SURVIVE A DOWNSWING PHASE
OF THE UNDERWRITING CYCLE

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AGENDA

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1. Introduction: underwriting cycles
due to random surrounding and due to competition

References (economics)

References (modelling)

Long-term variations called “business cycles”, are typically common for the most insurers and have several potential causes.

Understanding the driving forces of the underwriting cycles is a paramount theoretical and important practical problem.

- Cycles attributed to the fluctuations due to random surroundings, to volatile interest rates, or to random up- and down-swings of the risk exposure in the portfolio. Deficiencies are introduced by the exterior ambiguities limited by the so-called scenarios of nature.

- Such fluctuations can not be foreseen and their dynamics is known deficiently since its origin used to be exogenous with respect to the insurance industry.
- It causes inevitable errors in the rate making, and irregularly cyclic underwriting process ensues.

- Adaptive control strategies fighting back cycles due to scenarios of nature were proposed in the multiperiod framework

\[
\begin{align*}
\mathbf{w}_0 & \xrightarrow{\gamma_0} \mathbf{u}_0 \xrightarrow{\pi_1} \mathbf{w}_1 \cdots \xrightarrow{\pi_{k-1}} \mathbf{w}_{k-1} \xrightarrow{\gamma_{k-1}} \mathbf{u}_{k-1} \xrightarrow{\pi_k} \mathbf{w}_k \cdots .
\end{align*}
\]
Cycles attributed to the strategies of aggressive insurers seeking for greater market shares, and by the consequent industry response.

- At the first stage, the response lies in concerted reduction of the rates, sometimes below the real costs of insurance.
- This makes some companies ruined, and agrees with the observation that insurance cycles are correlated with clustered insolvencies.
- For instance (see [Feldblum 2007] with reference on Best’s Insolvency Study [Best’s 1991]), US industry-wide combined ratios peaked at 109% in 1975 and 117% in 1984. The insurance failure rate, or the ratio of insolvencies to total companies, peaked at 1.0% in 1975 and 1.4% in 1985.
  - Insolvencies appear a driving force behind the competition–originated cycles.
  - After elimination of the exceedingly aggressive and unwise agents, or just weaker carriers, the prices increase uniformly over the industry.
- The upswing phase of the cycle follows.
2. Price in the years of soft and hard market and portfolio size functions

- The insurance price $P^M$ prevailing in the market is called market price, or market price factor.
- The year of soft market occurs for a particular insurer when the market price factor is below the averaged losses $E_Y$, i.e. as $E_Y > P^M$. The year of hard market for a particular insurer occurs otherwise, i.e. as $E_Y < P^M$.
- The insurer applies maintaining market share control if $P = P^M$. The insurer applies conserving capital control if $P = E_Y$. The insurer applies mixed control, if $P^M < P < E_Y$, as $P^M < E_Y$ (soft market), and $E_Y < P < P^M$, as $E_Y < P^M$ (hard market).
- Without lack of generality, the set $\mathcal{P}$ of price controls introduced above may be written as

$$P_\gamma = \gamma P^M + (1 - \gamma)E_Y, \quad \gamma \in [0, 1],$$

with $P_1 = P^M$ and $P_0 = E_Y$.

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1 In the case of soft market (i.e., $E_Y > P^M$) prices $P$ below $P^M$ cause excessive danger of ruin, while prices $P$ above $E_Y$ yield excessively high rate of elimination of portfolio. Both are claimed unreasonable. The similar arguments are true in the case of hard market.
• For $\gamma \in [0, 1]$ and for the insurer’s price $P_\gamma \in \mathcal{P}$, the value
\[
d_\gamma = P_\gamma - P^M = (1 - \gamma)(EY - P^M)
\]
is called insurer’s price deficiency with respect to the market price $P^M$.

• For $\gamma \in [0, 1]$ and for the prices $P_\gamma \in \mathcal{P}$ with deficiency $d_\gamma = P_\gamma - P^M$, introduce the family
\[
\mathcal{L} = \{\lambda_{d_\gamma}(s), \ 0 \leq s \leq t\}
\]
of continuous non-negative functions of time, called portfolio size functions.

• Assume that $\lambda_{d_\gamma}(0) = \lambda$. The value $\lambda$ is referred to as the initial portfolio size.

• In the case of $d_\gamma = 0$ (neutral market or maintaining market share control, $P_1 = P^M$) set $\lambda_{d_\gamma}(s) \equiv \lambda$, $0 \leq s \leq t$.

