ASTIN Colloquium Helsinki 2009

Panjer‘s World

An all-in-one formula for the Poisson, Binomial, and Negative Binomial distribution

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Agenda

- Unite the Panjer class in one formula
- Attempt to order the confusing variety of Negative Binomial parametrisations
- Overview of everything

Question

What is the connection between Binomial and Negative Binomial?

The quick answer:

- Tossing coins (Bernoulli trial)
- 1st parameter: probability of success
- 2nd parameter: number of trials (total / successes)

Classical formulae

Displayed are: probability function, probability generating function, expected value, and dispersion VarN/E(N)

	pf	**pgf**	**E(N)**	**D(N)**
B1	$\binom{m}{k} p^k (1-p)^{m-k}$	$(1-p+pz)^m$	mp	$1-p$
NB1	$\binom{\alpha+k-1}{k} p^\alpha (1-p)^k$	$\left(\frac{1-(1-p)z}{p}\right)^{-\alpha}$	$\frac{\alpha(1-p)}{p}$	$\frac{1}{p}$

Better formulae: use the expectation

	pf	pgf	E(N)	D(N)
P	$\frac{\lambda^k}{k!}e^{-\lambda}$	$e^{\lambda(z-1)}$	λ	1
B2	$\binom{m}{k}\frac{\lambda^k(m-\lambda)^{m-k}}{m^m}$	$\left(1+\frac{\lambda}{m}(z-1)\right)^m$	λ	$1-\frac{\lambda}{m}$
NB2	$\binom{\alpha+k-1}{k}\left(\frac{\alpha}{\alpha+\lambda}\right)^\alpha\left(\frac{\lambda}{\alpha+\lambda}\right)^k$	$\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$	λ	$1+\frac{\lambda}{\alpha}$

Correspondence: α vs. $-m$

Panjer United

Ansatz: generalize **NB2** $\binom{\alpha+k-1}{k}\left(\frac{\alpha}{\alpha+\lambda}\right)^{\alpha}\left(\frac{\lambda}{\alpha+\lambda}\right)^{k}$ to get

	pf	**pgf**
PanU	$\left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{\alpha+i}{\alpha+\lambda}$	$\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$

Parameter space of α:

positive real	Negative Binomial
negative integer < –λ	Binomial
± infinite	Poisson

Moments

... follow from the Panjer recursive formula

$p_k = p_{k-1}\ (a + b/k)$ with $a = \dfrac{\lambda}{\alpha+\lambda}$, $b = \dfrac{(\alpha-1)\lambda}{\alpha+\lambda}$

$E(N) = \lambda$, $Var(N) = \lambda\left(1+\dfrac{\lambda}{\alpha}\right)$, $CV^2(N) = \dfrac{1}{\lambda}+\dfrac{1}{\alpha}$, $D(N) = 1+\dfrac{\lambda}{\alpha}$

Higher moments

... follow from the pgf and its derivatives:

n-th derivative of the pgf: $$\lambda^n \left(1 - \frac{\lambda}{\alpha}(z-1)\right)^{-\alpha-n} \prod_{i=0}^{n-1} \left(1 + \frac{i}{\alpha}\right)$$

Now transfer everything from NegBin, e.g. the

skewness formula $$\lambda\left(1 + \frac{\lambda}{\alpha}\right)\left(1 + \frac{2\lambda}{\alpha}\right)$$

Panjer United & contagion

Replace α by the inverse c, known as "contagion":

	pf	**pgf**
PanU*	$(1+c\lambda)^{-1/c}\,\dfrac{\lambda^k}{k!}\,\prod_{i=0}^{k-1}\dfrac{1+ci}{1+c\lambda}$	$(1-c\lambda(z-1))^{-1/c}$

Parameter space of c:

positive real	Negative Binomial
negative …	Binomial
zero	Poisson

$E(N) = \lambda$, $\quad Var(N) = \lambda + c\lambda^2$

The world of Negative Binomial

	pf	pgf	P(N = 0)
NB1	$\binom{\alpha+k-1}{k} p^{\alpha}(1-p)^{k}$	$\left(\frac{1-(1-p)z}{p}\right)^{-\alpha}$	p^{α}
NB1b	$\binom{\alpha+k-1}{k}(1-q)^{\alpha} q^{k}$	$\left(\frac{1-q}{1-qz}\right)^{\alpha}$	$(1-q)^{\alpha}$
NB2	$\binom{\alpha+k-1}{k}\left(\frac{\alpha}{\alpha+\lambda}\right)^{\alpha}\left(\frac{\lambda}{\alpha+\lambda}\right)^{k}$	$\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$	$\left(\frac{\alpha}{\alpha+\lambda}\right)^{\alpha}$
NB3	$\binom{\alpha+k-1}{k}\frac{\beta^{\alpha}}{(1+\beta)^{\alpha+k}}$	$\left(1-\frac{z-1}{\beta}\right)^{-\alpha}$	$\left(\frac{\beta}{1+\beta}\right)^{\alpha}$
NB4	$\binom{\alpha+k-1}{k}\frac{\xi^{k}}{(1+\xi)^{\alpha+k}}$	$(1-\xi(z-1))^{-\alpha}$	$(1+\xi)^{-\alpha}$

Where do they come from?

