

Optimal *per claim* reinsurance for dependent risks

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Problem

The direct insurer holds a portfolio of $k \geq 1$ risks:

Problem

Considering "all possible" per-claim reinsurance schemes, find the combination of reinsurance policies which maximizes the adjustment coefficient of the retained risk for the whole portfolio.

- Guerra, M. and Centeno, M.L. (2008), *Ins. Math. Eco.* 42 no. 2, 529–539.
 - Relationship between maximization of the adjustment coefficient and maximization of the expected utility of wealth;
 - Existence and uniqueness of solution;
 - Necessary condition for optimality;
 - Characterization of optimal policies for some premium calculation principles.
- Guerra, M. and Centeno, M.L. (submitted)
 - Numerical method for computation of optimal solutions;
 - Examples and comparisons.
- Centeno, M.L. and Guerra, M. (submitted).
 - Generalization of the previous results for "per-claim" reinsurance on a single risk;
 - Dependence of the structure of the optimal treaty on the distribution of the number of claims.

- $k \geq 1$ risks;
- $N_i, i = 1, 2, \dots, k$
Number of claims of risk i in one year;

$$N = (N_1, N_2, \dots, N_k)$$

- $Y_{i,j}, i = 1, 2, \dots, k, j = 1, 2, \dots, N_i$
Value of the j^{th} claim of risk i ;
- $Z_i(y)$ – Value ceded under the reinsurance policy in force for risk i , given a claim amount y ;

- \mathcal{Z}_i - the set of all possible reinsurance policies for risk i

$$\mathcal{Z}_i = \{ \zeta : [0, +\infty[\mapsto \mathbf{R} \mid \zeta \text{ is measurable and } 0 \leq \zeta(y) \leq y, \forall y \geq 0 \},$$

$$\mathcal{Z} = \bigotimes_{i=1}^k \mathcal{Z}_i, \quad Z = (Z_1, Z_2, \dots, Z_k) \in \mathcal{Z};$$

- \hat{Y}_i - Aggregate claim amount of risk i in one year,

$$\hat{Y}_i = \sum_{1 \leq j \leq N_i} Y_{i,j};$$

- \hat{Z}_i - Aggregate claim amount of risk i ceded under reinsurance policy Z_i ,

$$\hat{Z}_i = \sum_{1 \leq j \leq N_i} Z_i(Y_{i,j});$$

- $P_i(Z_i)$ – Reinsurance premium for the reinsurance policy Z_i (covering risk i);
- L_Z – Profit obtained in one year assuming the reinsurance policies Z_i , $i = 1, 2, \dots, k$:

$$\begin{aligned} L_Z &= c - \sum_{i=1}^k P_i(Z_i) - \sum_{i=1}^k (\hat{Y}_i - \hat{Z}_i) = \\ &= c - \sum_{i=1}^k P_i(Z_i) - \sum_{i=1}^k \left(\sum_{1 \leq j \leq N_i} Y_{i,j} - \sum_{1 \leq j \leq N_i} Z_i(Y_{i,j}) \right). \end{aligned}$$

Assumptions

A1 $\Pr \{L_Z < 0\} > 0$ holds for every $Z \in \mathcal{Z}$;

A2 For each $i \in \{1, 2, \dots, k\}$, all $Y_{i,j}, j \in \mathbf{N}$, are i.i.d. nonnegative continuous random variables with common density function f_i .

Y_i – generic r.v. with density f_i ;

A3 $N = (N_1, N_2, \dots, N_k)$ is an array of integer random variables with joint probability function

$$p(n) = p(n_1, n_2, \dots, n_k) = \Pr\{N_1 = n_1, N_2 = n_2, \dots, N_k = n_k\};$$

The moment-generating function of p is finite in some neighborhood of zero;

Assumptions

- A4** The random variables Y_i , $i = 1, 2, \dots, k$ are mutually independent and independent of the random vector N ;
- A5** All functionals $P_i : \mathcal{Z}_i \mapsto [0, +\infty]$ are convex, $P_i(0) = 0$, and are continuous in the mean-squared sense:

$$\lim_{m \rightarrow \infty} \int_0^{+\infty} (Z_{i,m}(y) - Z_i(y))^2 f_i(y) dy = 0$$

implies

$$\lim_{m \rightarrow \infty} P_i(Z_{i,m}) = P_i(Z_i).$$

We do not distinguish between functions $Z_i, Z'_i \in \mathcal{Z}_i$ which differ only on a set of zero probability with respect to the density f_i .

