Dividend problems in the dual risk model

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Abstract

We consider the dual risk model, dual to the well known classical risk model for insurance applications, where premiums are regarded as costs and claims are viewed as profits. The surplus can be interpreted as a venture capital like the capital of an economic activity involved in research and development. Like most authors, we consider an upper dividend barrier so that we model the gains of the venture capital and its return to the capital holders.

Based on the classical compound Poisson process, we show and explain clearly the dividends process dynamics, the properties of the different random quantities involved as well as their relations. The connections to the classical risk model together with the different variables involved are crucial in most of our developments. Using that connection, together with an additional upper absorbing barrier and allowing the process to continue after ruin, we derive several known and unknown results for the dual.

Some results about expected discounted dividends are known from the literature, several authors have addressed the problem. We go further. Based on some of the methods retrieved from the positive claims model, we address our study on different ruin and dividend probabilities. Such as the calculation of the probability of a dividend, number of dividends, expected and amount of dividends as well as the time of getting a dividend and inter-occurrence times.

We obtain some integro-differential equations for the above results and also Laplace transforms, then we can get either numerical or analytical results for cases where solutions and/or inversions are possible.

Keywords: Dual risk model; classical risk model; ruin probabilities; dividend probabilities; discounted dividends; dividend amounts; number of dividends.

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1 Introduction

We consider in this manuscript the dual risk model, as explained, for instance, by Avanzi et al. (2007). Using their language, the surplus or equity of a company is explained by the equation,

\[ U(t) = u - ct + S(t), \quad t \geq 0 \] (1.1)

where \( u \) is the initial surplus, \( c \) is a constant meaning the rate of expenses, \( \{S(t), t \geq 0\} \) is a compound Poisson process with parameter \( \lambda \) and density function \( p(x), x > 0 \), of the positive gains, with mean \( \mu_1 \) (we therefore assume that it exists). Its distribution function is denoted as \( P(x) \). The expected increase per unit time, given by \( \mu = E[S(1)] - c = \lambda p_1 - c \), is positive, that is \( c < \lambda p_1 \). All these quantities have a corresponding meaning in the classical continuous time risk model. For those used to work with the classical risk model the condition \( c < \lambda p_1 \) is reversed. A few authors have addressed this model, we can go back to Gerber (1979) who named it as the negative claims model, see pp. 136-138, also Bühlmann (1970). We can go even further back to authors like Cramér (1955), Takács (1967) and Seal (1969).

Avanzi et al. (2007), Section 1, explains well where applications of dual model are said to be appropriate, we just retain a simple but illustrative interpretation of the model, where the surplus can be considered as the capital of an economic activity like research and development where gains are random and at random instants and costs are certain. More precisely, The company pays expenses which occur continuously along time for the research activity and gets occasional profits of a random amount at random instants, according to a Poisson process. This model has been recently used by Bayraktar and Egami (2008) to model capital investments. Indeed, recently the model has been targeted with several developments, involving the present value of dividend and/or dividend strategies. We underline the cited work by Avanzi et al. (2007) and also Avanzi (2009), an excellent review paper. Other works by are important mentioning of which we comment appropriately some lines below.

Financial applications of this model, ruled by (1.1) are particularly important for modeling future dividends of investments. So, we add an upper barrier, the dividend barrier, noted as \( b (\geq u \geq 0) \). We refer to the upper graph in Figure 1 [see also Figure 1 of Avanzi et al. (2007)]. On the instant the surplus upcrosses de barrier a dividend is immediately paid and the process re-starts form level \( b \). We can also consider the case \( b < u \), however an immediate dividend is paid and the process re-starts from \( b \), see Avanzi et al. (2007). This makes the situation less interesting.

In this manuscript we are not interest on strategies of dividend payments but just the random amounts, once defined their level \( b \). We will consider the payments either discounted or not. Several papers have been published recently using this model considering an upper dividend barrier, where the calculation of expected amounts of the discounted paid dividends is the target. Higher moments have also been considered. See Avanzi et al. (2007), Avanzi and Gerber (2008), Cheung and Drekic (2008), Gerber and Smith (2008), Ng (2009) and Ng (2010). Yang and Zhu (2008) compute bounds for the ruin probability. Song et al. (2008) consider Laplace transforms for the calculation of the expected duration of negative surplus.

For those works as well as in ours where the dividend barrier \( b \) is the key point, it is important to emphasize two aspects: we are going to consider two absorbing barriers, the dividend barrier \( b \) and the ruin level “0”. However, these barriers lead to different actions. In the case of the upper barrier \( b \) the process restarts at level \( b \) if this is overtaken by a
claim. This is because an immediate amount of surplus in excess of \( b \) is paid in the form of a dividend, it’s a pay-back capital. It is not the case with the ruin level, where the process dies down definitely. To achieve a payable dividend the process must not be ruined previously. Furthermore, under the conditions stated the process, sooner or later, will be absorbed by one of the two barriers, we mean, with probability one the process will ultimately be absorbed.

In this paper we focus on the connection between the classical and the dual model and based on this we work on unknown problems, however having present some known results from a different viewpoint, which in some cases have interesting interpretations. We will underline these points appropriately. We base our research on the insights and ideas known from the classical risk model. This is a key point for our research. We first do a brief survey of the known results from the literature, then we make important connections between the classical and the dual model features. Afterwards, we make our own developments.

Let’s now consider some of the basic definitions and notation for the dual risk model, those which we address throughout this paper. Some specific quantities we will define and denote on the appropriate section only. First, consider the process as driven by Equation (1.1), free of the dividend barrier. Let

\[
\tau_x = \inf \{ t > 0 : U(t) = 0 | U(0) = x \}
\]

be the time to ruin, this is the usual definition for the model free of the dividend barrier (\( \tau_x = \infty \) if \( U(t) \geq 0 \ \forall t \geq 0 \)) and

\[
\psi(x, \delta) = E \left[ e^{-\delta \tau_x} I(\tau_x < \infty) | U(0) = x \right]
\]

where \( \delta \) is a non negative constant. \( \psi(u, \delta) \) is the Laplace transform of time to ruin \( \tau_x \). If \( \delta = 0 \) it reduces to the probability of ultimate ruin of the process free of the dividend barrier, when \( \delta > 0 \) we can see \( \psi(u, \delta) \) as the present value of a contingent claim of 1 payable at \( \tau_x \), evaluated under a given valuation force of interest \( \delta \) [see Ng (2010)].

Let’s now consider an upper level \( b \geq u \geq 0 \) in the model, see the upper graph of Figure 1, we don’t call it yet a dividend barrier. Let

\[
T_x = \inf \{ t > 0 : U(t) > b | U(0) = x \}
\]

be the time to reach an upper level \( b \geq x \geq 0 \) for the process which we allow to continue even if it crosses the “0”, or ruin, level. Due to the premium condition \( T_x \) is a proper random variable since the probability of crossing \( b \) is one. Dividend will only be due if \( T_x < \tau_x \) and ruin will only occur prior to that upcross otherwise. Whenever we refer to conditional random variables we will denote them by adding a “tilde”, like \( \tilde{T}_x \) for \( T_x | T_x < \tau_x \).

