Modelling and simulation of dependence structures in nonlife insurance with Bernstein copulas

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1. Introduction

- Classical copula concepts:
  - elliptical (Gauß, *t*, …)
  - Archimedian (Gumbel, Clayton, Frank, …)

- **Approximation theory:** Bernstein polynomials in one and more variables or in one and more dimensions (Bézier curves and surfaces) $\Rightarrow$ **Bernstein copulas:**

  - Bernstein copulas allow for a very flexible, non-parametric and essentially non-symmetric description of dependence structures also in higher dimensions

  - Bernstein copulas approximate any given copula arbitrarily well

  - Bernstein copula densities are given in an explicit form and can hence be easily used for Monte Carlo simulation studies.

2. Some simple mathematical facts on Bernstein polynomials and copulas

Lemma. Let $B(m,k,z)=\binom{m}{k}z^k(1-z)^{m-k},\ 0\le z\le 1,\ k=0,\cdots,m\in\mathbb{N}.$ Then we have

$$\int_0^1 m\,B(m-1,k,z)\,dz=1 \quad \text{for } k=0,\cdots,m-1.$$

Further,

$$\frac{d}{dz}B(m,k,z)=m\left[B(m-1,k-1,z)-B(m-1,k,z)\right] \quad \text{for } k=0,\cdots,m$$

with the convention $B(m-1,-1,z)=B(m-1,m,z)=0.$

Theorem. For $d \in \mathbb{N}$ let $\mathbf{U} = (U_1, \cdots, U_d)$ be a random vector whose marginal component U_i follows a **discrete uniform distribution** over $T_i := \{0, 1, \cdots, m_i - 1\}$ with $m_i \in \mathbb{N}$, $i = 1, \cdots, d$. Let further denote

$$p(k_1, \cdots, k_d) := P\left(\bigcap_{i=1}^{d} \{U_i = k_i\}\right) \text{ for all } (k_1, \cdots, k_d) \in \bigtimes_{i=1}^{d} T_i.$$

Then

$$c(u_1, \cdots, u_d) := \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_d=0}^{m_d-1} p(k_1, \cdots, k_d) \prod_{i=1}^{d} m_i B(m_i - 1, k_i, u_i), \ (u_1, \cdots, u_d) \in [0,1]^d$$

defines the density of a d-dimensional copula, called *Bernstein copula*. We call c the Bernstein copula density induced by $\mathbf{U}$.

By integration, we obtain the Bernstein copula C induced by $\mathbf{U}$ as

$$C(x_1,\cdots,x_d) := \int_0^{x_d}\cdots\int_0^{x_1} c(u_1,\cdots,u_d)\,du_1\cdots du_d = \sum_{k_1=0}^{m_1}\cdots\sum_{k_d=0}^{m_d} P\left(\bigcap_{i=1}^{d}\{U_i < k_i\}\right)\prod_{i=1}^{d} B(m_i,k_i,x_i),$$

for $(x_1,\cdots,x_d)\in[0,1]^d$.

Remark: if $\mathbf{V}=(V_1,\cdots,V_d)$ is a random vector with joint Bernstein copula density c then also any partial random vector $(V_{i_1},\cdots,V_{i_n})$ with $n<d$ and $1\le i_1<\cdots<i_n\le d$ possesses a Bernstein copula density $c^{[i_1,\cdots,i_n]}$ given by

$$c^{[i_1,\cdots,i_n]}(u_{i_1},\cdots u_{i_n}) = \sum_{k_{i_1}=0}^{m_{i_1}-1}\cdots\sum_{k_{i_n}=0}^{m_{i_n}-1} P\left(\bigcap_{\ell=1}^{n}\{U_{i_\ell}=k_{i_\ell}\}\right)\prod_{\ell=1}^{n} m_{i_\ell} B(m_{i_\ell}-1,k_{i_\ell},u_{i_\ell}),\ (u_{1_1},\cdots,u_{i_n})\in[0,1]^n.$$

Definition. Under the assumptions of the above theorem define the intervals $I_{k_1,\cdots,k_d} := \bigtimes_{j=1}^{d} \left(\frac{k_j}{m_j}, \frac{k_j+1}{m_j} \right]$ for all possible choices $(k_1,\cdots,k_d) \in \bigtimes_{i=1}^{d} T_i$. Then the function

$$c^* := \prod_{i=1}^{d} m_i \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_d=0}^{m_d-1} p(k_1,\cdots,k_d)\, \mathbb{1}_{I_{k_1,\cdots,k_d}}$$

is the density of a d-dimensional copula, called *grid-type* or *checkerboard copula* induced by $\mathbf{U}$. Here $\mathbb{1}_A$ denotes the indicator random variable of the set A, as usual.

Interpretation: $\mathbf{W} = (W_1,\cdots,W_d)$ follows a grid-type copula iff

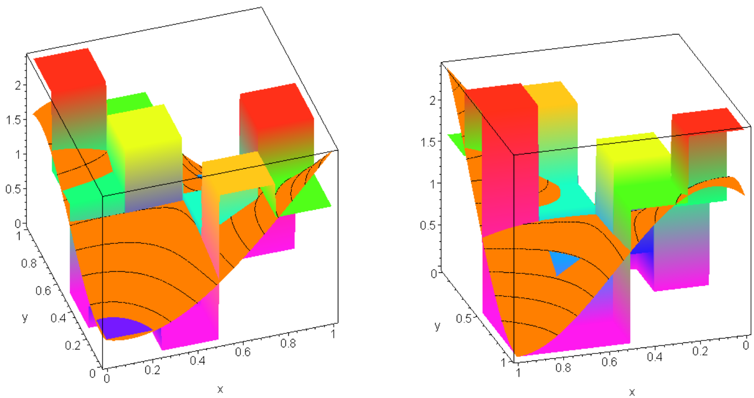
$$P^{\mathbf{W}}\left(\bullet \mid \mathbf{W} \in I_{k_1,\cdots,k_d}\right) = \mathcal{U}\left(I_{k_1,\cdots,k_d}\right) \text{ for all } (k_1,\cdots,k_d) \in \bigtimes_{i=1}^{d} T_i,$$

where $\mathcal{U}(\bullet)$ denotes the continuous uniform distribution over a Borel set with positive Lebesgue measure.