• When $d_\gamma > 0$ (soft market and $\gamma \in [0, 1)$), the portfolio size functions $\lambda_{d_\gamma}(s)$ must be monotone decreasing in $s$ and $\lambda_{d_\gamma_1}(s) < \lambda_{d_\gamma_2}(s)$ for all $0 \leq s \leq t$, as $d_\gamma_1 > d_\gamma_2$.

• When $d_\gamma < 0$ (hard market and $\gamma \in [0, 1)$), the portfolio size functions $\lambda_{d_\gamma}(s)$ must be monotone increasing in $s$ and $\lambda_{d_\gamma_1}(s) < \lambda_{d_\gamma_2}(s)$ for all $0 \leq s \leq t$, as $d_\gamma_1 > d_\gamma_2$. 
3. **Portfolio size models in the years of soft and hard market**

- Selecting \( L \), wise is to address to practice.
- [Subramanian 1998], p. 39:
  
  “Surveys of policyholders have consistently demonstrated some reluctance to switch insurers. In a survey of 2462 policyholders by Cummins et al. [Cummins et al. 1974], 54% of respondents confessed never to have shopped around for auto insurance prices. To the question “Which is the most important factor in your decision to buy insurance?”, 40% responded the company, 29% the agent, and only 27% the premium. A similar survey of 2004 Germans (see [Schlesinger et al. 1993]) indicated that, despite the fact that 67% of those responding knew that considerable price differences exist between automobile insurers, only 35% chose their carrier on the basis of their favorable premium. Therefore, we will assume that, given the opportunity to switch for a reduced premium, one-third of the policyholders will do so”.
Following that remark, assume that in the year of hard market, i.e. as \( d_\gamma > 0 \),
\[
\lambda_{d_\gamma}(s) = \lambda \cdot r_{d_\gamma}(s), \quad 0 \leq s \leq t, \quad \gamma \in [0, 1],
\]
where

- \( 0 \leq r_{d_\gamma}(s) \leq 1 \) is the rate of those who remained in the portfolio by time \( s \leq t \),
- \( m_{d_\gamma}(s) = 1 - r_{d_\gamma}(s) \) is the complementary rate function by time \( s \leq t \),
- \( m_{d_\gamma} = m_{d_\gamma}(+\infty) \) is the ultimate rate of migrants (which does not exceed one-third).

For example, introduce the rate function \( r_{d_\gamma}(s), 0 \leq s \leq t \),

- with exponential outgo of migrants,
  \[
  r_{d_\gamma}(s) = (1 - m_{d_\gamma}) + m_{d_\gamma} \cdot e^{-s} = 1 - m_{d_\gamma} \cdot (1 - e^{-s}),
  \]
  which yields
  \[
  \Lambda_{d_\gamma}(t) = \int_0^t \lambda_{d_\gamma}(s) ds = \lambda \cdot t \cdot (1 - m_{d_\gamma}) + \lambda \cdot m_{d_\gamma} \cdot (1 - e^{-t}).
  \]
In most cases the exponential outgo is unrealistically quick. Of more interest may be:

- the **power rate function**

\[
 r_{d\gamma}(s) = \left(1 - m_{d\gamma}\right) + m_{d\gamma} \cdot (1 + s)^{-k} = 1 - m_{d\gamma} \cdot \frac{\left(1 - (1 + s)^{-k}\right)}{k > 0},
\]

which yields

\[
 \Lambda_{d\gamma}(t) = \int_0^t \lambda_{d\gamma}(s) ds = \begin{cases} 
 \lambda t \left(1 - m_{d\gamma}\right) + \lambda m_{d\gamma} \left(1 - (t + 1)^{-k+1}\right)/(k - 1), & k \neq 1, \\
 \lambda t \left(1 - m_{d\gamma}\right) + \lambda m_{d\gamma} \ln(1 + t), & k = 1.
\end{cases}
\]

As \( k < 1 \), the migrating part in the portfolio is slow enough and still influences \( \Lambda_{d\gamma}(t) \) considerably.

- The concept of the set \( \mathcal{L} \) of portfolio size functions has to be further developed. For example, it may be sensible to allow dependence of the portfolio size functions on the initial risk reserve\(^2\).

\[^2\text{It is arguable that the outgo of insureds becomes more intensive from e.g., a smaller company, for not to mention such an abstract term as the initial risk reserve. That may be checked by means of a survey of policyholders.}\]
4. Annual risk reserve process and annual probabilities of ruin

Assume that fixed are the families $\mathcal{P}$ of the price controls and $\mathcal{L}$ of the portfolio size functions.