Now consider \( b \) as a dividend barrier and the ruin barrier, both absorbing, such that if the the process isn’t ruined it will be absorbed at the level \( b \) and vice versa. When it crosses \( b \) an immediate dividend is paid by an amount in excess of \( b \). Then the surplus is restored to level \( b \) and the process repeats. We will be mostly working the case \( 0 < u \leq b \), otherwise the process would immediately be set at level \( b \) where a dividend would be paid instantly. In that case then the process re-starts from level \( b \) and we are immediately re-set.

Let \( \xi(u, b) \) denote the probability of ruin before the process upcrosses the level \( b \) and \( \chi(u, b) \) denote the probability of upcrossing \( b \) before ruin occurring, for a process with initial surplus \( u \). Note that \( \xi(u, b) + \chi(u, b) = 1 \). We have \( \xi(u, b) = \Pr(T_x > \tau_x) \) and \( \chi(u, b) = \Pr(T_x < \tau_x) \).
Let \( M \) be the r.v. representing the number of dividends to be distributed. Let \( D_a = U(T_a) - b \) and \( T_u < \tau_u \) be the dividend amount and its distribution function be denoted as

\[
G(u, b; x) = \Pr(T_u < \tau_u \text{ and } U(T_u) \leq b + x) \big| (u, b)
\]

with density \( g(u, b; x) = \frac{d}{dx}G(u, b; x) \). \( G(u, b; x) \) is a defective distribution function, clearly \( G(u, b; \infty) = \chi(u, b) \).

We refer now to the upper graph in Figure 1. If the process crosses \( b \) for the first time before ruin at a random instant, say \( T(1) \), then a random amount, denoted as \( D(1) \) is paid. The process repeats, now from level \( b \). The random variables \( D(i) \) and \( T(i) \), \( i = 1, 2, \ldots \), respectively dividend amount \( i \) and waiting time \( T(i) \) until dividend \( i \), make a bivariate sequence of independent random variables \( \{(T(i), D(i))\}_{i=1}^{\infty} \). We mean, \( D(i) \) and \( T(i) \) are dependent in general but \( D(i) \) and \( T(j) \), \( i \neq j \), are independent. This follows from the Poisson process properties. Besides it’s easy to translate this from well known results concerning the classical risk model. Furthermore, if we take the subset \( \{(T(i), D(i))\}_{i=2}^{\infty} \) we now have a sequence of independent and jointly identically distributed random variables (and independent of \( (T(1), D(1)) \), the bivariate r.v. only have the same joint distribution if \( u = b \). To simplify notations we set that \( (T(i), D(i)) \) is distributed as \( (T_b, D_b) \), \( i = 2, 3, \ldots \), and \( (T(1), D(1)) \) is distributed as \( (T_u, D_a) \).

Let \( M(t) \) be the number of dividends to be distributed up to time \( t \) and \( M = M(\infty) \) the number of eventual dividends. Total amount of discounted dividends at a force of interest \( \delta > 0 \) is denoted as \( D(u, b, \delta) \) and \( D(u, b) = D(u, b, 0^+) \) is the undiscounted total amount. Their \( n \)-th moments are denoted as \( V_n(u; b, \delta) \) and \( V_n(u; b) \), respectively. For simplicity denote as \( V(u; b, \delta) = V_1(u; b, \delta) \).

The purpose of this work is to find new results for the different quantities of interest around the dividend problem in the dual risk model as well as to give a new insight in already known results. The key in our work is the interface we can establish between the classical and the dual model where, despite the reversed premium condition, many quantities can be characterized through features well known from the literature regarding the classical model. As far as the single dividend amount random variable is concerned it can be viewed as the severity of ruin from the classical model, although adapted to allow a second absorbing barrier.

The outline of the paper is what follows. In the next section we make an overview of the results and methods retrieved from the literature for the model. Section 3 makes the connection between the classical (positive claims model) and the dual risk model and considers the variables and features that can be used or transported from one to the other. Section 4 presents the new approach for the discounted dividends, particularly the expected discounted dividend amounts. The following section develops the probability of the dividend event as well as the number of dividends to occur and Section 6 deals with the distribution of the single dividend amount. Finally, in the last section we work illustrative examples.

## 2 Paper review and results

We present here known results from the literature particularly those related to our developments. We are interested on working on the different random variables defined in the previous section and expectations on dividends. It is not our concern dividend strategies, so
we omit findings related. Using a martingale argument Gerber (1979) showed that the ruin probability is given by

$$\psi(u) = \psi(u, 0) = e^{-Ru}$$ (2.1)

where $R$ is the unique positive root of the equation

$$\lambda \left( \int_0^\infty e^{-Rx} p(x) dx - 1 \right) + cR = 0. \quad (2.2)$$

We can use a standard probabilistic argument instead, we show it here as the method is going to be used later in the text for other purposes.

If there are no gains until $t_0 = u/c$ ruin level is crossed. By considering whether or not a gain occurs before time $t_0$, we have

$$\psi(u) = e^{-\lambda t_0} + \int_0^{t_0} \lambda e^{-\lambda t} \int_0^\infty p(x) \psi(u - ct + x) dx dt,$$

making $s = u - ct$ and rearranging we get

$$ce^{\lambda \frac{u}{c}} \psi(u) = c + \int_0^u \lambda e^{\lambda \frac{x}{c}} \int_0^\infty p(x) \psi(s + x) dx ds$$

Differentiating with respect to $u$ we get

$$ce^{\lambda \frac{u}{c}} \psi(u) \frac{\lambda}{c} + ce^{\lambda \frac{u}{c}} \frac{d}{du} \psi(u) = \lambda e^{\lambda \frac{u}{c}} \int_0^\infty p(x) \psi(u + x) dx$$

from which we get the following integro-differential equation

$$\lambda \psi(u) + c \frac{d}{du} \psi(u, 0) = \lambda \int_0^\infty p(x) \psi(u + x) dx$$

with the boundary conditions $\psi(0) = 1$ and $\psi(\infty) = 0$. Now, it’s easy to set that $\psi(u + x) = \psi(u) \psi(x)$ since ruin to occur from initial level $u + x$ must first cross level $u$ and from there get ruined (properties from the Poisson process apply). Hence,

$$c \frac{d}{du} \psi(u) = \lambda \psi(u, 0) \left( \int_0^\infty p(x) \psi(x, 0) dx - 1 \right)$$

$$\frac{d}{du} \log \psi(u, 0) = \frac{\lambda}{c} (A - 1)$$

$$\psi(u, 0) = e^{\lambda(A-1)u},$$

where $A = \int_0^\infty p(x) \psi(x) dx$ ($A$ is not dependent on $u$). If we set $R = -\frac{\lambda}{c} (A - 1)$, then we get (2.1).

