Optimal *per claim* reinsurance for dependent risks

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The direct insurer holds a portfolio of $k \geq 1$ risks:

*Problem*

*Considering "all possible" per-claim reinsurance schemes, find the combination of reinsurance policies which maximizes the adjustment coefficient of the retained risk for the whole portfolio.*
Previous work

  - Relationship between maximization of the adjustment coefficient and maximization of the expected utility of wealth;
  - Existence and uniqueness of solution;
  - Necessary condition for optimality;
  - Characterization of optimal policies for some premium calculation principles.

- Guerra, M. and Centeno, M.L. (submitted)
  - Numerical method for computation of optimal solutions;
  - Examples and comparisons.

  - Generalization of the previous results for "per-claim" reinsurance on a single risk;
  - Dependence of the structure of the optimal treaty on the distribution of the number of claims.
Notation

- $k \geq 1$ risks;
- $N_i, \ i = 1, 2, \ldots, k$
  Number of claims of risk $i$ in one year;
  \[ N = (N_1, N_2, \ldots, N_k) \]
- $Y_{i,j}, \ i = 1, 2, \ldots, k, \ j = 1, 2, \ldots, N_i$
  Value of the $j^{th}$ claim of risk $i$;
- $Z_i (y)$ – Value ceded under the reinsurance policy in force for risk $i$, given a claim amount $y$;
Notation

- $\mathcal{Z}_i$ - the set of all possible reinsurance policies for risk $i$
  \[
  \mathcal{Z}_i = \{\zeta : [0, +\infty[ \rightarrow \mathbb{R} | \zeta \text{ is measurable and } 0 \leq \zeta(y) \leq y, \forall y \geq 0\}, \]
  \[
  \mathcal{Z} = \bigotimes_{i=1}^{k} \mathcal{Z}_i, \quad Z = (Z_1, Z_2, \ldots, Z_k) \in \mathcal{Z};
  \]

- $\hat{Y}_i$ – Aggregate claim amount of risk $i$ in one year,
  \[
  \hat{Y}_i = \sum_{1 \leq j \leq N_i} Y_{i,j};
  \]

- $\hat{Z}_i$ – Aggregate claim amount of risk $i$ ceded under reinsurance policy $Z_i$,
  \[
  \hat{Z}_i = \sum_{1 \leq j \leq N_i} Z_i(Y_{i,j});
  \]
Notation

- $P_i(Z_i)$ – Reinsurance premium for the reinsurance policy $Z_i$ (covering risk $i$);

- $L_Z$ – Profit obtained in one year assuming the reinsurance policies $Z_i$, $i = 1, 2, ..., k$:

$$L_Z = c - \sum_{i=1}^{k} P_i(Z_i) - \sum_{i=1}^{k} \left( \hat{Y}_i - \hat{Z}_i \right) =$$

$$= c - \sum_{i=1}^{k} P_i(Z_i) - \sum_{i=1}^{k} \left( \sum_{1 \leq j \leq N_i} Y_{i,j} - \sum_{1 \leq j \leq N_i} Z_i(Y_{i,j}) \right).$$
Assumptions

**A1** \( \Pr \{ L_Z < 0 \} > 0 \) holds for every \( Z \in \mathcal{Z} \);

**A2** For each \( i \in \{1, 2, \ldots, k\} \), all \( Y_{i,j}, j \in \mathbb{N} \), are i.i.d. nonnegative continuous random variables with common density function \( f_i \).

\( Y_i \) – generic r.v. with density \( f_i \);

**A3** \( N = (N_1, N_2, \ldots, N_k) \) is an array of integer random variables with joint probability function

\[
p(n) = p(n_1, n_2, \ldots, n_k) = \Pr\{N_1 = n_1, N_2 = n_2, \ldots, N_k = n_k\};
\]

The moment-generating function of \( p \) is finite in some neighborhood of zero;
Assumptions

**A4** The random variables $Y_i$, $i = 1, 2, \ldots, k$ are mutually independent and independent of the random vector $N$;

