paid	$\text{msep}^{1/2}_{\text{CDR}}/\text{msep}^{1/2}_{\text{Ultimate}}$ PIC method Paid & Incurred
1	100.0%	100%	100%
2	100.0%	84.5%	87.6%
3	53.1%	52.7%	83.7%
4	91.5%	89.6%	62.4%
5	69.6%	69.1%	94.3%
6	65.5%	65.4%	80.8%
7	94.0%	93.9%	93.1%
8	70.1%	70.1%	70.3%
9	65.3%	65.3%	66.4%
Total	74.1%	74.3%	75.2%

Table 2: Ratio $\text{msep}^{1/2}_{\text{CDR}}/\text{msep}^{1/2}_{\text{Ultimate}}$ from the CLR method for claims payments and incurred losses (cf. Dahms [4]) and from the PIC method.

year CDR. Based on Theorem 4.2 applied for the case $u(\Theta|\mathcal{D}_J)$, Corollary 4.3 and Lemma 5.1 we can derive the full predictive distribution of the one-year CDR by means of a Monte Carlo simulation approach.

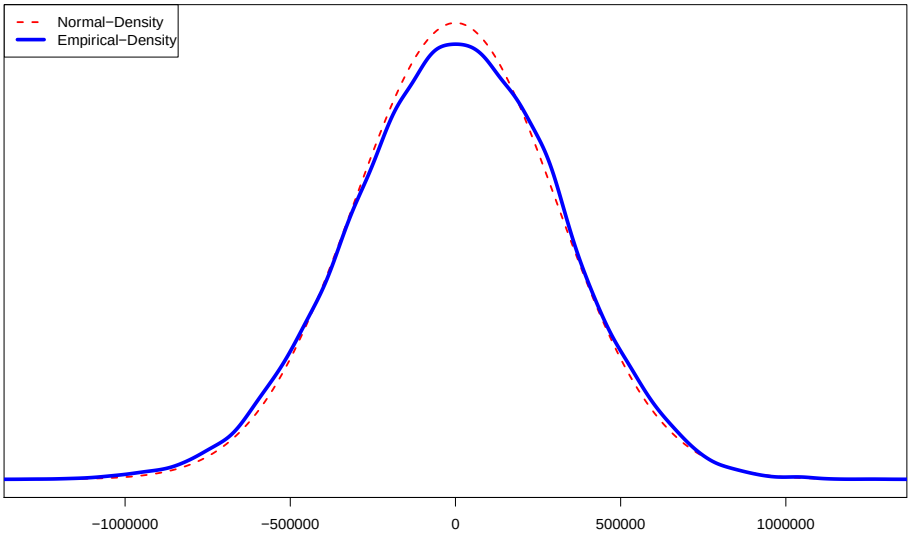

Figure 3: Empirical density for the one-year CDR from 100.000 simulations and fitted Gaussian density with mean 0 and standard deviation 292.879.

Firstly, we use Theorem 4.2 applied for the case $u(\Theta|\mathcal{D}_J)$ to generate Gaussian samples $\Theta^{(n)}$ with covariance matrix $T(\mathcal{D}_J)$ and mean $\nu(\mathcal{D}_J)$. Secondly, we generate independent two-dimensional Gaussians samples $(\log P_{i,J-i+1}, \log I_{i,J-i+1})_{\{\mathcal{D}_J,\Theta\}}$ and fill up the off-diagonal entries in the Paid and Incurred trapezoids (see Lemma 5.1). This way we obtain the data available at time $J+1$, i.e. $\mathcal{D}_{J+1}$, and can calculate $E\left[P_{i,J}|\mathcal{D}_{J+1}\right]$ by means of Corollary 4.3.