Ng (2009) generalized the above probability for a positive $\delta$, $\psi(u, \delta)$, which is given by

$$\psi(u, \delta) = e^{-R_{\delta}u},$$ (2.3)

where $R_{\delta}$ is the unique positive root such that

$$\lambda \left( \int_0^\infty e^{-R_{\delta}x} p(x) dx - 1 \right) + cR_{\delta} = \delta. \quad (2.4)$$
Results below concern expectations, moments, for discounted dividends by integro-differential equations which in some cases shown can be solved analytically. As far as expectations of total dividends is concerned, using a direct and standard approach, by considering possible single gains/jumps events over a small time interval, Avanzi \textit{et al.} (2007) found that 
\[ V(0; b) = 0, \]
since ruin is immediate if \( u = 0 \), and that,
\[ V(u; b) = u - b + V(b; b) \] if \( u > b \), and
\[ 0 = cV'(u; b) + (\lambda + \delta)V(u; b) - \lambda \int_0^{b-u} V(u + y, b)p(y)dy \] (2.5)
\[ - \lambda \int_{b-u}^{\infty} V(u - b + y, b)p(y)dy - \lambda V(b, b) [1 - P(b - u)] \], if \( 0 < u < b \).

Besides, Avanzi \textit{et al.} (2007) found solutions for equation (2.5) when exponential or mixtures of exponential gains size distributions are considered. Ng (2010) shows solutions when individual gains are phase-type distributed.

For higher moments of discounted dividends, Cheung and Drekic (2008) with a similar procedure show integro-differential equations similar to (2.5)
\[ V_n(u; b) = \sum_{j=0}^{n} \binom{n}{j} (u - b)^{n-j} V_j(b; b) \] if \( u > b \), and
\[ 0 = cV'_n(u; b) + (\lambda + n\delta)V_n(u; b) - \lambda \int_0^{b-u} V_n(u + y, b)p(y)dy \] (2.6)
\[ - \lambda \sum_{j=0}^{n} \binom{n}{j} V_j(b, b) \int_{b-u}^{\infty} [y - (b - u)]^{n-j} p(y)dy, \] if \( 0 < u < b \).

as well as solutions for combinations of exponentials distributed gains size. Also show approximation methods.

3 Connecting the classical and the dual model

In our further developments it’s crucial the connection between the classical and the dual model as we want to transport methods and results from the first to the second, which has had extensive treatment. So, we put aside together both two models, at a first stage we consider the models free of barriers. As widely known, the classical Cramér-Lundberg risk model is ruled by the equation
\[ U^*(t) = u^* + c^* t - S^*(t), \quad t \geq 0, \] (3.1)
where the quantities involved have similar characteristics (although different interpretations) to those correspondig to the dual model. To emphazise that we denote the corresponding quantities with the same letter but coming with a “*”. Apart form their application and interpretation the big difference between the two models comes with the premium condition \( c^* > \lambda p_1 \), whose condition is reversed. This conditions assures that the surplus ultimately tends to infinity with probably one. The reversed condition in the dual models is intended to achieve the same target, otherwise it would be of difficult application, investers wouldn’t get dividends as they could wish.
We can relate the two models in the following way and, we refer to Figure 1, free of the dividends barrier,

\[ U^*(t) = u^* + ct - S(t) = (b - u) + ct - S(t), \quad t \geq 0, \ b > u, \quad (3.2) \]

For our dividend problem, in order to relate these two models we need to set and comment on the barriers. In the dual model we consider the model with an upper dividend barrier and a ruin barrier. They are both absorbing barriers. In the corresponding positive claims model, the corresponding dividend barrier is now the ruin barrier from initial surplus \( b - u \). The other mentioned barrier usually is not considered in the positive claims model, and may corresponds just to an upper line at level \( b \). See again Figure 1.

We note that if the ruin level wasn’t an absorbing barrier, i.e., the process continuing even if ruin occurred, the upper level \( b \) would be reached with probability 1, due to the premium condition. This comes immediately from the knowledge we have from the classical risk model. However, we follow the model defined by Avanzi et al. (2007) that we should only pay dividends if the process isn’t ruined. Clearly, capital should not be rewarded in the case of ruin (perhaps negatively, if we could borrow). We are only interested working over the set of the sample paths of the surplus process that do not lead to ruin. We need to calculate the probability of the surplus process reaches the barrier \( b \) before crossing the level zero. This probability does not correspond to the survival probability, from initial level \( u \), even though the process either crosses the ruin level or a finite upper barrier \( b \) in the long run, if we consider a free, no barriers, process (with probability zero the process will travel indefinitely between the two levels). Survival means the non-ruin event. A simple argument is apparent: we can have a path, realization, of the process where ruin occurs with a previous upcross to the barrier level \( b \).

Hence, to get a dividend is necessary that the process attains the level \( b \) before getting ruined. The probability of that event is not evaluated, so far. The complementary probability is the probability of ruin before attaining level \( b \). We will come to this matter in Section 5.

In Figure 1, upper graph again, if we turn it upside down (rotate 180°) and look at it from right to left we get the classical model shape, where level “\( b \)” is the ruin level, “\( u \)” is the initial surplus and the level “\( 0 \)” is a upper barrier. \( \{D_{(i)}\}_{i=2}^\infty \) is viewed as a sequence of i.i.d. severity of ruin random variables from initial surplus zero and \( D_{(1)} \) the independent, but not identically distributed, severity of ruin random variable from initial surplus “\( b - u \)”.

Similarly, we have that \( \{T_{(i)}\}_{i=2}^\infty \) can be viewed as a sequence of i.i.d. random variables meaning time of ruin from initial surplus \( 0 \), independent of \( T_{(1)} \) which in turn represents the time of ruin from initial surplus \( b - u \). The connection between the two models is briefly mentioned by Avanzi (2009) (Section 3.1), but not clearly.

We need to consider some results on the severity of ruin (expectations on the discounted severity of ruin) from the classical risk model adapted to allow an absorbing upper barrier \( b > 0 \). The following reasoning follows from Dickson and Waters (2004), Section 4, adapted to set the barrier \( b \) (we refer to Figure 1, \( U^*(t) \) graph). We present some new definitions, valid for this section only.

Let \( Y_u \) denote the deficit at ruin and \( T^*_u \) the time of ruin given an initial surplus \( u \). We denote the defective distribution of the deficit \( Y_u \) as \( G^*(u; x) \) with density \( g^*(u; x) \) \( [G^*(u; \infty) = \psi^*(u), \ \text{no barrier } b \ \text{considered}] \). Now, define \( \phi(u^*, b) = E[e^{-\delta T^*_u} Y_u^*] \) as an expectation of a discounted power of the severity of ruin \( (\phi_1(u^*, b) \text{ is the expected discounted severity of ruin}) \). Let \( t_0 \) denote the time that the surplus takes to reach \( b \) if there are no claims, so that
Figure 1: Classical vs dual model
and by substituting ary condition Di↵erentiating and rearranging, we get the following integro-di↵erential equation, with bound-
u the amount of the …rst claim, below zero, dual model, it will become clear in the next section. This result is going to be used for computation of moments of times and dividends in the complementary comes in…nite time, or the total discounted dividends amounts, denoted simply as 

\[ V(u; b) = \sum_{i=1}^{\infty} e^{-\delta \sum_{j=1}^{i} T_{(j)}} D_{(i)} , \quad 0 \leq u \leq b . \]  

(4.1)

