Optimal *per claim* reinsurance for dependent risks

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Problem

The direct insurer holds a portfolio of $k \geq 1$ risks:

Problem

Considering "all possible" per-claim reinsurance schemes, find the combination of reinsurance policies which maximizes the adjustment coefficient of the retained risk for the whole portfolio.

Previous work

- Guerra, M. and Centeno, M.L. (2008), Ins. Math. Eco. 42 no. 2, 529–539.
  - Relationship between maximization of the adjustment coefficient and maximization of the expected utility of wealth;
  - Existence and uniqueness of solution;
  - Necessary condition for optimality;
  - Characterization of optimal policies for some premium calculation principles.
- Guerra, M. and Centeno, M.L. (submited)
  - Numerical method for computation of optimal solutions;
  - Examples and comparisons.
- Centeno, M.L. and Guerra, M. (submitted).
  - Generalization of the previous results for "per-claim" reinsurance on a single risk;
  - Dependence of the structure of the optimal treaty on the distribution of the number of claims.

Notation

- $k \geq 1$ risks;
- $N_i, \ i = 1, 2, \ldots, k$
  Number of claims of risk i in one year;

$$N = (N_1, N_2, \ldots, N_k)$$

- $Y_{i,j}, \ i = 1, 2, \ldots, k, \ \ j = 1, 2, \ldots, N_i$
  Value of the jth claim of risk i;
- $Z_i(y)$ – Value ceded under the reinsurance policy in force for risk i, given a claim amount y;

Notation

- $\mathcal{Z}_i$ - the set of all possible reinsurance policies for risk i

$$\mathcal{Z}_i = \{\zeta : [0, +\infty[\mapsto \boldsymbol{R} |\ \zeta \text{ is measurable and } 0 \leq \zeta(y) \leq y,\ \forall y \geq 0\},$$

$$\mathcal{Z} = \bigotimes_{i=1}^{k} \mathcal{Z}_i, \qquad Z = (Z_1, Z_2, \ldots, Z_k) \in \mathcal{Z};$$

- $\widehat{Y}_i$ – Aggregate claim amount of risk i in one year,

$$\widehat{Y}_i = \sum_{1 \leq j \leq N_i} Y_{i,j};$$

- $\widehat{Z}_i$ – Aggregate claim amount of risk i ceded under reinsurance policy Z_i,

$$\widehat{Z}_i = \sum_{1 \leq j \leq N_i} Z_i(Y_{i,j});$$

Notation

- $P_i(Z_i)$ – Reinsurance premium for the reinsurance policy Z_i (covering risk i);
- L_Z – Profit obtained in one year assuming the reinsurance policies Z_i, $i = 1, 2, ..., k$:

$$L_Z = c - \sum_{i=1}^{k} P_i\left(Z_i\right) - \sum_{i=1}^{k}\left(\widehat{Y}_i - \widehat{Z}_i\right) =$$

$$= \quad c - \sum_{i=1}^{k} P_i\left(Z_i\right) - \sum_{i=1}^{k}\left(\sum_{1\leq j\leq N_i} Y_{i,j} - \sum_{1\leq j\leq N_i} Z_i\left(Y_{i,j}\right)\right).$$

Assumptions

A1 $\Pr\{L_Z < 0\} > 0$ holds for every $Z \in \mathcal{Z}$;

A2 For each $i \in \{1, 2, \ldots, k\}$, all $Y_{i,j}$, $j \in \boldsymbol{N}$, are i.i.d. nonnegative continuous random variables with common density function f_i.

Y_i – generic r.v. with density f_i;

A3 $N = (N_1, N_2, \ldots, N_k)$ is an array of integer random variables with joint probability function

$$p(n) = p(n_1, n_2, \ldots, n_k) = \Pr\{N_1 = n_1, N_2 = n_2, \ldots, N_k = n_k\};$$

The moment-generating function of p is finite in some neighborhood of zero;

Assumptions

A4 The random variables Y_i, $i = 1, 2, \ldots, k$ are mutually independent and independent of the random vector N;

A5 All functionals $P_i : \mathcal{Z}_i \mapsto [0, +\infty]$ are convex, $P_i(0) = 0$, and are continuous in the mean-squared sense:

$$\lim_{m \to \infty} \int_0^{+\infty} \left(Z_{i,m}(y) - Z_i(y)\right)^2 f_i(y)\, dy = 0$$

implies

$$\lim_{m \to \infty} P_i\left(Z_{i,m}\right) = P_i\left(Z_i\right).$$

We do not distinguish between functions $Z_i, Z_i' \in \mathcal{Z}_i$ which differ only on a set of zero probability with respect to the density f_i.