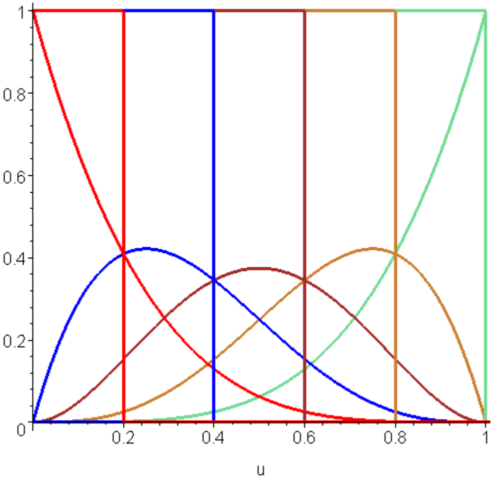
Hence the Bernstein copula induced by **U** can be regarded as a naturally smoothed version of the grid-type copula induced by **U**, replacing the indicator functions

$$\mathbb{1}_{I_{k_1,\cdots,k_d}}(u_1,\cdots,u_d)=\prod_{i=1}^{d}\mathbb{1}_{\left[\frac{k_i}{m_i},\frac{k_i+1}{m_i}\right]}(u_i) \text{ by the polynomials}$$

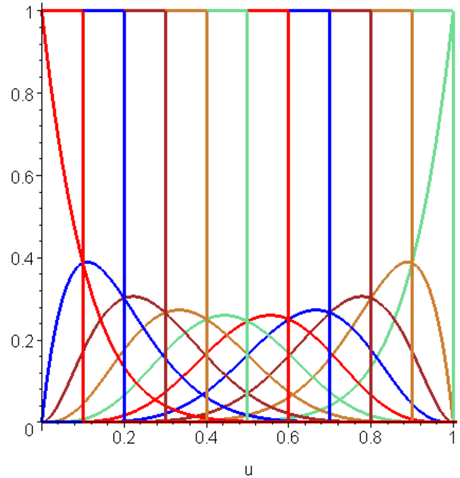
$$\prod_{i=1}^{d}B(m_i-1,k_i,u_i),\ (u_1,\cdots,u_d)\in[0,1]^d.$$



Example. The following graphs show the smoothing effect in case $d = 1$.



$m = 5$



$m = 10$

Natural generalizations of Bernstein and grid-type copulas are obtained if we look at suitable *partitions of unity*, i.e. families of non-negative functions $\{\phi(m,k,\bullet)|0 \le k \le m-1,\ m \in \mathbb{N}\}$ defined on the unit interval $[0,1]$ with the following properties:

- $\int_0^1 \phi(m,k,u)\,du = \frac{1}{m}$ for $k = 0,\cdots,m-1$
- $\sum_{k=0}^{m-1} \phi(m,k,\bullet) = 1$ for $m \in \mathbb{N}$.

In this case, a *d*-dimensional copula density c^{ϕ} induced by **U** is given by

$$c^{\phi}(u_1,\cdots,u_d) := \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_d=0}^{m_d-1} P\left(\bigcap_{i=1}^{d}\{U_i = k_i\}\right) \prod_{i=1}^{d} m_i \phi\,(m_i,k_i,u_i),\ (u_1,\cdots,u_d) \in [0,1]^d .$$

The copula itself is accordingly given by

$$C^{\phi}\left(u_1,\cdots,u_d\right):=\sum_{k_1=0}^{m_1-1}\cdots\sum_{k_d=0}^{m_d-1}P\left(\bigcap_{i=1}^{d}\left\{U_i<k_i\right\}\right)\prod_{i=1}^{d}\phi\ \left(m_i,k_i,u_i\right),\ \left(u_1,\cdots,u_d\right)\in\left[0,1\right]^d.$$

Note that

$$\phi(m,k,u)=B(m-1,k,u)=\binom{m-1}{k}u^k(1-u)^{m-1-k}$$

in case of Bernstein copulas and

$$\phi(m,k,u)=\mathbb{1}_{\left(\frac{k}{m},\frac{k+1}{m}\right]}(u)$$

for $0\leq k\leq m-1,\ m\in\mathbb{N}$ in case of grid-type copulas.

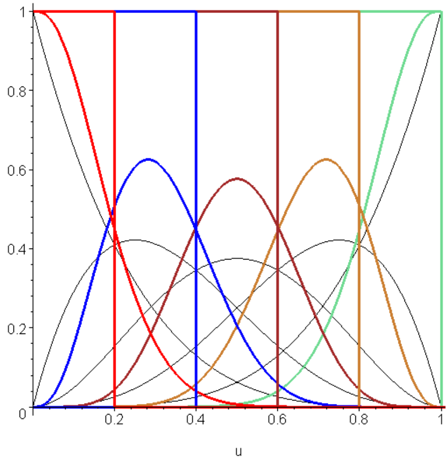
Note further that any such family of functions $\{\phi(m,k,\bullet)|0 \le k \le m-1,\ m \in \mathbb{N}\}$ induces immediately a new family $\{\phi_K(m,k,\bullet)|0 \le k \le m-1,\ m \in \mathbb{N}\}$ for arbitrary, but fixed $K \in \mathbb{N}$ with similar properties via