Its expected value comes, \( V(u; b) \),

\[ V(u; b) = \mathbb{E}[D(u, b, \delta)] = \mathbb{E} \left[ \sum_{i=1}^{\infty} e^{-\delta \sum_{j=1}^{i} T_{(j)}} D_{(i)} \right] = \sum_{i=1}^{\infty} \mathbb{E} \left[ e^{-\delta \sum_{j=1}^{i} T_{(j)}} D_{(i)} \right] \]

\[ = \mathbb{E} \left( e^{-\delta T_{(1)}} D_{(1)} \right) + \mathbb{E} \left( e^{-\delta T_{(1)}} \right) \mathbb{E} \left( e^{-\delta T_{(2)}} D_{(2)} \right) \]

\[ + \mathbb{E} \left( e^{-\delta T_{(1)}} \right) \mathbb{E} \left( e^{-\delta T_{(2)}} \right) \mathbb{E} \left( e^{-\delta T_{(3)}} D_{(3)} \right) + \ldots \]

because the pairs of random variables are independent \((T_{(i)}, D_{(i)}) \), \( i = 1, 2, \ldots \), two by two. Note that \( T_{(i)} \) and \( D_{(i)} \) are dependent in general, they have similar properties like time to
ruin and its severity in the classical case. Besides, \((T_i, D_i), i = 2, 3, \ldots,\) are also identically distributed, say \((T_b, D_b) \overset{d}{=} (T_i, D_i), i = 2, 3, \ldots,\) also \(T_i \overset{d}{=} T_b,\) also \(i = 2, 3, \ldots\) Set \((T_u, D_u) \overset{d}{=} (T_1, D_1),\) we can write

\[
V(u; b) = \mathbb{E}\left( e^{-\delta T_u} D_u \right) + \mathbb{E}\left( e^{-\delta T_u} \right) \mathbb{E}\left( e^{-\delta T_b} D_b \right) + \mathbb{E}\left( e^{-\delta T_u} \right)^2 \mathbb{E}\left( e^{-\delta T_b} D_b \right) + \ldots
\]

\[
= \mathbb{E}\left( e^{-\delta T_u} D_u \right) + \mathbb{E}\left( e^{-\delta T_u} \right) \mathbb{E}\left( e^{-\delta T_b} D_b \right) \left[ 1 + \mathbb{E}(e^{-\delta T_u}) + \mathbb{E}(e^{-\delta T_b})^2 + \ldots \right].
\]

Hence

\[
V(u; b) = \mathbb{E}\left( e^{-\delta T_u} D_u \right) + \mathbb{E}\left( e^{-\delta T_u} \right) \mathbb{E}\left( e^{-\delta T_b} D_b \right) \left[ \frac{1}{1 - \mathbb{E}(e^{-\delta T_b})} \right]. \tag{4.2}
\]

A similar expression can be found in Dickson and Waters (2004) relating discounted time and severity of ruin in the classical model with a dividend strategy. We only need to evaluate \(\mathbb{E}\left( e^{-\delta T_u} D_u \right), \mathbb{E}\left( e^{-\delta T_u} D_b \right), \mathbb{E}(e^{-\delta T_u})\) and \(\mathbb{E}(e^{-\delta T_b}).\)

To compute the above quantities, and therefore \(V(u; b),\) can make use of the Expression (3.3) and Equation (3.4) at the end of Section 3 \(\phi_n(u - b, b)\) for \(n = 0, 1\) and \(u^* = 0\) or \(u - b.\)

In the simpler case \(u = b,\) we have \((T_1, D_1) \overset{d}{=} (T_b, D_b)\) and \(T_1 \overset{d}{=} T_b,\) and the above formula simplifies to

\[
V(b; b) = \frac{E(e^{-\delta T_b} D_b)}{1 - E(e^{-\delta T_b})}. \tag{4.3}
\]

Then we have

\[
V(u; b) = u - b + \frac{E(e^{-\delta T_b} D_b)}{1 - E(e^{-\delta T_b})} \text{ if } u \geq b,
\]

because \(V(b; b)\) is (4.3).

Using the same method we can compute higher moments. For instance, if we want to compute the variance of the accumulated discounted dividends we need to compute \(V_2(u; b).\) Let \(Z_i\) be the discounted dividend \(i\) so that

\[
Z_i = e^{-\delta \sum_{j=1}^{i} T_{(j)}} D_{(i)}.
\]

\[
D(u, b, \delta) = \sum_{i=1}^{\infty} Z_i.
\]

Then,

\[
V_2(u; b) = \mathbb{E}[D(u, b, \delta)^2] = \sum_{i=1}^{\infty} \mathbb{E}[Z_i^2] + 2 \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \mathbb{E}[Z_i Z_j]
\]

Using the above properties on the distributions \((T_b, D_b) \overset{d}{=} (T_i, D_i), i = 2, 3, \ldots,\) and \((T_u, D_u) \overset{d}{=} (T_1, D_1)\) as well as independence in the sequence, we can write

\[
\mathbb{E}[Z_i^2] = \mathbb{E}\left( e^{-2\delta T_u} \right) \mathbb{E}\left( e^{-2\delta T_b} \right) i - 2 \mathbb{E}\left( e^{-2\delta T_b} D_b^2 \right), i = 2, 3, 4, \ldots
\]

\[
\mathbb{E}[Z_i^2] = \mathbb{E}\left( e^{-2\delta T_u} D_u^2 \right).
\]
Hence,
\[
\sum_{i=1}^{\infty} \mathbb{E}[Z_i^2] = E\left(e^{-\delta T_u} D_u\right) + \mathbb{E}\left(e^{-2\delta T_h} D_b^2\right) \sum_{i=1}^{\infty} \mathbb{E}\left(e^{-2\delta T_h}\right)^{i-2} \\
= E\left(e^{-\delta T_u} D_u\right) + \frac{\mathbb{E}\left(e^{-2\delta T_h}\right) \mathbb{E}\left(e^{-2\delta T_b} D_h^2\right)}{1 - \mathbb{E}\left(e^{-2\delta T_b}\right)}.
\]

Now,
\[
\sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \mathbb{E}[Z_i Z_j] = \sum_{j=2}^{\infty} \mathbb{E}[Z_1 Z_j] + \sum_{i=2}^{\infty} \sum_{j=i+1}^{\infty} \mathbb{E}[Z_i Z_j],
\]
with
\[
\mathbb{E}[Z_1 Z_j] = \mathbb{E}\left(e^{-2\delta T_u} D_u\right) \mathbb{E}\left(e^{-\delta T_h} D_b\right) \mathbb{E}\left(e^{-\delta T_h}\right)^{j-2} \\
\mathbb{E}[Z_i Z_j] = \mathbb{E}\left(e^{-2\delta T_u} D_u\right) \mathbb{E}\left(e^{-2\delta T_h} D_b\right) \mathbb{E}\left(e^{-\delta T_h} D_b\right) \mathbb{E}\left(e^{-\delta T_h}\right)^{i-2} \mathbb{E}\left(e^{-\delta T_h}\right)^{j-i-1},
\]
for \( i < j, i = 2, 3, \ldots \) Then
\[
\sum_{j=2}^{\infty} \mathbb{E}[Z_1 Z_j] = \frac{\mathbb{E}\left(e^{-2\delta T_u} D_u\right) \mathbb{E}\left(e^{-\delta T_h} D_b\right)}{1 - \mathbb{E}\left(e^{-\delta T_h}\right)} \\
\sum_{j=i+1}^{\infty} \mathbb{E}\left(e^{-\delta T_h}\right)^{j-(i+1)} = \sum_{k=0}^{\infty} \mathbb{E}\left(e^{-\delta T_h}\right)^k = \frac{1}{1 - \mathbb{E}\left(e^{-\delta T_h}\right)} \\
\sum_{i=2}^{\infty} \sum_{j=i+1}^{\infty} \mathbb{E}[Z_i Z_j] = \frac{\mathbb{E}\left(e^{-2\delta T_u} D_u\right) \mathbb{E}\left(e^{-2\delta T_h} D_b\right) \mathbb{E}\left(e^{-\delta T_h} D_b\right) \sum_{i=2}^{\infty} \mathbb{E}\left(e^{-2\delta T_h}\right)^{i-2} \mathbb{E}\left(e^{-\delta T_h}\right)^{j-i-1}}{1 - \mathbb{E}\left(e^{-\delta T_h}\right) [1 - \mathbb{E}\left(e^{-2\delta T_h}\right)]}
\]