Problem

- $G(R, Z) = E\left[e^{-RL_Z}\right], \qquad R \in [0, +\infty),\ Z \in \mathcal{Z}$
- R_Z - Adjustment coefficient of the retained risk for a particular combination of policies. R_Z solves equation

$$G(R, Z) = 1 \tag{1}$$

- $\mathcal{Z}^+ = \{Z \in \mathcal{Z} : (1) \text{ admits a positive solution}\}$

Problem

Find $(R^*, Z^*) \in (0, +\infty) \times \mathcal{Z}^+$ *such that*

$$R^* = R_{Z^*} = \max\left\{R_Z : Z \in \mathcal{Z}^+\right\}.$$

The combination of policies $Z^* \in \mathcal{Z}^+$ is said to be **optimal for the adjustment coefficient criterion** if (R_{Z^*,Z^*}) solves this problem.

Maximization of the expected utility of wealth

Exponential utility function with coefficient of risk aversion $R > 0$:

$$U_R(w) = -e^{-Rw}.$$

$$E[U_R(L_Z)] = -G(R, Z).$$

Problem

Find $Z^* \in \mathcal{Z}$, such that

$$E[U_R(L_{Z^*})] = \max\{E[U_R(L_Z)] : Z \in \mathcal{Z}\}.$$

Here $R > 0$ is a given constant (fixed).

A combination of policies $Z^* \in \mathcal{Z}$ is said to be **optimal for the expected utility criterion** with coefficient of risk aversion R if it solves this problem for that particular R.

Expected utility vs adjustment coefficient

Proposition

A pair $(R^*, Z^*) \in (0, +\infty) \times \mathcal{Z}^+$ *solves the adjustment coefficient problem if and only if it satisfies the following conditions:*

1. Z^* *is optimal for the expected utility criterion with coefficient of risk aversion* $R = R^*$;
2. $G(R^*, Z^*) = 1$.

The optimal policy for the adjustment coefficient criterion can be found in two steps:

1. For each $R \in (0, +\infty)$ find Z_R, the respective optimal policy for the expected utility criterion (find $Z_R = \arg\min \{G(R, Z) : Z \in \mathcal{Z}\}$);
2. Solve the equation with one single real variable

$$G(R, Z_R) = 1.$$

Existence and uniqueness of optimal policy

Theorem

For each $R \in (0, +\infty)$ there exists an optimal policy for the expected utility criterion.
There exists an optimal policy for the adjustment coefficient criterion.
If

- $\Pr\{N_i = m\} = 1$ *holds for some* $i \in \{1, 2, \ldots, k\}$, $m \in \boldsymbol{N}$ *and*
- $Z^* = (Z_1^*, Z_2^*, \ldots, Z_k^*) \in \mathcal{Z}$ *is optimal,*

the policy $Z_i^* \in \mathcal{Z}_i$ *can be replaced by a policy* $Z_i' \in \mathcal{Z}_i$ *if and only if*

$$\Pr\left\{ Z_i'(Y_i) - Z_i^*(Y_i) = \frac{P_i(Z_i') - P_i(Z_i^*)}{m} \right\} = 1.$$

If $\Pr\{N_i = m\} < 1$, $\forall i \in \{1, 2, \ldots, k\}, m \in \boldsymbol{N}$, *then the optimal combination of policies is unique.*

Needle-like perturbations

Fix $Z = (Z_1, Z_2, \ldots, Z_k) \in \mathcal{Z}$.

$$Z_i^{\alpha,v,\varepsilon}(y) = \begin{cases} Z_i(y), & \text{if } y \notin [v, v+\varepsilon]; \\ \alpha, & \text{if } y \in [v, v+\varepsilon]. \end{cases}$$

$$\Delta P_{i,Z_i}(v) = \lim_{\alpha \to Z_i(v),\ 0 \le \alpha \le v} \lim_{\varepsilon \to 0^+} \frac{P_i\left(Z_i^{\alpha,v,\varepsilon}\right) - P_i\left(Z_i\right)}{\varepsilon\left(\alpha - Z_i(v)\right)}, \qquad i = 1, 2, \ldots, k$$