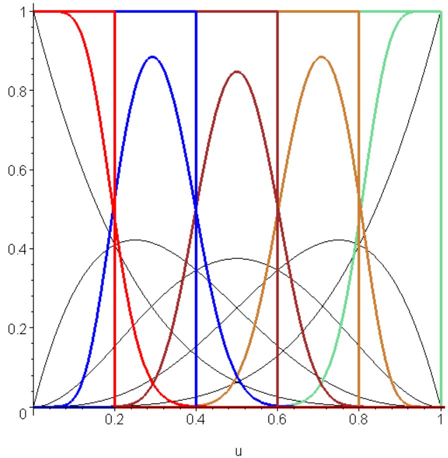
$$\phi_K(m,k,\bullet) := \sum_{j=0}^{K-1} \phi(K \cdot m, K \cdot k + j, \bullet) \quad \text{for } k = 0,\cdots,m-1$$

since obviously

- $\int_0^1 \phi_K(m,k,u)\,du = \sum_{j=0}^{K-1} \int_0^1 \phi(K \cdot m, K \cdot k + j, u)\,du = \sum_{j=0}^{K-1} \frac{1}{K \cdot m} = \frac{1}{m},\ k = 0,\cdots,m-1$
- $\sum_{k=0}^{m-1} \phi_K(m,k,\bullet) = \sum_{j=0}^{K-1} \sum_{k=0}^{m-1} \phi(K \cdot m, K \cdot k + j, \bullet) = \sum_{i=0}^{K \cdot m} \phi(K \cdot m, i, \bullet) = 1,\ m \in \mathbb{N}.$

For Bernstein copulas, this generalization has a direct impact on the smoothing effect pointed out in the above example. The following two graphs show this effect for $K = 3$ and $K = 10$. The case $K = 1$ is shown as a thin black line, for comparison.





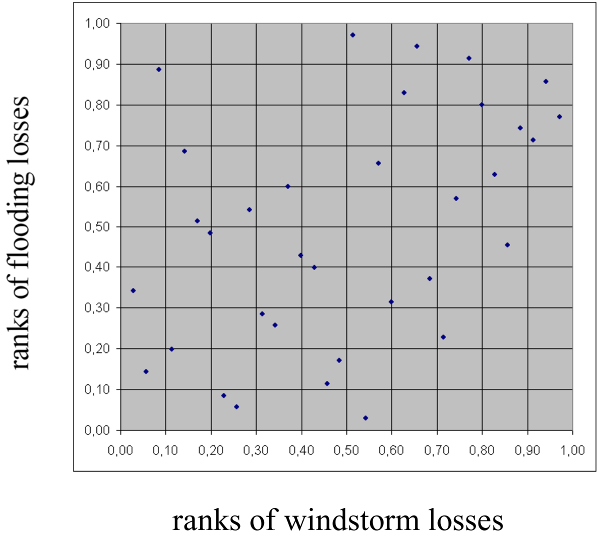
$K = 3$ $m = 5$ $K = 10$

3. Fitting empirical data to grid-type and Bernstein copulas

- In this section: case $d = 2$, for simplicity. However, the method proposed here works accordingly in any dimension d.

- Example data set: a 34-year time series of (economically adjusted) windstorm and flooding losses

One possible way to extract the dependence structure from the data is the *empirical copula scatterplot*, which is a plot of the joint relative ranks of the data. The following figure shows such a plot for a series of $n = 34$ observation years.



empirical copula scatter plot

- Fit these data to a grid-type copula with a given grid resolution, say $m_1 = m_2 = m = 10$, by counting the relative frequency of the data points in each of the $m_1 \times m_2 = 100$ target cells $\Rightarrow$ contingency table $\left[a_{ij}\right]$ (matrix notation: i = row index, j = column index; rounded to 3 decimal places).

upper cell boundary	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0	sum
1,0	0,000	0,000	0,000	0,000	0,000	0,029	0,029	0,029	0,000	0,000	0,009
0,9	0,029	0,000	0,000	0,000	0,000	0,000	0,029	0,000	0,000	0,029	0,009
0,8	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,029	0,029	0,059	0,012
0,7	0,000	0,029	0,000	0,000	0,000	0,029	0,000	0,000	0,029	0,000	0,009
0,6	0,000	0,029	0,029	0,029	0,000	0,000	0,000	0,029	0,000	0,000	0,012
0,5	0,000	0,029	0,000	0,029	0,000	0,000	0,000	0,000	0,029	0,000	0,009
0,4	0,029	0,000	0,000	0,000	0,029	0,029	0,029	0,000	0,000	0,000	0,012
0,3	0,000	0,000	0,000	0,059	0,000	0,000	0,000	0,029	0,000	0,000	0,009
0,2	0,029	0,029	0,000	0,000	0,059	0,000	0,000	0,000	0,000	0,000	0,012
0,1	0,000	0,000	0,059	0,000	0,000	0,029	0,000	0,000	0,000	0,000	0,009
sum	0,009	0,012	0,009	0,012	0,009	0,012	0,009	0,012	0,009	0,009	

- observed marginal sums are not equal to $\frac{1}{m} = \frac{1}{10}$ $\Rightarrow$ optimization problem:

$$\min! \sum_{i=1}^{m} \sum_{j=1}^{m} \left(x_{ij} - a_{ij} \right)^2 \text{ subject to}$$

$$\sum_{i=1}^{m} x_{ik} = \sum_{j=1}^{m} x_{\ell j} = \frac{1}{m} = \frac{1}{10} \text{ and } x_{\ell,k} \geq 0 \text{ for } k, \ell = 1, \cdots, m$$