Here we need to evaluate the following six different quantities \( \mathbb{E}\left(e^{-2\delta T_u}\right), \mathbb{E}\left(e^{-2\delta T_h}\right), \mathbb{E}\left(e^{-2\delta T_u} D_u\right), \mathbb{E}\left(e^{-2\delta T_h} D_b\right), \mathbb{E}\left(e^{-2\delta T_h} D_b^2\right), \mathbb{E}\left(e^{-2\delta T_u} D_u^2\right) \), apart from those four needed for the first moment \( \mathbb{E}\left(e^{-\delta T_u} D_u\right), \mathbb{E}\left(e^{-\delta T_h} D_b\right), \mathbb{E}\left(e^{-\delta T_u}\right) \) and \( \mathbb{E}\left(e^{-\delta T_h}\right) \).

5 On the number of dividends

As we said at the end of Section 1 to get a dividend is necessary that the process reaches/crosses the level \( b \) before ruin, which occurs with probability \( \chi(u, b) \). The complementary probability is \( \xi(u, b) \). That is
\[
\chi(u, b) = \Pr \left[ T_u < \tau_u \right] \text{ and } \xi(u, b) = \Pr \left[ T_u > \tau_u \right], \ u \leq b,
\]
\( \xi(u, b) + \chi(u, b) = 1 \), since the process does not travel indefinitely between levels \( b \) and 0. If a dividend is paid then the process restarts at level \( b \), and the process repeats, now from level \( b \).

Finding closed forms for \( \xi(u, b) \), or \( \chi(u, b) \), isn’t as straightforward as the similar quantities in the classical model referred at the end of Section 3. Using the usual approach, to reach
ruin level prior to dividend level is possible with or without a claim (at time $t_0 : u - ct_0 = 0$). Then, for $0 < u < b$:

$$
\xi(u, b) = e^{-\lambda t_0} + \int_0^{t_0} \lambda e^{-\lambda t} \int_0^{b-(u-ct)} p(x) \xi(u - ct + x, b) \, dx \, dt,
$$

from which we find

$$
\lambda \xi(u, b) + c \frac{d}{du} \xi(u, b) = \lambda \int_0^{b-u} p(x) \xi(u + x, b) \, dx
$$
or

$$
\lambda \xi(u, b) + c \frac{d}{du} \xi(u, b) = \lambda \int_u^b p(y - u) \xi(y, b) \, dy
$$

(5.1)

where the boundary conditions is $\xi(0, b) = 1$. Setting $\xi(u + x, b) = \xi(u, b - x) \xi(x, b) = \xi(x, b - u) \xi(u, b)$ we get

$$
\lambda \xi(u, b) + c \frac{d}{du} \xi(u, b) = \lambda \xi(u, b) \int_0^{b-u} p(x) \xi(x, b - u) \, dx
$$

$$
+ \frac{c}{\lambda} \frac{d}{du} \xi(u, b) = \xi(u, b) \left( \int_0^{b-u} p(x) \xi(x, b - u) \, dx - 1 \right)
$$

$$
\frac{d}{du} \log \xi(u, b) = \frac{\lambda}{c} \left( \int_0^{b-u} p(x) \xi(x, b - u) \, dx - 1 \right).
$$

Likewise, we can get

$$
\lambda \chi(u, b) + c \frac{d}{du} \chi(u, b) = \lambda \int_0^{b-u} p(x) \chi(u + x, b) \, dx + \lambda [1 - P(b - u)]
$$
or

$$
\lambda \chi(u, b) + c \frac{d}{du} \chi(u, b) = \lambda \int_u^b p(y - u) \chi(y, b) \, dy
$$

where the boundary conditions is $\chi(0, b) = 0$.

For $u = b$ we can write:

$$
\chi(b, b) = \int_0^b \lambda e^{-\lambda t} \left[ \int_0^{ct} p(x) \chi(b - ct + x, b) \, dx + \int_{ct}^\infty p(x) \, dx \right] \, dt
$$

We can compute Laplace transforms on Equation (5.1) as an alternative method to find $\xi(u, b)$. We can use a method of change of variable already used by Avanzi et al. (2007), Section 6, retrieved by Cheung and Drekic (2008) and mentioned in the review paper by Avanzi (2009). In that equation replace $u$ by $z = b - u$ and define $\mathcal{E}(z, b) = \xi(b-z, b) = \xi(u, b)$. This change of variable analytically is like setting the relation between the two models, classical and dual. Note that $\mathcal{E}(b, b) = \xi(0, b) = 1$. The corresponding integro-differential equation for $\mathcal{E}(z, b)$ is

$$
\lambda \mathcal{E}(z, b) - c \frac{\partial}{\partial z} \mathcal{E}(z, b) - \lambda \int_0^z p(z - y) \mathcal{E}(y, b) \, dy = 0.
$$
In function \( \mathcal{E}(z, b) \) extend the range of \( z \) from \( 0 \leq z \leq b \) to \( 0 \leq z \leq \infty \) and denote the resulting function by \( \epsilon(z) \), then compute its Laplace transform, denoted as \( \tilde{\epsilon}(s) \), so that

\[
\lambda \tilde{\epsilon}(s) - cs[\tilde{\epsilon}(s) - \epsilon(0)] - \lambda \tilde{\epsilon}(s) \tilde{p}(s) = 0,
\]

where \( \tilde{p}(s) \) is the Laplace transform of the single gains common density \( p(x) \). Hence,

\[
\tilde{\epsilon}(s) = \frac{c\epsilon(0)}{cs - \lambda + \lambda \tilde{p}(s)} \quad (5.2)
\]

where \( \epsilon(0) = \xi(b, b) \) (note that \( \epsilon(b) = \mathcal{E}(b, b) = \xi(0, b) = 1 \). When \( \tilde{p}(s) \) is a rational function we can invert \( \tilde{\epsilon}(s) \) to find a solution for \( \epsilon(z) \). Finally \( \xi(u, b) = \epsilon(b - u) \) for \( 0 \leq u \leq b \).