Necessary optimality conditions

Theorem

Let $Z \in \mathcal{Z}$ be optimal for the expected utility criterion.
There exist constants $C_i \in (0, +\infty)$, $i = 1, 2, \ldots, k$ such that:

$$\begin{cases} \Delta P_{i,Z_i}(y) \geq C_i e^{Ry} f_i(y), & \text{when } Z_i(y) = 0; \\ \Delta P_{i,Z_i}(y) = C_i e^{R(y - Z_i(y))} f_i(y), & \text{when } 0 < Z_i(y) < y; \\ \Delta P_{i,Z_i}(y) \leq C_i f_i(y), & \text{when } Z_i(y) = y. \end{cases}$$

$$C_i \geq \frac{\frac{\partial}{\partial x_i} \pi \left(E\left[e^{R(Y_1 - Z_1)}\right], E\left[e^{R(Y_2 - Z_2)}\right], \ldots, E\left[e^{R(Y_k - Z_k)}\right] \right)}{\pi \left(E\left[e^{R(Y_1 - Z_1)}\right], E\left[e^{R(Y_2 - Z_2)}\right], \ldots, E\left[e^{R(Y_k - Z_k)}\right] \right)}, \tag{2}$$

with equality in (2) if there exists some $x_i > E\left[e^{R(Y_i - Z_i)}\right]$ such that

$$\pi \left(E\left[e^{R(Y_1 - Z_1)}\right], \ldots, x_i, \ldots, E\left[e^{R(Y_k - Z_k)}\right] \right) < +\infty. \tag{3}$$

Necessary optimality conditions

Corollary

Let $Z \in \mathcal{Z}$ be optimal for the expected utility criterion, and suppose there exists $\tilde{Z}_i \in \mathcal{Z}_i$ such that:

$$\Pr\left\{\tilde{Z}_i \leq Z_i\right\} = 1, \qquad \Pr\left\{\tilde{Z}_i < Z_i\right\} > 0,$$
$$E\left[U_R\left(L_{(Z_1,\ldots,Z_{i-1},\tilde{Z}_i,Z_{i+1},\ldots,Z_k)}\right)\right] > -\infty.$$

Then, the following holds with probability equal to one:

$$\begin{cases} \Delta P_{i,Z_i}(y) \geq \frac{E[U_R(L_Z)|\ \exists m \in \{1,\ldots,N_i\}: Y_{i,m}=y]}{E[U_R(L_Z)]} f_i(y), & \text{when } Z_i(y) = 0; \\ \Delta P_{i,Z_i}(y) = \frac{E[U_R(L_Z)|\ \exists m \in \{1,\ldots,N_i\}: Y_{i,m}=y]}{E[U_R(L_Z)]} f_i(y), & \text{when } 0 < Z_i(y) < y; \\ \Delta P_{i,Z_i}(y) \leq \frac{E[U_R(L_Z)|\ \exists m \in \{1,\ldots,N_i\}: Y_{i,m}=y]}{E[U_R(L_Z)]} f_i(y), & \text{when } Z_i(y) = y. \end{cases}$$

The expected value principle

Suppose the premium calculation principle for risk i is the expected value principle:

$$P_i\left(Z_i\right) = \left(1+\beta_i\right) E[\hat{Z}_i] = \left(1+\beta_i\right) E\left[N_i\right] E\left[Z_i\right],$$

Corollary

The optimal treaty for the risk i is an excess of loss contract:

$$Z_i(y) = \begin{cases} 0, & \text{if } y \leq M_i; \\ y - M_i, & \text{if } y \geq M_i. \end{cases}$$

If (3) holds: $M_i = \frac{1}{R} \ln \frac{(1+\beta_i) E[N_i] \pi\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)}{\frac{\partial \pi}{\partial x_i}\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)}$;

If (3) fails: $M_i \leq \frac{1}{R} \ln \frac{(1+\beta_i) E[N_i] \pi\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)}{\frac{\partial \pi}{\partial x_i}\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)}$.

Variance-related principles

Suppose the premium calculation principle for risk i is a variance-related principle:

$$\begin{aligned} P_i\left(Z_i\right) &= E[\hat{Z}_i] + g_i\left(Var[\hat{Z}_i]\right) = \\ &= E[N_i]E[Z_i] + g_i\left(E[N_i]Var[Z_i] + Var[N_i]E[Z_i]^2\right), \end{aligned}$$

$g_i : [0, +\infty) \mapsto [0, +\infty)$ continuous, smooth in $(0, +\infty)$,

$$g_i(0) = 0, \qquad g_i'(t) > 0, \quad \forall t > 0.$$

Lemma

$P_i : \mathcal{Z}_i \mapsto [0, +\infty)$ is convex iff g_i satisfies

$$\frac{g_i''(t)}{g_i'(t)} \geq -\frac{1}{2t}, \qquad \forall t \in (0, B),$$

$B = \sup\left\{Var[\hat{Z}_i] : Z_i \in \mathcal{Z}_i\right\}$.