- NB1: Bernoulli, p = probability of success
- NB1b: Bernoulli, q = probability of failure
- N2: Expected value
- N3: Poisson-Gamma with expectation α/β
- N4: Poisson-Gamma with expectation $\alpha\xi$

  or: Generalized Binomial Formula

$$\left((1+\xi)-\xi\right)^{-\alpha} = \sum_{k=0}^{\infty} \binom{-\alpha}{k} (1+\xi)^{-\alpha-k} (-\xi)^{k}$$

Overview

	pf	pgf	P(N = 0)	E(N)	Var(N)	CV²(N)	D(N)	a	b	
P	$\frac{\lambda^k}{k!}e^{-\lambda}$	$e^{\lambda(z-1)}$	$e^{-\lambda}$	λ	λ	$\frac{1}{\lambda}$	1	0	λ	**P**
B1	$\binom{m}{k}p^k(1-p)^{m-k}$	$(1-p+pz)^m$	$(1-p)^m$	mp	$mp(1-p)$	$\frac{1-p}{mp}$	$1-p$	$\frac{-p}{1-p}$	$\frac{(m+1)p}{1-p}$	**B1**
B2	$\binom{m}{k}\frac{\lambda^k(m-\lambda)^{m-k}}{m^m}$	$\left(1+\frac{\lambda}{m}(z-1)\right)^m$	$\left(1-\frac{\lambda}{m}\right)^m$	λ	$\lambda\left(1-\frac{\lambda}{m}\right)$	$\frac{1}{\lambda}-\frac{1}{m}$	$1-\frac{\lambda}{m}$	$\frac{-\lambda}{m-\lambda}$	$\frac{(m+1)\lambda}{m-\lambda}$	**B2**
NB1	$\binom{\alpha+k-1}{k}p^\alpha(1-p)^k$	$\left(\frac{1-(1-p)z}{p}\right)^{-\alpha}$	p^α	$\frac{\alpha(1-p)}{p}$	$\frac{\alpha(1-p)}{p^2}$	$\frac{1}{\alpha(1-p)}$	$\frac{1}{p}$	$1-p$	$(\alpha-1)(1-p)$	**NB1**
NB1b	$\binom{\alpha+k-1}{k}(1-q)^\alpha q^k$	$\left(\frac{1-q}{1-qz}\right)^\alpha$	$(1-q)^\alpha$	$\frac{\alpha q}{1-q}$	$\frac{\alpha q}{(1-q)^2}$	$\frac{1}{\alpha q}$	$\frac{1}{1-q}$	q	$(\alpha-1)q$	**NB1b**
NB2	$\binom{\alpha+k-1}{k}\left(\frac{\alpha}{\alpha+\lambda}\right)^\alpha\left(\frac{\lambda}{\alpha+\lambda}\right)^k$	$\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$	$\left(\frac{\alpha}{\alpha+\lambda}\right)^\alpha$	λ	$\lambda\left(1+\frac{\lambda}{\alpha}\right)$	$\frac{1}{\lambda}+\frac{1}{\alpha}$	$1+\frac{\lambda}{\alpha}$	$\frac{\lambda}{\alpha+\lambda}$	$\frac{(\alpha-1)\lambda}{\alpha+\lambda}$	**NB2**
NB3	$\binom{\alpha+k-1}{k}\frac{\beta^\alpha}{(1+\beta)^{\alpha+k}}$	$\left(1-\frac{z-1}{\beta}\right)^{-\alpha}$	$\left(\frac{\beta}{1+\beta}\right)^\alpha$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta}\left(1+\frac{1}{\beta}\right)$	$\frac{1+\beta}{\alpha}$	$1+\frac{1}{\beta}$	$\frac{1}{1+\beta}$	$\frac{\alpha-1}{1+\beta}$	**NB3**
NB4	$\binom{\alpha+k-1}{k}\frac{\xi^k}{(1+\xi)^{\alpha+k}}$	$(1-\xi(z-1))^{-\alpha}$	$(1+\xi)^{-\alpha}$	$\alpha\xi$	$\alpha\xi(1+\xi)$	$\frac{1+\xi}{\alpha\xi}$	$1+\xi$	$\frac{\xi}{1+\xi}$	$\frac{(\alpha-1)\xi}{1+\xi}$	**NB4**
PanU	$\left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{\alpha+i}{\alpha+\lambda}$	$\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$	$\left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}$	λ	$\lambda\left(1+\frac{\lambda}{\alpha}\right)$	$\frac{1}{\lambda}+\frac{1}{\alpha}$	$1+\frac{\lambda}{\alpha}$	$\frac{\lambda}{\alpha+\lambda}$	$\frac{(\alpha-1)\lambda}{\alpha+\lambda}$	**PanU**
PanU*	$(1+c\lambda)^{-1/c}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{1+ci}{1+c\lambda}$	$(1-c\lambda(z-1))^{-1/c}$	$(1+c\lambda)^{-1/c}$	λ	$\lambda+c\lambda^2$	$\frac{1}{\lambda}+c$	$1+c\lambda$	$\frac{c\lambda}{1+c\lambda}$	$\frac{(1-c)\lambda}{1+c\lambda}$	**PanU***

Conclusion

- Any of the existing parametrisations has a specific advantage in a specific situation.
- The coexistence of so many variants is an important source of errors / misunderstandings.
- It might be a good idea to agree on a standard among actuaries, or at least to name the variants.

Thanks

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