**A5** All functionals $P_i : \mathcal{Z}_i \mapsto [0, +\infty]$ are convex, $P_i(0) = 0$, and are continuous in the mean-squared sense:

$$\lim_{m \to \infty} \int_0^{+\infty} (Z_{i,m}(y) - Z_i(y))^2 f_i(y) \, dy = 0$$

implies

$$\lim_{m \to \infty} P_i(Z_{i,m}) = P_i(Z_i) .$$

We do not distinguish between functions $Z_i, Z_i' \in \mathcal{Z}_i$ which differ only on a set of zero probability with respect to the density $f_i$. 
Problem

- \( G(R, Z) = E[e^{-RLZ}] \), \( R \in [0, +\infty), \ Z \in \mathcal{Z} \)

- \( R_Z \) - Adjustment coefficient of the retained risk for a particular combination of policies. \( R_Z \) solves equation

\[
G(R, Z) = 1
\]

- \( \mathcal{Z}^+ = \{ Z \in \mathcal{Z} : (1) \text{ admits a positive solution} \} \)

Problem

Find \((R^*, Z^*) \in (0, +\infty) \times \mathcal{Z}^+ \) such that

\[
R^* = R_{Z^*} = \max \{ R_Z : Z \in \mathcal{Z}^+ \}.
\]

The combination of policies \( Z^* \in \mathcal{Z}^+ \) is said to be optimal for the adjustment coefficient criterion if \((R_{Z^*}, Z^*)\) solves this problem.
Maximization of the expected utility of wealth

Exponential utility function with coefficient of risk aversion $R > 0$:

$$U_R(w) = -e^{-Rw}.$$  

$$E[U_R(L_Z)] = -G(R, Z).$$

Problem

Find $Z^* \in \mathcal{Z}$, such that

$$E[U_R(L_{Z^*})] = \max \{ E[U_R(L_Z)] : Z \in \mathcal{Z} \}.$$  

Here $R > 0$ is a given constant (fixed).

A combination of policies $Z^* \in \mathcal{Z}$ is said to be optimal for the expected utility criterion with coefficient of risk aversion $R$ if it solves this problem for that particular $R$.  

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A pair \((R^*, Z^*) \in (0, +\infty) \times \mathbb{Z}^+\) solves the adjustment coefficient problem if and only if it satisfies the following conditions:

1. \(Z^*\) is optimal for the expected utility criterion with coefficient of risk aversion \(R = R^*\); 
2. \(G(R^*, Z^*) = 1\).

The optimal policy for the adjustment coefficient criterion can be found in two steps:

1. For each \(R \in (0, +\infty)\) find \(Z_R\), the respective optimal policy for the expected utility criterion (find \(Z_R = \arg\min \{G(R, Z) : Z \in \mathbb{Z}\}\));
2. Solve the equation with one single real variable 
\[
G(R, Z_R) = 1.
\]
Existence and uniqueness of optimal policy

**Theorem**

For each $R \in (0, +\infty)$ there exists an optimal policy for the expected utility criterion.

There exists an optimal policy for the adjustment coefficient criterion.

If

- $\Pr\{N_i = m\} = 1$ holds for some $i \in \{1, 2, \ldots, k\}$, $m \in \mathbb{N}$ and
- $Z^* = (Z_1^*, Z_2^*, \ldots, Z_k^*) \in \mathcal{Z}$ is optimal,

the policy $Z_i^* \in \mathcal{Z}_i$ can be replaced by a policy $Z_i' \in \mathcal{Z}_i$ if and only if

$$\Pr\left\{ Z_i' (Y_i) - Z_i^* (Y_i) = \frac{P_i(Z_i') - P_i(Z_i^*)}{m} \right\} = 1.$$  

If $\Pr\{N_i = m\} < 1$, $\forall i \in \{1, 2, \ldots, k\}$, $m \in \mathbb{N}$, then the optimal combination of policies is unique.

Needle-like perturbations

Fix \( Z = (Z_1, Z_2, \ldots, Z_k) \in \mathcal{Z} \).

\[
Z_i^{\alpha,v,\varepsilon}(y) = \begin{cases} 
Z_i(y), & \text{if } y \notin [v, v + \varepsilon]; \\
\alpha, & \text{if } y \in [v, v + \varepsilon].
\end{cases}
\]

\[
\Delta P_{i,Z_i}(v) = \lim_{\alpha \to Z_i(v), \ 0 \leq \alpha \leq v} \lim_{\varepsilon \to 0^+} \frac{P_i(Z_i^{\alpha,v,\varepsilon}) - P_i(Z_i)}{\varepsilon (\alpha - Z_i(v))}, \quad i = 1, 2, \ldots, k
\]
Necessary optimality conditions