The explicit solution of such a problem is in general not straightforward to find, although there exists a solution due to the *Karush-Kuhn-Tucker theorem* from optimization theory. Using a suitable software package like *octave* (a public domain computer algebra system), we obtain the following solution (rounded to 3 decimal places); see the code listing in the Appendix of the paper.

upper cell boundary	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0	sum
1,0	0,003	0,000	0,002	0,000	0,003	0,027	0,032	0,027	0,003	0,003	0,1
0,9	0,032	0,000	0,001	0,000	0,002	0,000	0,031	0,000	0,002	0,031	0,1
0,8	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,020	0,025	0,055	0,1
0,7	0,003	0,027	0,002	0,000	0,003	0,027	0,003	0,000	0,032	0,003	0,1
0,6	0,000	0,025	0,029	0,021	0,000	0,000	0,000	0,025	0,000	0,000	0,1
0,5	0,003	0,028	0,002	0,025	0,003	0,000	0,003	0,000	0,032	0,003	0,1
0,4	0,027	0,000	0,000	0,000	0,027	0,021	0,026	0,000	0,000	0,000	0,1
0,3	0,003	0,000	0,002	0,054	0,003	0,000	0,003	0,028	0,003	0,003	0,1
0,2	0,025	0,020	0,000	0,000	0,055	0,000	0,000	0,000	0,000	0,000	0,1
0,1	0,003	0,000	0,061	0,000	0,003	0,026	0,002	0,000	0,002	0,002	0,1
sum	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	

optimal resulting contingency table

A more pragmatic way to find at least a good **suboptimal solution** that can be easily implemented e.g. in spreadsheets is as follows. Consider the above optimization problem without the non-negativity conditions first. The equivalent Lagrange problem (which leads to a system of linear equations) is easy to solve and gives the (general) solution

$$x_{ij} = a_{ij} - \frac{a_{\bullet j}}{m} - \frac{a_{i\bullet}}{m} + \frac{2}{m^2} \quad \text{for} \quad i, j = 1, \cdots, m,$$

where the index $\bullet$ means summation, as usual. For the data set above, we thus obtain

upper cell boundary	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0	sum
1,0	0,002	**-0,001**	0,002	**-0,001**	0,002	0,029	0,032	0,029	0,002	0,002	0,1
0,9	0,032	**-0,001**	0,002	**-0,001**	0,002	**-0,001**	0,032	**-0,001**	0,002	0,032	0,1
0,8	**-0,001**	**-0,004**	**-0,001**	**-0,004**	**-0,001**	**-0,004**	**-0,001**	0,026	0,029	0,058	0,1
0,7	0,002	0,029	0,002	**-0,001**	0,002	0,029	0,002	**-0,001**	0,032	0,002	0,1
0,6	**-0,001**	0,026	0,029	0,026	**-0,001**	**-0,004**	**-0,001**	0,026	**-0,001**	**-0,001**	0,1
0,5	0,002	0,029	0,002	0,029	0,002	**-0,001**	0,002	**-0,001**	0,032	0,002	0,1
0,4	0,029	**-0,004**	**-0,001**	**-0,004**	0,029	0,026	0,029	**-0,004**	**-0,001**	**-0,001**	0,1
0,3	0,002	**-0,001**	0,002	0,058	0,002	**-0,001**	0,002	0,029	0,002	0,002	0,1
0,2	0,029	0,026	**-0,001**	**-0,004**	0,058	**-0,004**	**-0,001**	**-0,004**	**-0,001**	**-0,001**	0,1
0,1	0,002	**-0,001**	0,061	**-0,001**	0,002	0,029	0,002	**-0,001**	0,002	0,002	0,1
sum	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	

- "solution" is not feasible since it contains negative entries

$\Rightarrow$ cell-wise additive correction with $a := -\min\left\{x_{ij} \mid 1 \leq i, j \leq m\right\}$ and consecutive norming

upper cell boundary	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0	sum
1,0	0,004	0,002	0,004	0,002	0,004	0,024	0,026	0,024	0,004	0,004	0,1
0,9	0,026	0,002	0,004	0,002	0,004	0,002	0,026	0,002	0,004	0,026	0,1
0,8	0,002	0,000	0,002	0,000	0,002	0,000	0,002	0,022	0,024	0,046	0,1
0,7	0,004	0,024	0,004	0,002	0,004	0,024	0,004	0,002	0,026	0,004	0,1
0,6	0,002	0,022	0,024	0,022	0,002	0,000	0,002	0,022	0,002	0,002	0,1
0,5	0,004	0,024	0,004	0,024	0,004	0,002	0,004	0,002	0,026	0,004	0,1
0,4	0,024	0,000	0,002	0,000	0,024	0,022	0,024	0,000	0,002	0,002	0,1
0,3	0,004	0,002	0,004	0,046	0,004	0,002	0,004	0,024	0,004	0,004	0,1
0,2	0,024	0,022	0,002	0,000	0,046	0,000	0,002	0,000	0,002	0,002	0,1
0,1	0,004	0,002	0,048	0,002	0,004	0,024	0,004	0,002	0,004	0,004	0,1
sum	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	

final suboptimal contingency table $\left[y_{ij}\right] = \left[\dfrac{x_{ij} + a}{1 + m^2 \cdot a}\right]$