Problem

- $G(R, Z) = E[e^{-RL_Z}]$, $R \in [0, +\infty)$, $Z \in \mathcal{Z}$
- R_Z - Adjustment coefficient of the retained risk for a particular combination of policies. R_Z solves equation

$$G(R, Z) = 1 \quad (1)$$

- $\mathcal{Z}^+ = \{Z \in \mathcal{Z} : (1) \text{ admits a positive solution}\}$

Problem

Find $(R^*, Z^*) \in (0, +\infty) \times \mathcal{Z}^+$ such that

$$R^* = R_{Z^*} = \max \{R_Z : Z \in \mathcal{Z}^+\}.$$

The combination of policies $Z^* \in \mathcal{Z}^+$ is said to be **optimal for the adjustment coefficient criterion** if (R_{Z^*}, Z^*) solves this problem.

Maximization of the expected utility of wealth

Exponential utility function with coefficient of risk aversion $R > 0$:

$$U_R(w) = -e^{-Rw}.$$

$$E[U_R(L_Z)] = -G(R, Z).$$

Problem

Find $Z^* \in \mathcal{Z}$, such that

$$E[U_R(L_{Z^*})] = \max\{E[U_R(L_Z)] : Z \in \mathcal{Z}\}.$$

Here $R > 0$ is a given constant (fixed).

A combination of policies $Z^* \in \mathcal{Z}$ is said to be **optimal for the expected utility criterion** with coefficient of risk aversion R if it solves this problem for that particular R .

Proposition

A pair $(R^*, Z^*) \in (0, +\infty) \times \mathcal{Z}^+$ solves the adjustment coefficient problem if and only if it satisfies the following conditions:

- 1 Z^* is optimal for the expected utility criterion with coefficient of risk aversion $R = R^*$;
- 2 $G(R^*, Z^*) = 1$.

The optimal policy for the adjustment coefficient criterion can be found in two steps:

- 1 For each $R \in (0, +\infty)$ find Z_R , the respective optimal policy for the expected utility criterion (find $Z_R = \arg \min \{G(R, Z) : Z \in \mathcal{Z}\}$);
- 2 Solve the equation with one single real variable

$$G(R, Z_R) = 1.$$

Theorem

For each $R \in (0, +\infty)$ there exists an optimal policy for the expected utility criterion.

*There exists an optimal policy for the adjustment coefficient criterion.
If*

- $\Pr\{N_i = m\} = 1$ holds for some $i \in \{1, 2, \dots, k\}$, $m \in \mathbf{N}$ and
- $Z^* = (Z_1^*, Z_2^*, \dots, Z_k^*) \in \mathcal{Z}$ is optimal,

the policy $Z_i^ \in \mathcal{Z}_i$ can be replaced by a policy $Z_i' \in \mathcal{Z}_i$ if and only if*

$$\Pr \left\{ Z_i' (Y_i) - Z_i^* (Y_i) = \frac{P_i (Z_i') - P_i (Z_i^*)}{m} \right\} = 1.$$

If $\Pr\{N_i = m\} < 1$, $\forall i \in \{1, 2, \dots, k\}$, $m \in \mathbf{N}$, then the optimal combination of policies is unique.