- For $P_\gamma \in \mathcal{P}$ with deficiency $d_\gamma$ and for the corresponding portfolio size function $\lambda_{d_\gamma} \in \mathcal{L}$, assume that the claim number process is a non-homogeneous Poisson process $\nu_\gamma(s)$, $0 \leq s \leq t$, with the yield (intensity) function
  \[ \Lambda_{d_\gamma}(s) = \int_0^s \lambda_{d_\gamma}(z)dz, \quad 0 \leq s \leq t. \]

- Assume that $Y_i$, $i = 1, 2, \ldots$, are i.i.d. and independent on the claim number process $\nu_\gamma(s)$, $0 \leq s \leq t$. The claim outcome process associated with the portfolio size function $\lambda_{d_\gamma} \in \mathcal{L}$ is the compound non-homogeneous Poisson process
  \[ \sum_{i=1}^{\nu_\gamma(s)} Y_i, \]
  as $\nu_\gamma(s) > 0$, or zero, as $\nu_\gamma(s) = 0$, $0 \leq s \leq t$. 
• The premium income process associated with the portfolio size function \( \lambda_{d\gamma} \in \mathcal{L} \) and with the premium factor \( P_\gamma \) is the non-random process
\[
P_\gamma \Lambda_{d\gamma}(s) = P_\gamma \int_0^s \lambda_{d\gamma}(z)dz, \quad 0 \leq s \leq t.
\]

• The risk reserve process generated by the premium income process and claim outcome processes is the random process
\[
R_{u,\gamma}(s) = u + P_\gamma \Lambda_{d\gamma}(s) - \sum_{i=1}^{\nu_\gamma(s)} Y_i,
\]
as \( \nu_\gamma(s) > 0 \), or \( u + P_\gamma \Lambda_{d\gamma}(s) \), as \( \nu_\gamma(s) = 0 \), \( 0 \leq s \leq t \). The value \( u > 0 \) is called the initial risk reserve.

**Lemma 1.** For a homogeneous Poisson process \( N_\lambda(s), 0 \leq s \leq t \), with intensity \( \lambda > 0 \),
\[
R_{u,\gamma}(s) = \hat{R}_{u,\gamma}(\tau(s)), \quad 0 \leq s \leq t,
\]
where \( \tau(s) = \Lambda_{d\gamma}(s)/\lambda, 0 \leq s \leq t \), is the operational time, and where
\[
\hat{R}_{u,\gamma}(s) = u + [P_\gamma \lambda]s - \sum_{i=1}^{N_\lambda(s)} Y_i, \quad 0 \leq s \leq \Lambda_{d\gamma}(t)/\lambda.
\]
The probability
\[
P\{ \inf_{0 \leq s \leq t} R_{u,\gamma}(s) < 0 \}
\]
is called **annual probability of ruin**, or **probability of ruin** within time \( t \).

**Theorem 1.** *In the year of soft market* (i.e., as \( EY > PM \)) *the probability*
\[
P\{ \inf_{0 \leq s \leq t} R_{u,\gamma}(s) < 0 \}
\]
is monotone increasing, as \( \gamma \) increases.

• Since \( \inf_{0 \leq s \leq t} R_{u,\gamma}(s) = \inf_{0 \leq s \leq \Lambda_{d\gamma}(t)/\lambda} \hat{R}_{u,\gamma}(s) \), one has
\[
P\{ \inf_{0 \leq s \leq t} R_{u,\gamma}(s) < 0 \} = P\{ \inf_{0 \leq s \leq \Lambda_{d\gamma}(t)/\lambda} \hat{R}_{u,\gamma}(s) < 0 \}
= P\{ \inf_{0 \leq s \leq \Lambda_{d\gamma}(t)/\lambda} \left( u + \underbrace{\left[ EY - \gamma(\underbrace{EY - PM}_{P_{\gamma}}) \right]}_{>0} \lambda s - \sum_{i=1}^{N_{\lambda}(s)} Y_i \right) < 0 \}.
\]

• In the year of soft market \( P_{\gamma} \) is monotone decreasing, as \( \gamma \) increases, from \( P_0 = EY \) to \( P_1 = PM \), with \( P_0 > P_1 \), and \( \Lambda_{d\gamma}(t) \) is monotone increasing, as \( \gamma \) increases. Both factors contribute to a monotone growth of \( P\{ \inf_{0 \leq s \leq t} R_{u,\gamma}(s) < 0 \} \), as \( \gamma \) increases.
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Theorem 2. Assume that \( Y_i, \ i = 1, 2, \ldots \), are i.i.d. exponential with intensity \( \mu \) (i.e., \( 1/\mu = EY \)) and denote by \( I_n(z) \) the modified Bessel function of \( n \)th order, \( z \) real and \( n = 0, 1, 2, \ldots \) In that model

\[
P\{ \inf_{0 \leq s \leq t} R_{u,\gamma}(s) < 0 \} = e^{-u\mu} \sum_{n \geq 0} \frac{(u\mu)^n}{n!} (P_\gamma \mu)^{-(n+1)/2} \times \int_0^{\Lambda_{d}(t)} \frac{n + 1}{x} e^{-(1+P_\gamma \mu)x} I_{n+1}(2x \sqrt{P_\gamma \mu}) \, dx.
\]