For dimension $d > 2$, with the index sets $I^d := \{1, \cdots, m\}^d$ and, for $i \in \{1, \cdots, m\}$ and $k = 1, \cdots, d$, $I_k^d(i) := \{1, \cdots, m\}^{k-1} \times \{i\} \times \{1, \cdots, m\}^{d-k}$, the corresponding Lagrange optimization problem

$$\text{min! } \sum_{(i_1 \cdots i_d) \in I^d} \left(x_{i_1 \cdots i_d} - a_{i_1 \cdots i_d} \right)^2 \text{ subject to}$$

$$x_{\bullet[k]}(i_k) := \sum_{(i_1 \cdots i_d) \in I_k^d(i_k)} x_{i_1 \cdots i_d} = \frac{1}{m} \text{ for } i_k \in \{1, \cdots, m\}, k = 1, \cdots, d \qquad (*)$$

has the solution

$$x_{i_1 \cdots i_d} = a_{i_1 \cdots i_d} - \frac{1}{m^{d-1}} \sum_{k=1}^{d} a_{\bullet[k]}(i_k) + \frac{d}{m^d} \text{ for } (i_1, \cdots, i_d) \in \{1, \cdots, m\}^d .$$

Proof: putting the gradient of the Lagrange function

$$L = \sum_{(i_1 \cdots i_d) \in I^d} \left(x_{i_1 \cdots i_d} - a_{i_1 \cdots i_d} \right)^2 + 2 \sum_{k=1}^{d} \sum_{i_k=1}^{m} \lambda_{k,i_k} \left(x_{\bullet[k]}(i_k) - \frac{1}{m} \right)$$

to zero results in the m^d additional equations (besides the side conditions (*))

$$\frac{\partial L}{\partial x_{i_1 \cdots i_d}} = 2\left(x_{i_1 \cdots i_d} - a_{i_1 \cdots i_d} \right) + 2 \sum_{k=1}^{d} \lambda_{k,i_k} = 0 \text{ for all } \left(i_1 \cdots i_d \right) \in I^d. \qquad (**)$$

These two sets of equations are solved by

$$\lambda_{k,i_k} = \frac{a_{\bullet[k]}(i_k)}{m^{d-1}} - \frac{1}{m^d} \text{ and } x_{i_1 \cdots i_d} = a_{i_1 \cdots i_d} - \sum_{k=1}^{d} \lambda_{k,i_k} \quad \text{for } i_k \in \{1, \cdots, m\},\ k = 1, \cdots, d.$$

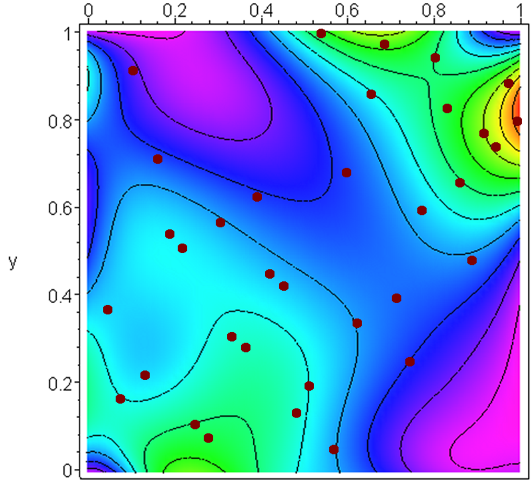
Note also that in the special case $d = 2$, we have

$$a_{\bullet[1]}(i) = a_{i\bullet} \text{ and } a_{\bullet[2]}(j) = a_{\bullet j} \text{ for } i, j \in \{1, \cdots, m\}.$$

The above solution can be used as an initial solution for either the multidimensional Karush-Kuhn-Tucker approach or the simplified version described above, giving

$$y_{i_1 \cdots i_d} = \frac{x_{i_1 \cdots i_d} + a}{1 + m^d a} \text{ with } a := -\min\left\{x_{i_1 \cdots i_d} \mid 1 \le i_1, \cdots, i_d \le m\right\}.$$

Any of the contingency tables above can be used to define the joint distribution of the discrete random vector $\mathbf{U} = (U_1, U_2)$ inducing the grid-type and Bernstein copulas.



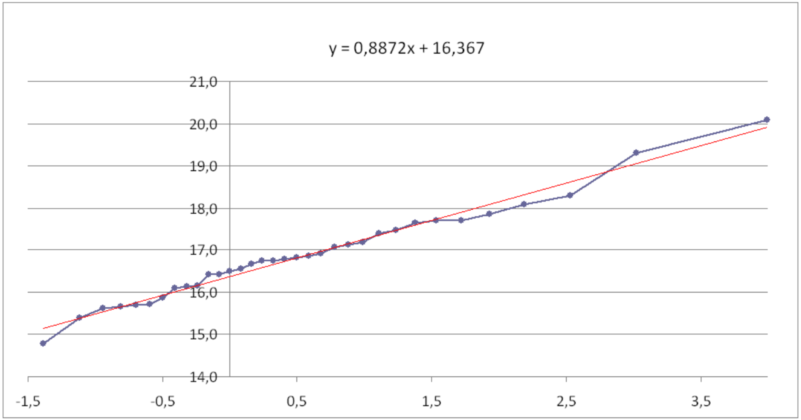
contour plot of the (suboptimal) Bernstein copula density,
with empirical copula scatterplot superimposed