Needle-like perturbations

Fix $Z = (Z_1, Z_2, \dots, Z_k) \in \mathcal{Z}$.

$$Z_i^{\alpha, v, \varepsilon}(y) = \begin{cases} Z_i(y), & \text{if } y \notin [v, v + \varepsilon]; \\ \alpha, & \text{if } y \in [v, v + \varepsilon]. \end{cases}$$

$$\Delta P_{i, Z_i}(v) = \lim_{\alpha \rightarrow Z_i(v)} \lim_{0 \leq \alpha \leq v \quad \varepsilon \rightarrow 0^+} \frac{P_i(Z_i^{\alpha, v, \varepsilon}) - P_i(Z_i)}{\varepsilon(\alpha - Z_i(v))}, \quad i = 1, 2, \dots, k$$

Theorem

Let $Z \in \mathcal{Z}$ be optimal for the expected utility criterion.

There exist constants $C_i \in (0, +\infty)$, $i = 1, 2, \dots, k$ such that:

$$\begin{cases} \Delta P_{i,Z_i}(y) \geq C_i e^{Ry} f_i(y), & \text{when } Z_i(y) = 0; \\ \Delta P_{i,Z_i}(y) = C_i e^{R(y-Z_i(y))} f_i(y), & \text{when } 0 < Z_i(y) < y; \\ \Delta P_{i,Z_i}(y) \leq C_i f_i(y), & \text{when } Z_i(y) = y. \end{cases}$$

$$C_i \geq \frac{\frac{\partial}{\partial x_i} \pi \left(E \left[e^{R(Y_1-Z_1)} \right], E \left[e^{R(Y_2-Z_2)} \right], \dots, E \left[e^{R(Y_k-Z_k)} \right] \right)}{\pi \left(E \left[e^{R(Y_1-Z_1)} \right], E \left[e^{R(Y_2-Z_2)} \right], \dots, E \left[e^{R(Y_k-Z_k)} \right] \right)}, \quad (2)$$

with equality in (2) if there exists some $x_i > E \left[e^{R(Y_i-Z_i)} \right]$ such that

$$\pi \left(E \left[e^{R(Y_1-Z_1)} \right], \dots, x_i, \dots, E \left[e^{R(Y_k-Z_k)} \right] \right) < +\infty. \quad (3)$$

Corollary

Let $Z \in \mathcal{Z}$ be optimal for the expected utility criterion, and suppose there exists $\tilde{Z}_i \in \mathcal{Z}_i$ such that:

$$\Pr \{ \tilde{Z}_i \leq Z_i \} = 1, \quad \Pr \{ \tilde{Z}_i < Z_i \} > 0,$$

$$E \left[U_R(L_{(Z_1, \dots, Z_{i-1}, \tilde{Z}_i, Z_{i+1}, \dots, Z_k)}) \right] > -\infty.$$

Then, the following holds with probability equal to one:

$$\begin{cases} \Delta P_{i,Z_i}(y) \geq \frac{E[U_R(L_Z) | \exists m \in \{1, \dots, N_i\}: Y_{i,m}=y]}{E[U_R(L_Z)]} f_i(y), & \text{when } Z_i(y) = 0; \\ \Delta P_{i,Z_i}(y) = \frac{E[U_R(L_Z) | \exists m \in \{1, \dots, N_i\}: Y_{i,m}=y]}{E[U_R(L_Z)]} f_i(y), & \text{when } 0 < Z_i(y) < y; \\ \Delta P_{i,Z_i}(y) \leq \frac{E[U_R(L_Z) | \exists m \in \{1, \dots, N_i\}: Y_{i,m}=y]}{E[U_R(L_Z)]} f_i(y), & \text{when } Z_i(y) = y. \end{cases}$$

The expected value principle

Suppose the premium calculation principle for risk i is the expected value principle:

$$P_i(Z_i) = (1 + \beta_i) E[\hat{Z}_i] = (1 + \beta_i) E[N_i] E[Z_i],$$

Corollary

The optimal treaty for the risk i is an excess of loss contract:

$$Z_i(y) = \begin{cases} 0, & \text{if } y \leq M_i; \\ y - M_i, & \text{if } y \geq M_i. \end{cases}$$