**Theorem**

Let \( Z \in Z \) be optimal for the expected utility criterion. There exist constants \( C_i \in (0, +\infty), i = 1, 2, \ldots, k \) such that:

\[
\begin{align*}
\Delta P_{i,Z_i}(y) &\geq C_i e^{Ry} f_i(y), & \text{when } Z_i(y) = 0; \\
\Delta P_{i,Z_i}(y) & = C_i e^{R(y-Z_i(y))} f_i(y), & \text{when } 0 < Z_i(y) < y; \\
\Delta P_{i,Z_i}(y) &\leq C_i f_i(y), & \text{when } Z_i(y) = y.
\end{align*}
\]

\[
C_i \geq \frac{\frac{\partial}{\partial x_i} \pi \left( E \left[ e^{R(Y_1-Z_1)} \right], E \left[ e^{R(Y_2-Z_2)} \right], \ldots, E \left[ e^{R(Y_k-Z_k)} \right] \right)}{\pi \left( E \left[ e^{R(Y_1-Z_1)} \right], E \left[ e^{R(Y_2-Z_2)} \right], \ldots, E \left[ e^{R(Y_k-Z_k)} \right] \right)},
\]

(2)

with equality in (2) if there exists some \( x_i > E \left[ e^{R(Y_i-Z_i)} \right] \) such that

\[
\pi \left( E \left[ e^{R(Y_1-Z_1)} \right], \ldots, x_i, \ldots, E \left[ e^{R(Y_k-Z_k)} \right] \right) < +\infty.
\]

(3)
Corollary

Let \( Z \in \mathcal{Z} \) be optimal for the expected utility criterion, and suppose there exists \( \tilde{Z}_i \in \mathcal{Z}_i \) such that:

\[
\begin{align*}
\Pr \{ \tilde{Z}_i \leq Z_i \} &= 1, \\
\Pr \{ \tilde{Z}_i < Z_i \} &> 0, \\
E \left[ U_R(L(Z_1, \ldots, Z_{i-1}, \tilde{Z}_i, Z_{i+1}, \ldots, Z_k)) \right] &> -\infty.
\end{align*}
\]

Then, the following holds with probability equal to one:

\[
\begin{align*}
\Delta P_{i, Z_i} (y) &\geq \frac{E[U_R(L_Z) | \exists m \in \{1, \ldots, N_i\}: Y_{i,m} = y]}{E[U_R(L_Z)]} f_i (y), \\
\text{when } Z_i(y) &= 0; \\
\Delta P_{i, Z_i} (y) &= \frac{E[U_R(L_Z) | \exists m \in \{1, \ldots, N_i\}: Y_{i,m} = y]}{E[U_R(L_Z)]} f_i (y), \\
\text{when } 0 &< Z_i(y) < y; \\
\Delta P_{i, Z_i} (y) &\leq \frac{E[U_R(L_Z) | \exists m \in \{1, \ldots, N_i\}: Y_{i,m} = y]}{E[U_R(L_Z)]} f_i (y), \\
\text{when } Z_i(y) &= y.
\end{align*}
\]
The expected value principle

Suppose the premium calculation principle for risk \( i \) is the expected value principle:

\[
P_i(Z_i) = (1 + \beta_i) E[\hat{Z}_i] = (1 + \beta_i) E[N_i] E[Z_i],
\]

Corollary

*The optimal treaty for the risk \( i \) is an excess of loss contract:*

\[
Z_i(y) = \begin{cases} 
0, & \text{if } y \leq M_i; \\
y - M_i, & \text{if } y \geq M_i.
\end{cases}
\]

*If (3) holds:*

\[
M_i = \frac{1}{R} \ln \frac{(1+\beta_i)E[N_i] \pi (E[e^{R(Y_1-Z_1)}],...,E[e^{R(Y_k-Z_k)}])}{\frac{\partial \pi}{\partial x_i} (E[e^{R(Y_1-Z_1)}],...,E[e^{R(Y_k-Z_k)}])};
\]