4. Simulating from Bernstein copulas

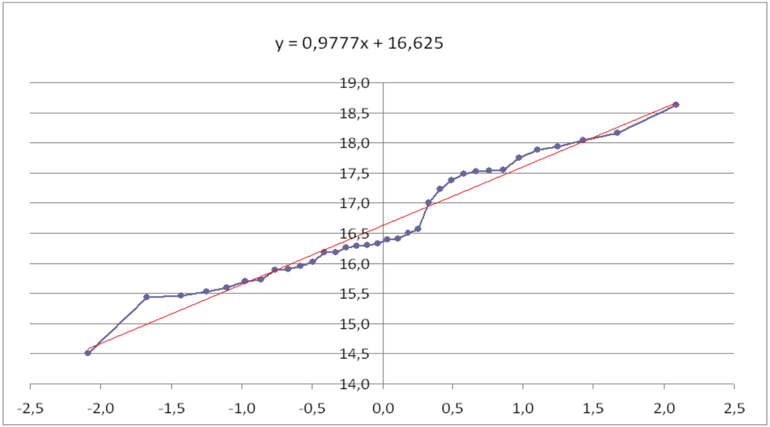
- Bernstein copula densities are polynomials, hence bounded over the unit cube $[0,1]^d$ by a constant $M > 0 \quad \Rightarrow \quad$ multivariate acceptance-rejection method
- average rate of samples obtained by this procedure is $1/M$

- <u>Step 1:</u>
  generate $d+1$ independent uniformly distributed random numbers $u_1, \cdots, u_{d+1}$.
- <u>Step 2:</u>
  check whether $c(u_1, \cdots, u_d) > M\, u_{d+1}$. If so, go to Step 3, otherwise go to Step 1.
- <u>Step 3:</u>
  use $(u_1, \cdots, u_d)$ as a sample from the Bernstein copula.

- in the windstorm / flooding example: $M = 2,35$
- Q-Q-plots from the 34 year time series of the logarithms of windstorm and flooding losses ($\mu =$ location parameter, $\sigma =$ scale parameter):



windstorm



y = 0,9777x + 16,625
19,0
18,5
18,0
17,5
17,0
16,5
16,0
15,5
15,0
14,5
14,0
-2,5
-2,0
-1,5
-1,0
-0,5
0,0
0,5
1,0
1,5
2,0
2,5

flooding

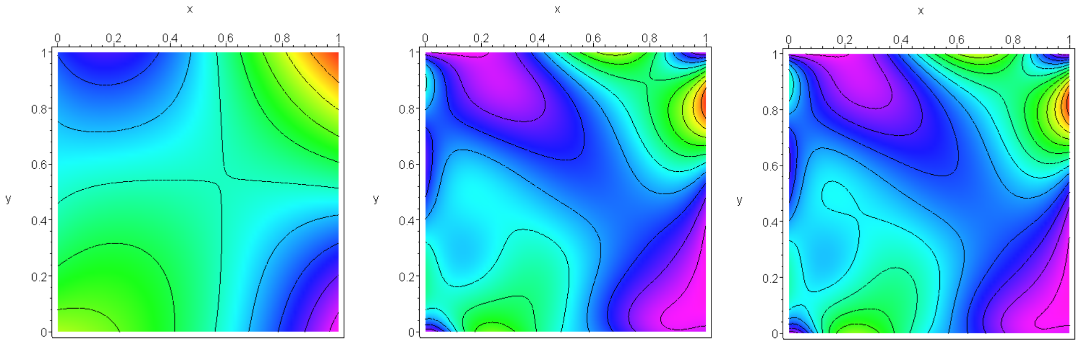
	Log windstorm losses	Log flooding losses
Distribution	Gumbel	Normal
Parameters	$\mu = 16,367$	$\mu = 16,625$
	$\sigma = 0,8872$	$\sigma = 0,9777$

- windstorm losses are considered to follow a **Fréchet distribution** with extremal index $\alpha = 1/\sigma = 1,1271$

- flooding losses are considered to follow a **lognormal distribution**

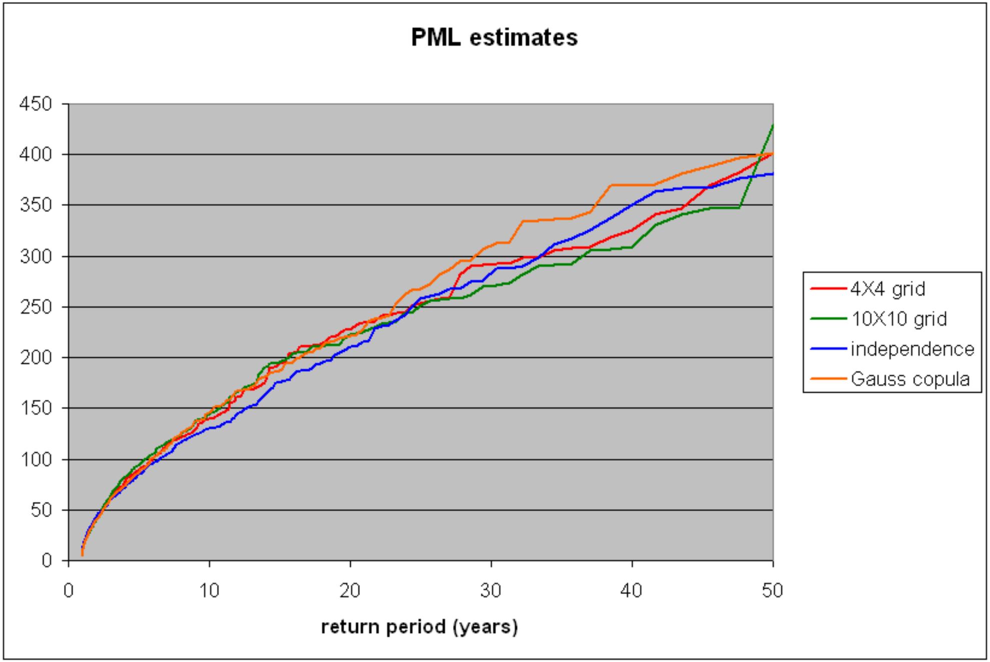
The following graphs show the results of a fourfold Monte Carlo simulation for the aggregate risk (windstorm and flooding) on the basis of 1000 pairs of points simulated from Bernstein copulas and the marginal distributions specified above. The four cases considered are:

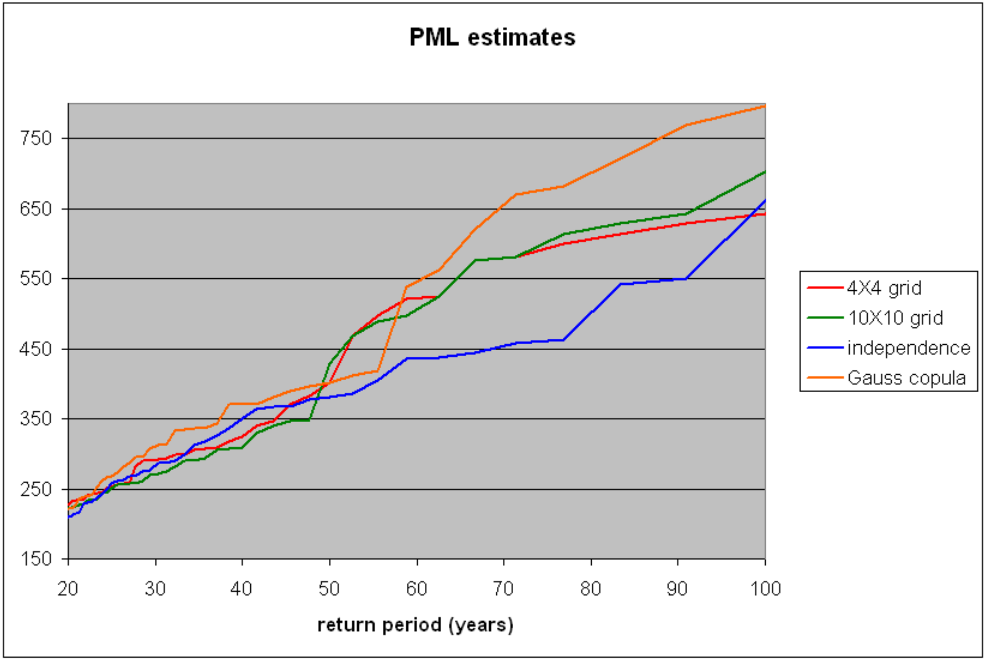
- **red line: Bernstein copula on the basis of a 4 x 4 grid**
- **green line: Bernstein copula on the basis of a 10 x 10 grid**
- **blue line: independence case**
- **orange line: Gaussian copula estimated from original data**



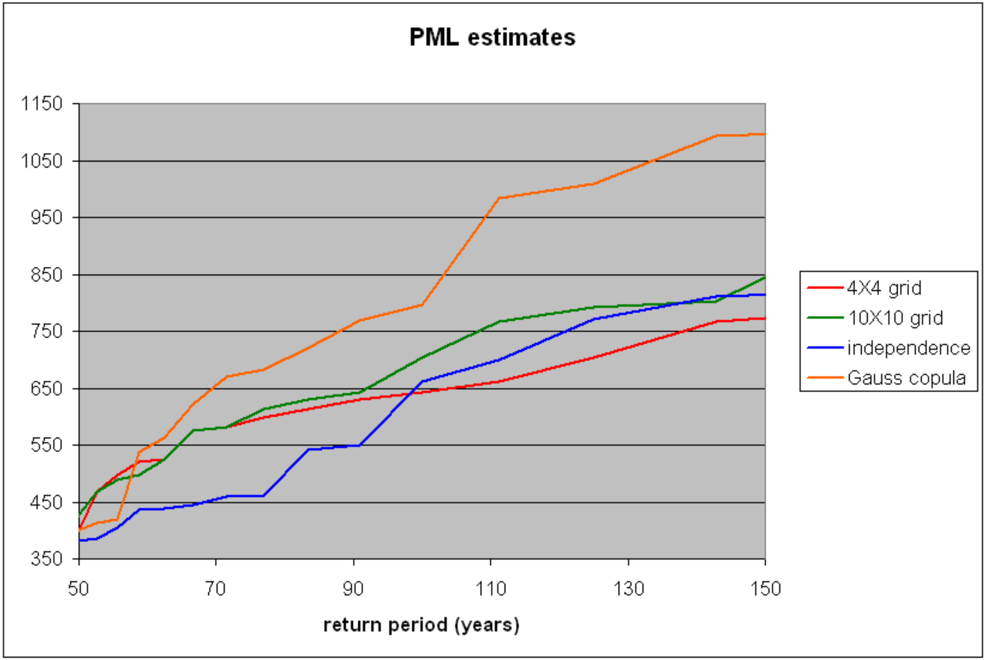
4x4 grid, approx. sol. 10x10 grid, approx. sol. 10x10 grid, opt. sol.