$$\text{If (3) holds: } M_i = \frac{1}{R} \ln \frac{(1 + \beta_i) E[N_i] \pi(E[e^{R(Y_1 - Z_1)}], \dots, E[e^{R(Y_k - Z_k)}])}{\frac{\partial \pi}{\partial x_i}(E[e^{R(Y_1 - Z_1)}], \dots, E[e^{R(Y_k - Z_k)}])};$$

$$\text{If (3) fails: } M_i \leq \frac{1}{R} \ln \frac{(1 + \beta_i) E[N_i] \pi(E[e^{R(Y_1 - Z_1)}], \dots, E[e^{R(Y_k - Z_k)}])}{\frac{\partial \pi}{\partial x_i}(E[e^{R(Y_1 - Z_1)}], \dots, E[e^{R(Y_k - Z_k)}])}.$$

Variance-related principles

Suppose the premium calculation principle for risk i is a variance-related principle:

$$\begin{aligned}P_i(Z_i) &= E[\hat{Z}_i] + g_i(\text{Var}[\hat{Z}_i]) = \\ &= E[N_i]E[Z_i] + g_i(E[N_i]\text{Var}[Z_i] + \text{Var}[N_i]E[Z_i]^2),\end{aligned}$$

$g_i : [0, +\infty) \mapsto [0, +\infty)$ continuous, smooth in $(0, +\infty)$,

$$g_i(0) = 0, \quad g_i'(t) > 0, \quad \forall t > 0.$$

Lemma

$P_i : \mathcal{Z}_i \mapsto [0, +\infty)$ is convex iff g_i satisfies

$$\frac{g_i''(t)}{g_i'(t)} \geq -\frac{1}{2t}, \quad \forall t \in (0, B),$$

$$B = \sup \{ \text{Var}[\hat{Z}_i] : Z_i \in \mathcal{Z}_i \}.$$

Corollary

If g_i' is bounded in a neighborhood of zero, then there are constants $\alpha_{i,1} > 0$, $\alpha_{i,2} \in \mathbf{R}$ such that the optimal treaty for the risk i satisfies

$$y \leq \frac{1}{R} \ln \frac{-\alpha_{i,2}}{\alpha_{i,1}}, \text{ when } Z_i(y) = 0;$$

$$y = Z_i(y) + \frac{1}{R} \ln \frac{Z_i(y) - \alpha_{i,2}}{\alpha_{i,1}}, \text{ when } 0 < Z_i(y) < y;$$

$$y \leq \alpha_{i,1} + \alpha_{i,2}, \text{ when } Z_i(y) = y.$$

If g_i' is unbounded in any neighborhood of zero, then the optimal treaty must be either a function of the type above or $Z_i \equiv 0$ (no reinsurance at all).

If (3) holds: $\alpha_{i,1} = \frac{\frac{\partial \pi}{\partial x_i} (E[e^{R(Y_1-Z_1)}], \dots, E[e^{R(Y_k-Z_k)}])}{E[N_i] \pi(E[e^{R(Y_1-Z_1)}], \dots, E[e^{R(Y_k-Z_k)}]) 2g'_i(\text{Var}[\hat{Z}_i])}$;

If (3) fails: $\alpha_{i,1} \geq \frac{\frac{\partial \pi}{\partial x_i} (E[e^{R(Y_1-Z_1)}], \dots, E[e^{R(Y_k-Z_k)}])}{E[N_i] \pi(E[e^{R(Y_1-Z_1)}], \dots, E[e^{R(Y_k-Z_k)}]) 2g'_i(\text{Var}[\hat{Z}_i])}$;

$$\alpha_{i,2} = \frac{E[N_i] - \text{Var}[N_i]}{E[N_i]} E[Z_i] - \frac{1}{2g'_i(\text{Var}[\hat{Z}_i])}.$$

Example:

We consider an insurance company dealing with two risks.

- $Y_i \sim \text{Pareto}(a_i, b_i)$, i.e.

$$F_i(y) = 1 - \left(\frac{b_i}{b_i + y} \right)^{a_i}, \quad i = 1, 2$$

with parameters $a_i > 2$, $b_i > 0$.