Now let’s consider the multiple dividend situations and let \( M \) be the random variable representing the number of dividends to be claimed, or the number of times the process upcrosses the upper level \( b \). Then, \( \Pr[M = 0] = \xi(u, b) \) and \( \Pr[M \geq 1] = 1 - \xi(u, b) = \chi(u, b) \). Besides, \( \Pr[M = 1] = \chi(u, b) \xi(b, b) \), first the process crosses \( b \) a dividend is paid, then restarts from \( b \) and after that is ruined. \( \Pr[M = 2] = \chi(u, b) \xi(b, b) \xi(b, b) \), and so on. In summary, we get

\[
\begin{align*}
\Pr[M = 0] &= \xi(u, b) \\
\Pr[M = k] &= \chi(u, b) \xi(b, b)^{k-1} \xi(b, b), \ k = 1, 2, ... 
\end{align*}
\]

\( M \) follows a zero-modified geometric distribution. If \( u = b \) we get a geometric distribution with \( \Pr[M = k] = \xi(b, b)^k \xi(b, b), \ k = 0, 1, 2, ... \) Its moment generation function of \( M \) is

\[
\varphi_M(t) = \xi(u, b) + \chi(u, b) \frac{\xi(b, b)e^t}{1 - \xi(b, b)}. 
\]

Then, the total dividend gains (not discounted) follows a compound zero-delayed geometric distribution.

## 6 On the dividend amount distribution

Now we are going to work on the distribution of the random variable \( D_u \), distribution and density functions denoted as \( G(u, b; x) \) and \( g(u, b; x) \), respectively (see Section 1). First we will set the probability \( \xi(u, b) \) and \( g(u, b; x) \). Consider the process free of the absorbing barrier \( b \), just consider \( b \geq u \) as a fixed level. We can write, considering that ruin can occur before or after crossing \( b \),

\[
\begin{align*}
\psi(u) &= \xi(u, b) + \int_0^\infty g(u, b; x)\psi(b + x)dx = \xi(u, b) + \psi(b) \int_0^\infty g(u, b; x)\psi(x)dx \\
\xi(u, b) &= e^{-Ru} - e^{-Rb} \bar{g}(u, b; R) \quad (6.1)
\end{align*}
\]

setting \( \psi(x) = e^{-Rx} \) and rearranging, where \( \bar{g}(u, b; R) \) is the Laplace transform of the density \( g(u, b; x) \) evaluated at \( R \).

Back to the usual model, we can compute an integro-differential equation for \( G(u, b; x) \) using the standard procedure. Conditioning on the first claim we get, where \( t_0 \) is such that \( u - ct_0 = 0 \),

\[
G(u, b; x) = \int_0^{t_0} \lambda e^{-\lambda t} \left[ \int_0^{b-(u-ct)} p(y)G(u - ct + y, b; x)dy + \int_{b-(u-ct)+x}^{b-(u-ct)+x} p(y) \right] dt.
\]

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Rearranging and differentiating with respect to \( u \), we obtain the following integro-differential equation

\[
\lambda G(u, b; x) + c \frac{\partial}{\partial u} G(u, b; x) = \lambda \int_u^b p(y-u)G(y, b; x) \, dy + \lambda [P(b-u+x) - P(b-u)], \tag{6.2}
\]

with boundary condition \( G(0, b; x) = 0 \). Similarly, we get

\[
\lambda g(u, b; x) + c \frac{\partial}{\partial u} g(u, b; x) = \lambda \int_0^{b-u} p(y)g(u+y, b; x) \, dy + \lambda p(b-u+x). 
\]

We can compute Laplace transforms for \( G(u, b; x) \) by methods similar to those used for (5.2). Let \( G(z, b; x) = G(bz, b; x) \). Then \( G(b, b; x) = G(0, b; x) = 0 \) Then, from (6.2) we get

\[
\lambda G(z, b; x) - c \frac{\partial}{\partial z} G(z, b; x) - \lambda \int_z^x p(z-y)G(y, b; x) \, dy - \lambda (P(z+x) - P(z)) = 0
\]

Let \( \rho(z, x) \) be the correspondent function arising from extending the range of \( z \). Taking Laplace Transforms we get easily,

\[
\lambda \tilde{\rho}(s, x) - c [s \tilde{\rho}(s, x) - \rho(0, x)] - \lambda \tilde{\rho}(s, x)\tilde{\rho}(s) + \lambda \left[ \frac{\tilde{\rho}(s)}{s} - \tilde{\rho}(s, x) \right] = 0,
\]

where

\[
\frac{\tilde{\rho}(s)}{s} = \int_0^\infty e^{-sz} \rho(z) \, dz
\]

\[
\tilde{\rho}(s, x) = \int_0^\infty e^{-sz} \rho(z, x) \, dz
\]

\[
\tilde{\rho}(s, x) = \int_0^\infty e^{-sz} \rho(z+x) \, dz = e^{sx} \int_x^\infty e^{-sy} \rho(y) \, dy. \tag{6.3}
\]

Hence,

\[
\tilde{\rho}(s, x) = \frac{cp(0, x) + \lambda [\tilde{\rho}(s)/s - \tilde{\rho}(s, x)]}{cs - \lambda + \lambda \tilde{\rho}(s)} \tag{6.4}
\]

Likewise, for the density \( g(u, b; x) \) from (6) setting \( \gamma(z, b; x) = g(b-z, b; x) \).

\[
\lambda \gamma(z; x) - c \frac{\partial}{\partial z} \gamma(z; x) - \lambda \int_0^z p(z-y)\gamma(y; x) \, dy - \lambda p(z+x) = 0
\]

from which we get the Laplace transform for \( \gamma(z; x) \)

\[
\tilde{\gamma}(s; x) = \frac{c\gamma(0; x) - \lambda \tilde{\rho}(s, x)}{cs - \lambda + \lambda \tilde{\rho}(s)}
\]

with

\[
\tilde{\rho}(s, x) = \int_0^\infty e^{-sz} \rho(z+x) \, dz = e^{sx} \int_x^\infty e^{-sy} \rho(y) \, dy.
\]

Note that

\[
\frac{\partial}{\partial x} \tilde{\rho}(s, x) = \tilde{\gamma}(s; x)
\]
With a probability argument we can write
\[ G(u, b; x) = \int_0^{b-u} g(u, y)G(u + y, b; x) \, dy + \int_{b-u}^{b-x} g(u, y) \, dy \]
\[ g(u, b; x) = \int_0^{b-u} g(u, y)(u + y, b; x) \, dy + g(u, b - u + x) \]

Let’s now consider the process continuing even if ruin occurs. The process can cross for the first time the upper dividend level before or after having ruined. Then we can write the (proper) distribution of the amount by which the process first upcrosses 0, as
\[ H(u, b; x) = H(u, b; x|T_u < \tau_u)\chi(u, b) + H(u, b; x|T_u > \tau_u)\xi(u, b) \]
\[ = G(u, b; x) + \xi(u, b)H(0, b; x). \]

Clearly, \( H(u, b; x) = \Pr [U(T_u) \leq b + x] \). The second equation above simply means that the probability of the amount by which the process first upcrosses \( b \) is less or equal than \( x \), equals the probability of a dividend claim less or equal than \( x \) plus the probability of a similar amount but in that case it cannot be a dividend as the process first crosses the ruin line. This second probability can be computed through the probability of first reaching the level 0, \( \xi(u, b) \), times the probability of an upcrossing of level \( b \) by the same amount \( \leq x \) but restarting from 0. We can compute \( H(u, b; x) \), \( u \) through expressions known for the distribution of the severity of ruin obtained from the positive claims model (recall that the premium condition is reversed, and then it is a proper distribution function). Then we get
\[ G^*(b - u; x) = G(u, b; x) + \xi(u, b)G^*(b; x) \]
equivalent to
\[ G(u, b; x) = G^*(b - u; x) - \xi(u, b)G^*(b; x). \]