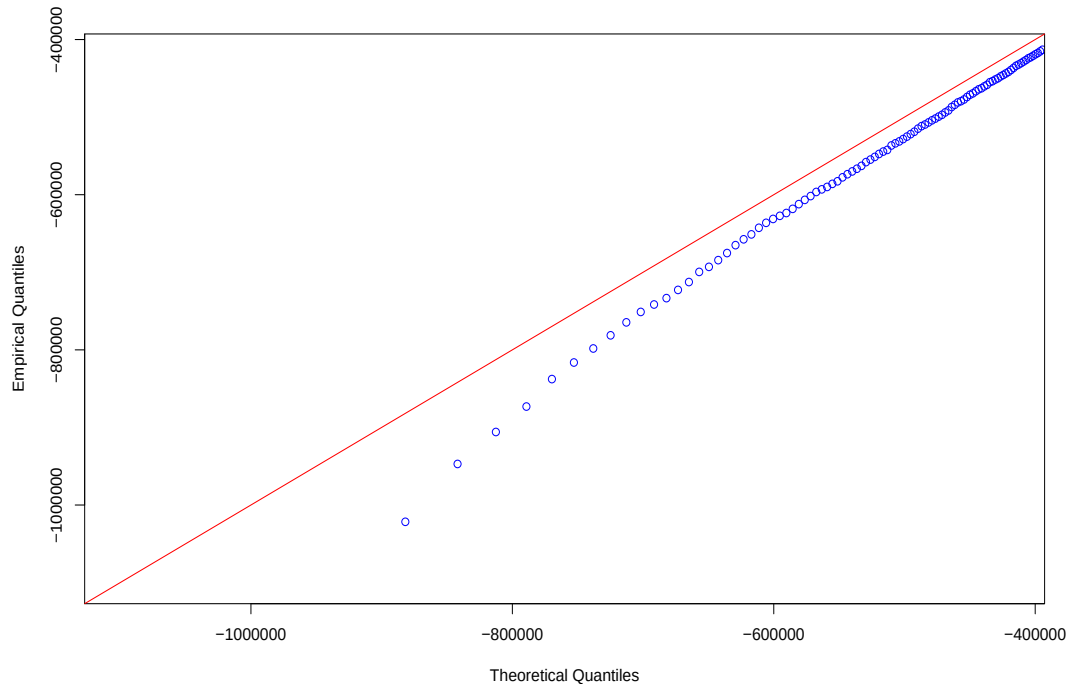

Figure 4: QQ-plot for lower quantiles $q \in (0, 0.1)$ to compare the left tail of the empirical density for the one-year CDR with the left tail of the fitted Gaussian density with mean 0 and standard deviation 292.879.

In Figure 3 we compare the empirical density from 100.000 simulations to the Gaussian density with the same mean ($\mu = 0$) and the same standard deviation ($\sigma = 292.879$) (cf. Table 1). We observe that these two densities look similar, but the Gaussian density has less probability mass on the left tail and therefore underestimates the shortfall risk of the one-year CDR. To get a closer look on the left tail of the empirical density for the one-year CDR we plot a QQ-plot for quantiles $q \in (0, 0.1)$. We observe a fatter left tail for the empirical density of the one-year CDR than for the fitted Gaussian density with mean 0 and standard deviation 292.879 (see Figure 4). This means that using a Gaussian approximation for the density of the one-year CDR leads to an underestimation of the shortfall risk in the one-year CDR.

A Appendix

Proof of Theorem 4.2: From (4) immediately follows that the posterior distribution $u(\Theta|\mathcal{D}_{J+1})$ is a multivariate Gaussian distribution. Therefore, it remains to calculate the first two moments of $u(\Theta|\mathcal{D}_{J+1})$. This is done by squaring out all terms. $\square$

Proof of Proposition 5.5:

We have

$$(L_i + L_j)X|_{\{\Theta,\mathcal{D}_J\}} \sim N((L_i + L_j)\mu, (L_i + L_j)\Sigma(L_i + L_j)')$$

for $i, j = 1, \ldots, J$. This implies

$$E\left[\exp\{(L_i + L_j)X|\Theta, \mathcal{D}_J\right] = \exp\{(L_i + L_j)\mu + (L_i + L_j)\Sigma(L_i + L_j)'/2\}$$

and using $\mu = \Gamma\Theta' + \gamma$ this leads to

$$\begin{aligned} E\left[\exp\{(L_i + L_j)X|\mathcal{D}_J\right] = E\left[\exp\{L_i X\}|\mathcal{D}_J\right]\ E\left[\exp\{L_j X\}|\mathcal{D}_J\right] \\ \times \exp\left\{L_i \Sigma L_j'\right\} \times \exp\left\{L_i \Gamma T(\mathcal{D}_J)\Gamma' L_j'\right\} \end{aligned}$$

for $i, j = 1, \ldots, J$. $\square$

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