The alternative expression is

\[
P\{ \inf_{0 \leq s \leq t} R_{u,\gamma}(s) < 0 \} = -\frac{1}{\pi} \int_0^\pi f_t(x, u) \, dx + \begin{cases} (1/P_\gamma \mu) \exp\{-u\mu(1 - 1/P_\gamma \mu)\}, & P_\gamma \mu > 1, \\ 1, & P_\gamma \mu \leq 1, \end{cases}
\]

where

\[
f_t(x, u) = (P_\gamma \mu)^{-1} \left( 1 + (P_\gamma \mu)^{-1} - 2(P_\gamma \mu)^{-1/2} \cos x \right)^{-1} \times \exp \left\{ u\mu \left( (P_\gamma \mu)^{-1/2} \cos x - 1 \right) - \Lambda_{d}(t) P_\gamma \mu \left( 1 + (P_\gamma \mu)^{-1} - 2(P_\gamma \mu)^{-1/2} \cos x \right) \right\} \times \left[ \cos \left( u\mu(P_\gamma \mu)^{-1/2} \sin x \right) - \cos \left( u\mu(P_\gamma \mu)^{-1/2} \sin x + 2x \right) \right].
\]
5. Admissible risk reserve and premium controls

- In the year of soft market, admissible are those controls which do not compel (A) the annual probability of ruin be larger than a prescribed value $\alpha \in (0, 1)$, and (B) the year-end portfolio size be less than a prescribed lower limit $L$.

$$\mathbf{w}_0 \xrightarrow{\gamma_0} \mathbf{u}_0 \xrightarrow{\pi_1} \mathbf{w}_1 \cdots \xrightarrow{\pi_{k-1}} \mathbf{w}_{k-1} \xrightarrow{\gamma_{k-1}} \mathbf{u}_{k-1} \xrightarrow{\pi_k} \mathbf{w}_k \cdots$$

1st year, $P^M_1, \alpha_1$

$k$th year, $P^M_k, \alpha_k$

- Admissible risk reserve (annual) controls

- Admissible premium (annual) controls, the solvency point of view (A)

**Theorem 3.** For sufficiently small $\alpha \in (0, 1)$, for the initial risk reserve $u$ and for the family $L$, in the year of soft market allowed are the price controls $P_\gamma \in \mathcal{P}$, $\gamma \in [0, \gamma_{t,u|L}(\alpha)]$, where $\gamma_{t,u|L}(\alpha)$ is the unique solution of the equation

$$P\{\inf_{0 \leq s \leq t} R_{u,\gamma}(s) < 0\} = \alpha,$$

as $P\{\inf_{0 \leq s \leq t} R_{u,1}(s) < 0\} \geq \alpha$, and $\gamma_{t,u|L}(\alpha) = 1$, as $P\{\inf_{0 \leq s \leq t} R_{u,1}(s) < 0\} < \alpha.$
• Put $\gamma_{t,\alpha}$ for $\gamma_{t,u}|L(\alpha)$, set $P\{\inf_{0 \leq s \leq t} R_{u,\gamma}(s) < 0\} = \psi_t(\gamma)$ and note that in the year of soft market $\psi_+^{\infty}(\gamma) = 1$.

**Theorem 4.** For $\tau_\gamma = -\gamma(EY - P^M)/EY, \gamma \in (0, 1]$, assume that $\tau_\gamma < 0$. Then

$$\sup_{t \in \mathbb{R}^+} \left| \psi_t(\gamma) - \Phi_{\{0,1\}}((\Lambda_{d,\gamma}(t) - M_{\tau,\gamma}u\mu)/(S_{\tau,\gamma}(u\mu)^{1/2})) \right| = \mathcal{O}(u^{-1/2}), \quad \text{as} \quad u \to \infty,$$

where $M_{\tau,\gamma} = -1/\tau_\gamma, S_{\tau,\gamma}^2 = -2/\tau_\gamma^3$.