For \( u = b \) we have
\[ H(b, b; x) = G(b, b; x) + \xi(b, b)H(0, b; x) \]
\[ G(b, b; x) = G^*(0; x) - \xi(b, b)G^*(b; x). \]

From here, take Laplace transforms and evaluate at \( R \)
\[ g(b, b; x) = g^*(0; x) - \xi(b, b)g^*(b; x) \]
\[ \bar{g}(b, b; R) = \bar{g}^*(0; R) - \xi(b, b)\bar{g}^*(b; R), \]
then use (6.1) to get
\[ \xi(b, b) = \frac{[1 - \bar{g}^*(0; R)] e^{-Rb}}{1 - \bar{g}^*(b; R)e^{-Rb}}. \]

\( g^*(0; x) = p_1^{-1} [1 - P(x)] \) is the severity density in the positive claims model whose Laplace transform is
\[ \bar{g}^*(0; R) = \frac{1}{Rp_1} \left( 1 - \int_0^{\infty} e^{-Rx} p(x) \, dx \right) = \frac{c}{\lambda p_1} \]
using (2.2). We still need to compute \( \bar{g}^*(b; R) \), clearly, it is not a trivial calculation since it’s a Laplace transform of the severity of ruin with a positive initial surplus in the positive claims model. If \( p(x) \) is exponential then \( \bar{g}^*(0; R) = \bar{g}^*(u; R) \), that is the trivial example.
7 Illustrations

7.1 Exponential jumps

We consider the case when claim amounts are exponentially distributed, that is \( p(y) = \alpha e^{-\alpha y}, \) \( y > 0, \) with \( \alpha > 0. \) Using Equation (3.4), making \( u^* = v \)

\[
\frac{d}{dv} \phi_n(v, b) = \frac{\lambda + \delta}{c} \phi_n(v, b) - \frac{\lambda}{c} \int_0^v \phi_n(v - y, b)\alpha e^{-\alpha y} dy - \frac{\lambda}{c} \int_v^\infty (y - v) n e^{-\alpha y} dy
\]

\[
= \frac{\lambda + \delta}{c} \phi_n(v, b) - \frac{\lambda}{c} e^{-\alpha u} \left( \int_0^v \phi_n(x, b)\alpha e^{\alpha x} dx + \frac{n!}{\alpha^n} \right) \tag{7.1}
\]

Rearranging and differentiating, we get

\[
\frac{d^2}{dv^2} \phi_n(v, b) + \left( \frac{\lambda + \delta}{c} - \frac{\lambda}{c} \right) \frac{d}{dv} \phi_n(v, b) - \frac{\alpha \delta}{c} \phi_n(v, b) = 0
\]

which has the general solution

\[
\phi_n(v, b) = h_1 e^{r_1 v} + h_2 e^{r_2 v}
\]

where \( r_1 \) and \( r_2 \) are the roots of the characteristic equation

\[
s^2 + \left( \frac{\lambda + \delta}{c} - \frac{\lambda}{c} \right) s - \frac{\alpha \delta}{c} = 0 \tag{7.2}
\]

so that

\[
r_1, r_2 = -\frac{(\alpha c - \lambda - \delta) \pm \sqrt{(\alpha c - \lambda - \delta)^2 + 4\alpha \delta c}}{2c}
\]

with \( r_1 > 0 \) and \( r_2 < 0. \) These roots verify

\[
r_2 \times r_1 = -\alpha \delta / c \text{ and } r_1 + r_2 = -\alpha + (\lambda + \delta) / c. \tag{7.3}
\]

Inserting \( \phi_n(v, b) \) in (7.1), we have

\[
h_1 r_1 e^{r_1 v} + h_2 r_2 e^{r_2 v} = \frac{\lambda + \delta}{c} \left( h_1 e^{r_1 v} + h_2 e^{r_2 v} \right) - \frac{\lambda}{c} e^{-av} \left( \alpha \int_0^v h_1 e^{(r_1 + \alpha) x} + h_2 e^{(r_2 + \alpha) x} dx + \frac{n!}{\alpha^n} \right)
\]

\[
= \frac{\lambda + \delta}{c} \left( h_1 e^{r_1 v} + h_2 e^{r_2 v} \right)
\]

\[
- \frac{\lambda}{c} \left( \frac{\alpha h_1}{r_1 + \alpha} \left( e^{r_1 v} - e^{-av} \right) + \frac{\alpha h_2}{r_2 + \alpha} \left( e^{r_2 v} - e^{-av} \right) + \frac{n!}{\alpha^n} e^{-av} \right),
\]

from where we get the following equations,

\[
h_1 \left( r_1 - \frac{\lambda + \delta}{c} + \frac{\lambda}{c} \frac{\alpha}{r_1 + \alpha} \right) = 0
\]

\[
h_2 \left( r_2 - \frac{\lambda + \delta}{c} + \frac{\lambda}{c} \frac{\alpha}{r_2 + \alpha} \right) = 0
\]

\[
\frac{\alpha h_1}{r_1 + \alpha} + \frac{\alpha h_2}{r_2 + \alpha} = \frac{n!}{\alpha^n} \tag{7.4}
\]

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so that
\[
\begin{align*}
    r_1 - \frac{\lambda + \delta}{c} + \frac{\lambda}{c} \frac{\alpha}{r_1 + \alpha} &= 0 \\
    r_2 - \frac{\lambda + \delta}{c} + \frac{\lambda}{c} \frac{\alpha}{r_2 + \alpha} &= 0
\end{align*}
\]

The boundary condition gives
\[
\phi_n(b, b) = 0 \Leftrightarrow h_1 e^{r_1 b} + h_2 e^{r_2 b} = 0 \Leftrightarrow \frac{h_1}{h_2} = -e^{(r_2 - r_1) b}
\]

Dividing equation (7.4) by \(h_2\), leads to
\[
\frac{n!}{\alpha^n h_2} = -e^{(r_2 - r_1) b} \frac{\alpha}{r_1 + \alpha} + \frac{\alpha}{r_2 + \alpha} = -\alpha e^{(r_2 - r_1) b} \frac{(r_2 + \alpha) + \alpha (r_1 + \alpha)}{(r_1 + \alpha)(r_2 + \alpha)}
\]

so that
\[
h_2 = \frac{n!}{\alpha^{n+1}} \frac{(r_1 + \alpha)(r_2 + \alpha)}{(r_1 + \alpha) - (r_2 + \alpha)e^{(r_2 - r_1) b}}
\]

Due to (7.3) we have
\[
(r_1 + \alpha)(r_2 + \alpha) = \alpha^2 + \alpha(r_1 + r_2) + r_1 r_2 = \frac{\alpha \lambda}{c}.
\]

Then
\[
h_1 = -\frac{\lambda n!}{\alpha^nc} e^{(r_2 - r_1)b}
\]

Therefore
\[
\phi_n(v, b) = \frac{\lambda n!}{\alpha^n c} \frac{e^{r_2 b + r_1 v} - e^{r_2 b + r_1 v}}{(r_1 + \alpha) - (r_2 + \alpha)e^{(r_2 - r_1) b}}.
\]

Hence, (4.2) comes in this case
\[
V(u; b) = \frac{\phi_1(b - u, b) + \phi_0(b - u, b)}{1 - \phi_0(b, b)} - \frac{\phi_1(b, b)}{1 - \phi_0(b, b)}
\]

after some simplification. We can show that it coincides with the solution given by Avanzi et al. (2007)

\[
V(u, b) = \frac{\lambda}{\alpha (c r + \delta) e^{r b} - (c s + \delta) e^{s b}}
\]

where \(s\) and \(r\) are the solutions of the equation \(c y^2 + (\lambda + \delta - \alpha c) y - \alpha \delta = 0\).