*If (3) fails:*

\[
M_i \leq \frac{1}{R} \ln \frac{(1+\beta_i)E[N_i] \pi (E[e^{R(Y_1-Z_1)}],...,E[e^{R(Y_k-Z_k)}])}{\frac{\partial \pi}{\partial x_i} (E[e^{R(Y_1-Z_1)}],...,E[e^{R(Y_k-Z_k)}])}.
\]
Variance-related principles

Suppose the premium calculation principle for risk $i$ is a variance-related principle:

$$P_i(Z_i) = E[\hat{Z}_i] + g_i(\text{Var}[\hat{Z}_i]) =$$
$$= E[N_i]E[Z_i] + g_i(E[N_i]\text{Var}[Z_i] + \text{Var}[N_i]E[Z_i]^2),$$

$g_i : [0, +\infty) \mapsto [0, +\infty)$ continuous, smooth in $(0, +\infty)$,

$$g_i(0) = 0, \quad g_i'(t) > 0, \quad \forall t > 0.$$

Lemma

$P_i : Z_i \mapsto [0, +\infty)$ is convex iff $g_i$ satisfies

$$\frac{g_i''(t)}{g_i'(t)} \geq -\frac{1}{2t}, \quad \forall t \in (0, B),$$

$B = \sup \{ \text{Var}[\hat{Z}_i] : Z_i \in Z_i \}.$
Corollary

If $g_i^1$ is bounded in a neighborhood of zero, then there are constants $\alpha_{i,1} > 0$, $\alpha_{i,2} \in \mathbb{R}$ such that the optimal treaty for the risk $i$ satisfies

$$y \leq \frac{1}{R} \ln \frac{-\alpha_{i,2}}{\alpha_{i,1}}, \text{ when } Z_i(y) = 0;$$

$$y = Z_i(y) + \frac{1}{R} \ln \frac{Z_i(y) - \alpha_{i,2}}{\alpha_{i,1}}, \text{ when } 0 < Z_i(y) < y;$$

$$y \leq \alpha_{i,1} + \alpha_{i,2}, \text{ when } Z_i(y) = y.$$

If $g_i^1$ is unbounded in any neighborhood of zero, then the optimal treaty must be either a function of the type above or $Z_i \equiv 0$ (no reinsurance at all).
Variance-related principles

If (3) holds: \( \alpha_{i,1} = \frac{\frac{\partial \pi}{\partial x_i} \left( E[e^R(Y_1-Z_1)],...,E[e^R(Y_k-Z_k)] \right)}{E[N_i] \pi \left( E[e^R(Y_1-Z_1)],...,E[e^R(Y_k-Z_k)] \right) 2g_i \left( \text{Var}[\hat{Z}_i] \right)}; \)

If (3) fails: \( \alpha_{i,1} \geq \frac{\frac{\partial \pi}{\partial x_i} \left( E[e^R(Y_1-Z_1)],...,E[e^R(Y_k-Z_k)] \right)}{E[N_i] \pi \left( E[e^R(Y_1-Z_1)],...,E[e^R(Y_k-Z_k)] \right) 2g_i \left( \text{Var}[\hat{Z}_i] \right)}; \)

\( \alpha_{i,2} = \frac{E[N_i] - \text{Var}[N_i]}{E[N_i]} E[Z_i] - \frac{1}{2g_i \left( \text{Var}[\hat{Z}_i] \right)}. \)
Example:

We consider an insurance company dealing with two risks.

- $Y_i \sim \text{Pareto}(a_i, b_i)$, i.e.
  \[
  F_i(y) = 1 - \left( \frac{b_i}{b_i + y} \right)^{a_i}, \quad i = 1, 2
  \]
  with parameters $a_i > 2, b_i > 0$.

- \[
  \Pr\{N_i = n|\Theta = \theta\} = e^{-\theta \lambda_i} \frac{\theta^{n_i}}{n_i!}, \quad i = 1, 2; \]
  \[
  \Pr\{N_1 = n_1, N_2 = n_2\} = \int_0^{+\infty} e^{-\theta (\lambda_1 + \lambda_2)} \frac{(\theta \lambda_1)^{n_1}(\theta \lambda_2)^{n_2}}{n_1!n_2!} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta \theta} d\theta,
  \]
  with parameters $\alpha > 0, \beta > 0, \lambda_1 > 0, \lambda_2 > 0$. The probability generating function for the number of claims in this model is
  \[
  \pi(x_1, x_2) = \left( \frac{\beta}{\beta - \lambda_1(x_1 - 1) - \lambda_2(x_2 - 1)} \right)^\alpha,
  \]
  i.e. $(N_1, N_2)$ follows a bivariate negative binomial.
Example