• Introduce $\phi_t(\gamma) = \psi_+^{\infty}(\gamma) - \psi_t(\gamma) = 1 - \psi_t(\gamma)$ the probability of ultimate ruin after time $t$, and rewrite $\phi_t(\gamma_{t,\alpha}) = 1 - \psi_t(\gamma_{t,\alpha}) = 1 - \alpha$, which yields

$$\gamma_{t,\alpha} = \phi_t^{-1}(1 - \alpha).$$

**Theorem 5.** For $\tau_\gamma = -\gamma(EY - P^M)/EY, \gamma \in (0, 1]$, set $a_\gamma = (1 - \sqrt{1 + \tau_\gamma})^2$ and $b_\gamma = 1/\sqrt{1 + \tau_\gamma}$. In the framework of Theorem 2, one has $\tau_\gamma < 0$ and

$$\phi_t(\gamma) = \frac{b_{\gamma}^{3/2}(b_{\tau}u\mu + 1)}{2\sqrt{\pi}a_\gamma(\Lambda_{d,\gamma}(t))^{3/2}} e^{-u^2(1-b_\gamma)} e^{-a_\gamma\Lambda_{d,\gamma}(t)} \exp \left\{ - \frac{b_{\gamma}^3(u\mu)^2}{4\Lambda_{d,\gamma}(t)} \right\} \left\{ 1 + \mathcal{O}(\Lambda_{d,\gamma}^{-1/2}(t)) \right\}$$

for $u \leq \mathcal{O}(\Lambda_{d,\gamma}^{1/2}(t))$, as $t \to \infty$.

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3 Under rather general regularity conditions. The result is suitable to apply for $u \geq \mathcal{O}(\Lambda_{d,\gamma}^{1/2}(t))$, as $t \to \infty$. 
• Admissible premium (annual) controls, the portfolio size point of view (B)

**Theorem 6.** For sufficiently small $\alpha \in (0, 1)$, for the initial risk reserve $u$ and for the family $\mathcal{L}$, in the year of soft market allowed are the price controls $P_\gamma \in \mathcal{P}$, $\gamma \in [\gamma_L, 1]$, where

$$\gamma_L = \inf\{\gamma \in [0, 1] : \lambda_{d\gamma}(t) = L\} > 0,$$

as $\lambda_{d0}(t) < L$, and $\gamma_L = 0$, as $\lambda_{d0}(t) \geq L$.

• Theorems 3–6 yield the set of the annual price controls allowed both from (A) solvency and (B) portfolio size points of view. This set is

$$P_\gamma \in \mathcal{P}, \quad \gamma \in [0, \gamma_{t,u|\mathcal{L}}(\alpha)] \cap [\gamma_L, 1] = [\gamma_L, \gamma_{t,u|\mathcal{L}}(\alpha)].$$
6. Conclusion: a strategy beating the downswing phase of the cycle

For the family $\mathcal{L}$ and for a sequence $u, w_1, \ldots, w_{k-1}$ of the initial risk reserve values, as the $(i - 1)$st year-end risk reserve is assumed equal to the initial risk reserve in $i$th year $(i = 2, \ldots, k)$, the adaptive control strategy beating the downswing phase of the insurance cycle with the period $k$, generated by the market prices $P_1^M > \cdots > P_k^M > 0$, all below the average risk $E_Y$, is

\[
P_1(u) = P_\gamma, \quad \gamma \in [\gamma_L, \gamma_t, u|\mathcal{L}(\alpha_1)], \quad \text{if} \quad [\gamma_L, \gamma_t, u|\mathcal{L}(\alpha_1)] \neq \emptyset,
\]

\[
P_2(w_1) = P_\gamma, \quad \gamma \in [\gamma_L, \gamma_t, w_1|\mathcal{L}(\alpha_2)], \quad \text{if} \quad [\gamma_L, \gamma_t, w_1|\mathcal{L}(\alpha_2)] \neq \emptyset,
\]

\[
\cdots
\]

\[
P_k(w_{k-1}) = P_\gamma, \quad \gamma \in [\gamma_L, \gamma_t, w_{k-1}|\mathcal{L}(\alpha_k)], \quad \text{if} \quad [\gamma_L, \gamma_t, w_{k-1}|\mathcal{L}(\alpha_k)] \neq \emptyset.
\]

Recall that $\alpha_1, \ldots, \alpha_k$ in

\[
\begin{align*}
    w_0 & \xrightarrow{\gamma_0} u_0 \xrightarrow{\pi_1} w_1 \cdots \xrightarrow{\pi_{k-1}} w_{k-1} \xrightarrow{\gamma_{k-1}} u_{k-1} \xrightarrow{\pi_k} w_k \cdots,
\end{align*}
\]

are the allowed levels or ruin within the downswing phase of the underwriting cycle.