Change of Measures for Frequency and Severity

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Pricing Issues

- Lines of Business
- Data captured by the insurer
- Actuarial/Statistical models
- Adjustment of raw data
- Incomplete data
- Calibration of alternative models
- Deployment phase

Pure Premium=Lost Cost

Loss Cost = Frequency × Severity

In a world without risk, changes:

$$dLC(F,S) = \frac{\partial LC}{\partial F} dF + \frac{\partial LC}{\partial S} dS$$

Accounting for <u>both</u> Frequency (F) & Severity (S)

- Analyze F & S separately
- Zero Adjusted “Severity” model
- Compound Poisson process

Probabilistic models for Frequency & Severity

Frequency: Poisson with mean λ per unit time

Severity: random variable X

Compound Poisson Process: combining F & S

Poisson process: $N(t) = \sum_{n=1}^{\infty} I(T_n \le t)$

Counting Process: $\{T_n, n \ge 1\}$.

Marked Point Process, MPP: $\{(T_n, X_n), n \ge 1\}$

Compound Poisson Process: $Y(t)$

Alternative expressions for compound Poisson process

$$Y(t) = \sum_{i=1}^{\infty} X_i \, I(T_i \le t)$$

$$= \sum_{i=1}^{\infty} \{ \sum_{j=1}^{m} x_j \, I(X_i = x_j)\} \, I(T_i \le t)$$

$$= \sum_{j=1}^{m} x_j \sum_{i=1}^{\infty} I(T_i \le t \, , X_i \, \varepsilon A_j)$$

$$= \sum_{j=1}^{m} x_j \, N(t, A_j)$$

$$= \sum_{j=1}^{m} x_j \, N_j(t)$$

Change of measure: Radon-Nikodym Theorem

P: “initial” measure

Q: “final” measure

Change of measure: $P \to Q$

Absolute continuity: $Q << P$

$$Q(A) = \int_A Z \, dP = E_P(Z \, I(A))$$

R-N derivative : $Z = \frac{dQ}{dP}$

R-N derivative : $Z = \frac{dQ}{dP}$

Explicit Z vs. Implicit Z

Denumerable case for Severity X only!

$$Z_j = \frac{\Pr_Q(X = x_j)}{\Pr_P(X = x_j)} = \frac{Q(X = x_j)}{P(X = x_j)}, j = 1, 2, \cdots$$

Change of measure for Frequency only!

$$\lambda \to \lambda'$$

$$Z(t) = e^{(\lambda - \lambda')t} \left(\frac{\lambda'}{\lambda} \right)^{N(t)}$$

Poisson: r.v. vs. process

Assigning probabilities to a Y(t) Compound Poisson Process

$$\Pr(Y(t) = y)$$

$$y = x_1 N_1(t) + x_2 N_2(t) + \cdots + x_m N_m(t),$$

$$N(t) = N_1(t) + N_2(t) + \cdots + N_m(t)$$

$$\binom{k}{k_1 \quad k_2 \quad . \quad . \quad . \quad k_m} p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$$

Assigning probabilities to a Y(t) Compound Poisson Process

$$\Pr_{(\lambda', \underline{p})}(N_1(t) = k_1, N_2(t) = k_2, \cdots, N_m(t) = k_m)$$

$$= \prod_{j=1}^{m} \Pr_{(\lambda', \underline{p})}(N_j(t) = k_j)$$

$$= \prod_{j=1}^{m} Z_j(t) \Pr_{(\lambda, p)}(N_j(t) = k_j)$$

Radon-Nikodym derivative $Z_j(t)$ for $N_j(t)$

$$Z_j(t) = \exp(\lambda p_j - \lambda' q_j)\left(\frac{\lambda' q_j}{\lambda p_j}\right)^{N_j(t)}$$

$$\Pr_{(\lambda',Q)}(N_1(t) = k_1, N_2(t) = k_2, \cdots, N_m(t) = k_m)$$

$$= \{ e^{(\lambda-\lambda')t} \prod_{j=1}^{m} \left(\frac{\lambda' q_j}{\lambda p_j}\right)^{k_j} \} \Pr_{(\lambda,P)}(N_1(t) = k_1, N_2(t) = k_2, \cdots, N_m(t) = k_m)$$

Re-writing:

$$\prod_{j=1}^{m}\left(\frac{\lambda' q_j}{\lambda p_j}\right)^{k_j} = \prod_{i=1}^{k_1+k_2+\cdots+k_m}\left(\frac{\lambda' q(X_i)}{\lambda p(X_i)}\right)$$

R-N derivative for compound Poisson process

$$Z(t) = e^{(\lambda-\lambda')t}\prod_{i=1}^{N(t)}\left(\frac{\lambda' q(X_i)}{\lambda p(X_i)}\right)$$

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