We set the following parameter values:

\[ a_1 = 3, \quad a_2 = 4, \quad b_1 = \frac{1}{2}, \quad b_2 = \frac{9}{20}, \]
\[ \lambda_1 = 1, \quad \lambda_2 = 5, \quad \alpha = \beta = 1.89898 \]

\[ E[\hat{Y}_1] = \frac{1}{4}, \quad E[\hat{Y}_2] = \frac{3}{4} \]

which gives the moments

\[ E[N_1] = 1, \quad E[N_2] = 5, \]
\[ Var[N_1] = 1.52660, \quad Var[N_2] = 18.165, \quad Corr(N_1, N_2) = 0.5, \]
\[ E[Y_1] = 0.25, \quad E[Y_2] = 0.15, \quad Var[Y_1] = 0.1875, \quad Var[Y_2] = 0.045, \]
\[ Var[\hat{Y}_1] = 0.282912, \quad Var[\hat{Y}_2] = 0.633712. \]
We assume that

\[ P_i(Z_i) = E[\hat{Z}_i] + 0.3 \sqrt{\text{Var}[\hat{Z}_i]}, \quad i = 1, 2. \]

and that \( c = 1.19919 \).
Table 1: **Parameters of optimal treaties** (adjustment coefficient criterion)

<table>
<thead>
<tr>
<th></th>
<th>Independent $N_1, N_2$</th>
<th>Corr($N_1, N_2$) = 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>best treaties</strong></td>
<td><strong>best XL</strong></td>
<td><strong>best treaties</strong></td>
</tr>
<tr>
<td>$\alpha_{1,1} = 0.487313$</td>
<td>$M_1 = 8.94428$</td>
<td>$\alpha_{1,1} = 0.580562$</td>
</tr>
<tr>
<td>$\alpha_{1,2} = -0.487313$</td>
<td>$M_2 = 15.8155$</td>
<td>$\alpha_{1,2} = -0.540189$</td>
</tr>
<tr>
<td>$\alpha_{2,1} = 0.342036$</td>
<td></td>
<td>$\alpha_{2,1} = 0.231556$</td>
</tr>
<tr>
<td>$\alpha_{2,2} = -0.342036$</td>
<td></td>
<td>$\alpha_{2,2} = -0.229717$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$M_1 = 11.7585$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$M_2 = 21.0894$</td>
</tr>
</tbody>
</table>
Table 2: **Optimal treaties in the independent case vs the dependent case** (adjustment coefficient criterion)

<table>
<thead>
<tr>
<th></th>
<th>Independent $N_1, N_2$</th>
<th>Corr($N_1, N_2$) = 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>best treaties</td>
<td>best XL</td>
</tr>
<tr>
<td>$R$</td>
<td>0.311772</td>
<td>0.284421</td>
</tr>
<tr>
<td>$E[\hat{Z}_1]$</td>
<td>0.037230</td>
<td>0.000701</td>
</tr>
<tr>
<td>$E[\hat{Z}_2]$</td>
<td>0.015349</td>
<td>$3.176 \times 10^{-6}$</td>
</tr>
<tr>
<td>$E[\hat{Z}_1]/E[\hat{Y}_1]$</td>
<td>0.148920</td>
<td>0.002803</td>
</tr>
<tr>
<td>$E[\hat{Z}_2]/E[\hat{Y}_2]$</td>
<td>0.020465</td>
<td>$4.235 \times 10^{-6}$</td>
</tr>
<tr>
<td>$P_1(Z_1)$</td>
<td>0.079324</td>
<td>0.035215</td>
</tr>
<tr>
<td>$P_2(Z_2)$</td>
<td>0.103890</td>
<td>0.004838</td>
</tr>
</tbody>
</table>

If we compute the adjustment coefficient of the retained risk under treaties $Z_1^i, Z_2^i$ (i for independent) but the risks were dependent the answer is $R_{Z_1^i, Z_2^i} = 0.258863$ (versus $R_{Z_1^d, Z_2^d} = 0.262623$).