contour plot of Bernstein copula densities

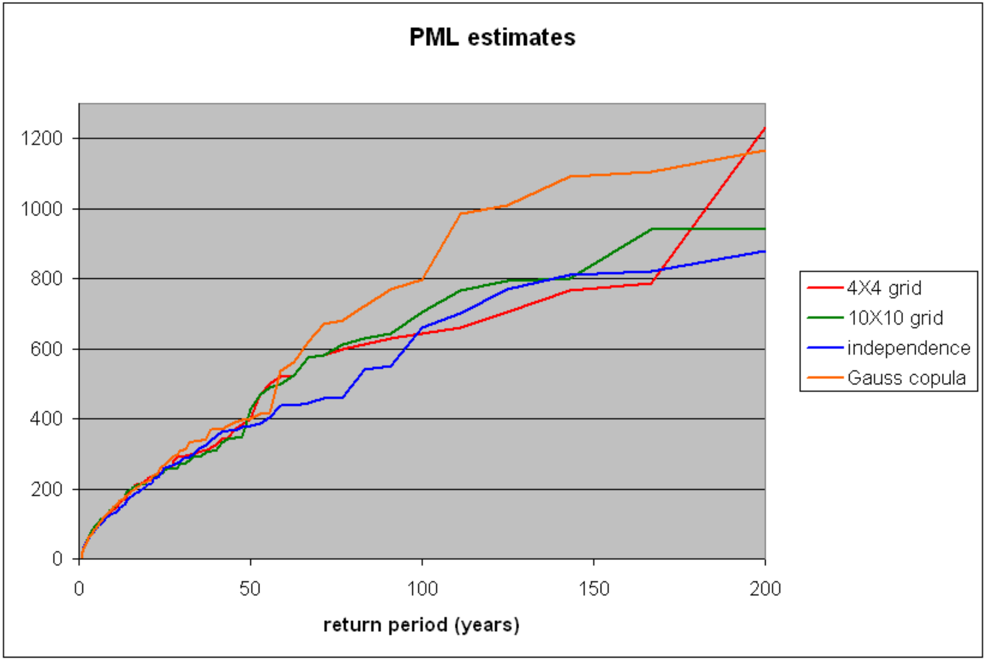

PML estimates
450
400
350
300
250
200
150
100
50
0
0
10
20
30
40
50
return period (years)
4X4 grid
10X10 grid
independence
Gauss copula

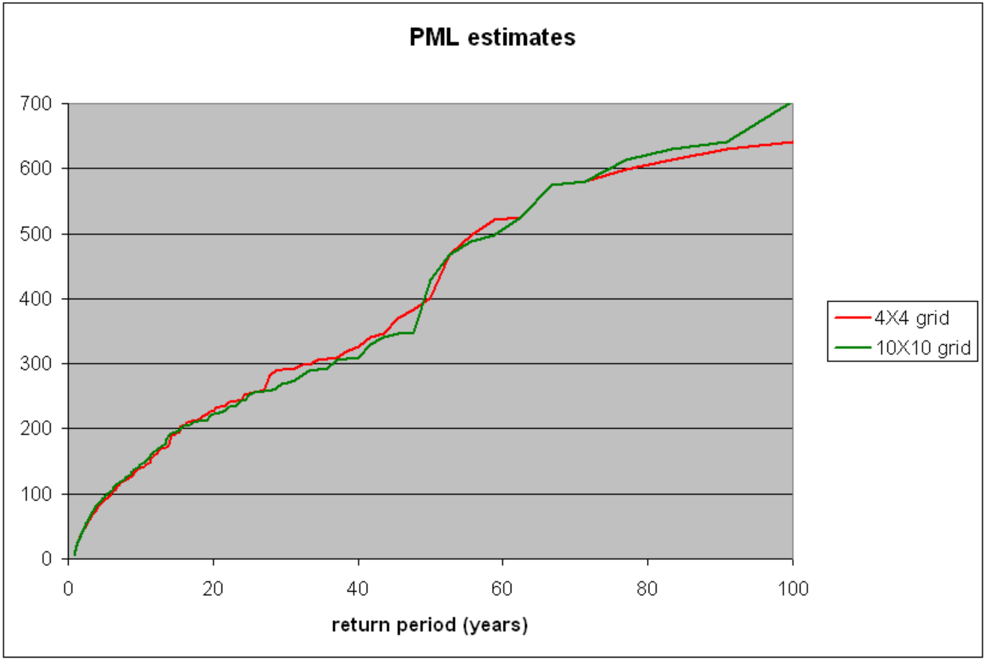


PML estimates
750
650
550
450
350
250
150
20
30
40
50
60
70
80
90
100
return period (years)
4X4 grid
10X10 grid
independence
Gauss copula



PML estimates
1150
1050
950
850
750
650
550
450
350
50
70
90
110
130
150
return period (years)
4X4 grid
10X10 grid
independence
Gauss copula


PML estimates
1200
1000
800
600
400
200
0
0
50
100
150
200
return period (years)
4X4 grid
10X10 grid
independence
Gauss copula


PML estimates
700
600
500
400
300
200
100
0
0
20
40
60
80
100
return period (years)
4X4 grid
10X10 grid

References

[1] T. BOUEZMARNI, J. V.K. ROMBOUTS, A. TAAMOUTI (2008): *Asymptotic properties of the Bernstein density copula for dependent data*. CORE discussion paper 2008/45, Leuven University, Belgium.

[2] V. DURRLEMAN, A. NIKEGHBALI, T. RONCALLI (2000): *Copulas approximation and new families*. Groupe de Recherche Opérationelle, Crédit Lyonnais, France, Working Paper.

[3] T. KULPA (1999): *On approximation of copulas*. Internat. J. Math. & Math. Sci. 22, 259 – 269.

[4] X. LI, P. MIKUSIŃSKI, H. SHERWOOD, M.D. TAYLOR (1997): *On approximation of copulas*. In: V. Beneš and J. Štěpán (Eds.), Distributions with Given Marginals and Moment Problems, Kluwer Academic Publishers, Dordrecht.

[5] A. SANCETTA, S.E. SATCHELL (2004): *The Bernstein copula and its applications to modelling and approximations of multivariate distributions*. Econometric Theory 20(3), 535 – 562.

[6] M. SALMON, C. SCHLEICHER (2007): *Pricing multivariate currency options with copulas*. In: Copulas. From Theory to Application in Finance, J. Rank (ed.), Risk Books, London, 219 – 232.