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$$\Pr\{N_i = n | \Theta = \theta\} = e^{-\theta\lambda_i} \frac{(\theta\lambda_i)^n}{n!}, \quad i = 1, 2;$$

$$\Pr\{N_1 = n_1, N_2 = n_2\} = \int_0^{+\infty} e^{-\theta(\lambda_1 + \lambda_2)} \frac{(\theta\lambda_1)^{n_1} (\theta\lambda_2)^{n_2}}{n_1! n_2!} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta,$$

with parameters $\alpha > 0$, $\beta > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$. The probability generating function for the number of claims in this model is

$$\pi(x_1, x_2) = \left(\frac{\beta}{\beta - \lambda_1(x_1 - 1) - \lambda_2(x_2 - 1)} \right)^\alpha,$$

i.e. (N_1, N_2) follows a bivariate negative binomial.

Example

We set the following parameter values:

$$\begin{aligned}a_1 &= 3, & a_2 &= 4, & b_1 &= \frac{1}{2}, & b_2 &= \frac{9}{20}, \\ \lambda_1 &= 1, & \lambda_2 &= 5, & \alpha &= \beta &= 1.89898 \\ E[\hat{Y}_1] &= \frac{1}{4}, & E[\hat{Y}_2] &= \frac{3}{4}\end{aligned}$$

which gives the moments

$$\begin{aligned}E[N_1] &= 1, & E[N_2] &= 5, \\ \text{Var}[N_1] &= 1.52660, & \text{Var}[N_2] &= 18.165, & \text{Corr}(N_1, N_2) &= 0.5, \\ E[Y_1] &= 0.25, & E[Y_2] &= 0.15, & \text{Var}[Y_1] &= 0.1875, & \text{Var}[Y_2] &= 0.045, \\ \text{Var}[\hat{Y}_1] &= 0.282912, & \text{Var}[\hat{Y}_2] &= 0.633712.\end{aligned}$$

Example

We assume that

$$P_i(Z_i) = E[\hat{Z}_i] + 0.3\sqrt{\text{Var}[\hat{Z}_i]}, \quad i = 1, 2.$$

and that $c = 1.19919$.

Table 1: **Parameters of optimal treaties** (adjustment coefficient criterion)

Independent N_1, N_2		$\text{Corr}(N_1, N_2) = 0.5$	
best treaties	best XL	best treaties	best XL
$\alpha_{1,1} = 0.487313$	$M_1 = 8.94428$ $M_2 = 15.8155$	$\alpha_{1,1} = 0.580562$	$M_1 = 11.7585$ $M_2 = 21.0894$
$\alpha_{1,2} = -0.487313$		$\alpha_{1,2} = -0.540189$	
$\alpha_{2,1} = 0.342036$		$\alpha_{2,1} = 0.231556$	
$\alpha_{2,2} = -0.342036$		$\alpha_{2,2} = -0.229717$	

Table 2: **Optimal treaties in the independent case vs the dependent case** (adjustment coefficient criterion)

	Independent N_1, N_2		$\text{Corr}(N_1, N_2) = 0.5$	
	best treaties	best XL	best treaties	best XL
R	0.311772	0.284421	0.262623	0.238882
$E[\hat{Z}_1]$	0.037230	0.000701	0.067591	0.000416
$E[\hat{Z}_2]$	0.015349	3.176×10^{-6}	0.010809	1.368×10^{-6}
$\frac{E[\hat{Z}_1]}{E[\hat{Y}_1]}$	0.148920	0.002803	0.270363	0.001664
$\frac{E[\hat{Z}_2]}{E[\hat{Y}_2]}$	0.020465	4.235×10^{-6}	0.014412	1.824×10^{-6}
$P_1(Z_1)$	0.079324	0.035215	0.113004	0.030710
$P_2(Z_2)$	0.103890	0.004838	0.072157	0.003648

If we compute the adjustment coefficient of the retained risk under treaties Z_1^i, Z_2^i (i for independent) but the risks were dependent the answer is $R_{Z_1^i, Z_2^i} = 0.258863$ (versus $R_{Z_1^d, Z_2^d} = 0.262623$).