Let us now consider the computation of \(\chi(u, b)\) and \(\xi(u, b)\). From (6.1) we have

\[
\begin{align*}
    \frac{d}{du} \xi(u, b) &= \frac{\lambda}{c} \left(\lambda \int_u^b \alpha e^{-\alpha(y - u)} e^{\gamma y} dy \right) \frac{d}{du} \\
    \frac{d^2}{du^2} \xi(u, b) &= \alpha \frac{d}{du} \xi(u, b)
\end{align*}
\]
A general solution is of the form
\[ \xi(u, b) = h_1 e^{r_1 u} + h_2 e^{r_2 u} \]
where \( r_1 \) and \( r_2 \) are the roots of the characteristic equation of (7.6)
\[ s^2 + \left( \frac{\lambda}{c} - \alpha \right) s = 0 \iff s \left( s + \left( \frac{\lambda}{c} - \alpha \right) \right) = 0 \]
so that \( r_1 = 0 \) and \( r_2 = \alpha - \lambda/c = -R \). From the boundary condition \( \xi(0, b) = 1 \) we have \( h_1 + h_2 = 1 \). Setting \( h = h_2 \), we have \( \xi(u, b) = (1 - h) + he^{-Ru} \) Replacing it in (7.5) we get
\[ \lambda(1 - h) + \lambda he^{-Ru} - cRhe^{-Ra} = \lambda e^{au} \int_{u}^{b} e^{-ay} (1 - h + he^{-Ry}) dy, \]
equivalent to
\[ h \left( e^{-Ru} \left( \lambda - cR - \frac{\alpha \lambda}{\alpha + R} \right) - \lambda e^{-\alpha(b-u)} + \frac{\alpha \lambda e^{-(a+R)b+au}}{\alpha + R} \right) = -\lambda e^{-\alpha(b-u)} \]
Since \( R = \lambda/c - \alpha \) we finally get
\[ h = \frac{\lambda e^{-\alpha(b-u)}}{\lambda e^{-\alpha(b-u)} - \alpha e^{-(a(b-u)-Rb)}} = \frac{\lambda}{\lambda - \alpha e^{-Rb}} \]
Therefore
\[ \xi(u, b) = \frac{\lambda e^{-Ru} - \alpha e^{-Rb}}{\lambda - \alpha e^{-Rb}} \]
\[ \chi(u, b) = \frac{\lambda - \lambda e^{-Ru}}{\lambda - \alpha e^{-Rb}}. \]
It is much easier to find \( \xi(u, b) \) by using (5.2). We have
\[ \epsilon(s) = \frac{ce(0)}{cs - \lambda + \frac{\alpha \lambda}{\alpha + s}} = \frac{(\alpha + s) e(0)}{s(s - R)} = e(0) \left( \frac{A}{s} + \frac{B}{s - R} \right), \]
using partial fractions method. Then \( A = -\alpha/R \) and \( B = (\alpha + R)/R \). Therefore
\[ \epsilon(u) = \frac{e(0)}{R} \left( (\alpha + R)e^{Ru} - \alpha \right) \]
Since \( \epsilon(b) = 1 \) we get
\[ \epsilon(0) = \frac{R}{(\alpha + R)e^{Rb} - \alpha} \]
giving
\[ \epsilon(u) = \frac{(\alpha + R)e^{Ru} - \alpha}{(\alpha + R)e^{Rb} - \alpha}. \]
Thus,
\[ \xi(u, b) = \epsilon(b - u) = \frac{(\alpha + R)e^{-Ru} - \alpha e^{-Rb}}{(\alpha + R) - \alpha e^{-Rb}} = \frac{\lambda e^{-Ru} - \alpha e^{-Rb}}{\lambda - \alpha e^{-Rb}}. \]
Now we find a solution for the distribution of a single amount of dividend, \( G(u; b; x) \), by using the Laplace transforms dealt in Section 6. For (6.3) and (6.4) we have

\[
\tilde{p}(s, x) = \frac{\alpha + s(1 - e^{-\alpha x})}{s(s + \alpha)}
\]

\[
\tilde{\rho}(s, x) = \frac{\rho(0, x)(\alpha + s) - (\alpha + R)(1 - e^{-\alpha x})}{s(s - R)} = \left( \frac{A_2}{s} + \frac{B_2}{s - R} \right).
\]

using partial fractions method. Then \( A_2 = (1 - e^{-\alpha x}) - [\alpha(\rho(0, x) - 1 + e^{-\alpha x})]/R \) and \( B_2 = (\alpha + R)(\rho(0, x) - 1 + e^{-\alpha x})/R \). Therefore

\[
\rho(u, x) = 1 - e^{-\alpha x} - \frac{\alpha(\rho(0, x) - 1 + e^{-\alpha x})}{R} + \frac{(\alpha + R)(\rho(0, x) - 1 + e^{-\alpha x})e^{Ru}}{R}.
\]

Since \( \rho(b, x) = 0 \), we get

\[
(1 - e^{-\alpha x})R = (\rho(0, x) - 1 + e^{-\alpha x})(\alpha - (\alpha + R)e^{Rb})
\]

\[
\rho(0, x) - 1 + e^{-\alpha x} = \frac{(1 - e^{-\alpha x})R}{\alpha - (\alpha + R)e^{Rb}}.
\]

Hence

\[
\rho(u, x) = 1 - e^{-\alpha x} - \frac{\alpha(1 - e^{-\alpha x})}{\alpha - (\alpha + R)e^{Rb}} + \frac{(\alpha + R)(1 - e^{-\alpha x})e^{Ru}}{\alpha - (\alpha + R)e^{Rb}}
\]

\[
= (1 - e^{-\alpha x}) \left[ 1 + \frac{\alpha - (\alpha + R)e^{Rb}}{(\alpha + R)e^{Rb} - \alpha} \right].
\]

Finally we get, simplifying

\[
G(u, b; x) = \rho(b - u, x) = (1 - e^{-\alpha x}) \frac{\lambda - \lambda e^{-Rb}}{\lambda - \alpha e^{-Rb}}.
\]

Note that we have

\[
\frac{G(u, b; x)}{\chi(u, b)} = 1 - e^{-\alpha x}.
\]

There is a correspondence to the classical model. Due to the memoryless property of the exponential distribution, the conditional distribution of the single dividend amount has the same distribution of the single gains amount, conditional on the upcross of level \( b \) prior to ruin.

References