Variance-related principles

Corollary

If g_i' is bounded in a neighborhood of zero, then there are constants $\alpha_{i,1} > 0$, $\alpha_{i,2} \in \boldsymbol{R}$ such that the optimal treaty for the risk i satisfies

$$y \leq \frac{1}{R} \ln \frac{-\alpha_{i,2}}{\alpha_{i,1}}, \text{when } Z_i(y) = 0;$$
$$y = Z_i(y) + \frac{1}{R} \ln \frac{Z_i(y) - \alpha_{i,2}}{\alpha_{i,1}}, \text{when } 0 < Z_i(y) < y;$$
$$y \leq \alpha_{i,1} + \alpha_{i,2}, \text{when } Z_i(y) = y.$$

If g_i' is unbounded in any neighborhood of zero, then the optimal treaty must be either a function of the type above or $Z_i \equiv 0$ (no reinsurance at all).

Variance-related principles

If (3) holds: $\alpha_{i,1} = \frac{\frac{\partial \pi}{\partial x_i}\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)}{E[N_i]\pi\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)2g_i'\left(Var[\hat{Z}_i]\right)}$;

If (3) fails: $\alpha_{i,1} \geq \frac{\frac{\partial \pi}{\partial x_i}\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)}{E[N_i]\pi\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)2g_i'\left(Var[\hat{Z}_i]\right)}$;

$$\alpha_{i,2} = \frac{E[N_i]-Var[N_i]}{E[N_i]}E[Z_i] - \frac{1}{2g_i'\left(Var[\hat{Z}_i]\right)}.$$

Example:

We consider an insurance company dealing with two risks.

- $Y_i \sim Pareto(a_i, b_i)$, i.e.

$$F_i(y) = 1 - \left(\frac{b_i}{b_i + y}\right)^{a_i}, \quad i = 1, 2$$

with parameters $a_i > 2$, $b_i > 0$.

- $$\Pr\{N_i = n | \Theta = \theta\} = e^{-\theta\lambda_i} \frac{(\theta\lambda_i)^n}{n!}, \qquad i = 1, 2;$$

$$\Pr\{N_1 = n_1, N_2 = n_2\} = \int_0^{+\infty} e^{-\theta(\lambda_1+\lambda_2)} \frac{(\theta\lambda_1)^{n_1}(\theta\lambda_2)^{n_2}}{n_1! n_2!} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta,$$

with parameters $\alpha > 0$, $\beta > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$. The probability generating function for the number of claims in this model is

$$\pi(x_1, x_2) = \left(\frac{\beta}{\beta - \lambda_1(x_1 - 1) - \lambda_2(x_2 - 1)}\right)^\alpha,$$

i.e. (N_1, N_2) follows a bivariate negative binomial.

Example

We set the following parameter values:

$$a_1 = 3, \qquad a_2 = 4, \qquad b_1 = \frac{1}{2}, \qquad b_2 = \frac{9}{20},$$
$$\lambda_1 = 1, \qquad \lambda_2 = 5, \qquad \alpha = \beta = 1.89898$$
$$E[\hat{Y}_1] = \frac{1}{4}, \qquad E[\hat{Y}_2] = \frac{3}{4}$$

which gives the moments

$$E[N_1] = 1, \qquad E[N_2] = 5,$$
$$Var[N_1] = 1.52660, \qquad Var[N_2] = 18.165, \qquad \mathrm{Corr}(N_1, N_2) = 0.5,$$
$$E[Y_1] = 0.25, \quad E[Y_2] = 0.15, \quad \mathrm{Var}[Y_1] = 0.1875, \quad \mathrm{Var}[Y_2] = 0.045,$$
$$\mathrm{Var}[\hat{Y}_1] = 0.282912, \qquad \mathrm{Var}[\hat{Y}_2] = 0.633712.$$

Example

We assume that

$$P_i(Z_i) = E[\hat{Z}_i] + 0.3\sqrt{\mathrm{Var}[\hat{Z}_i]}, \qquad i = 1, 2.$$

and that $c = 1.19919$.

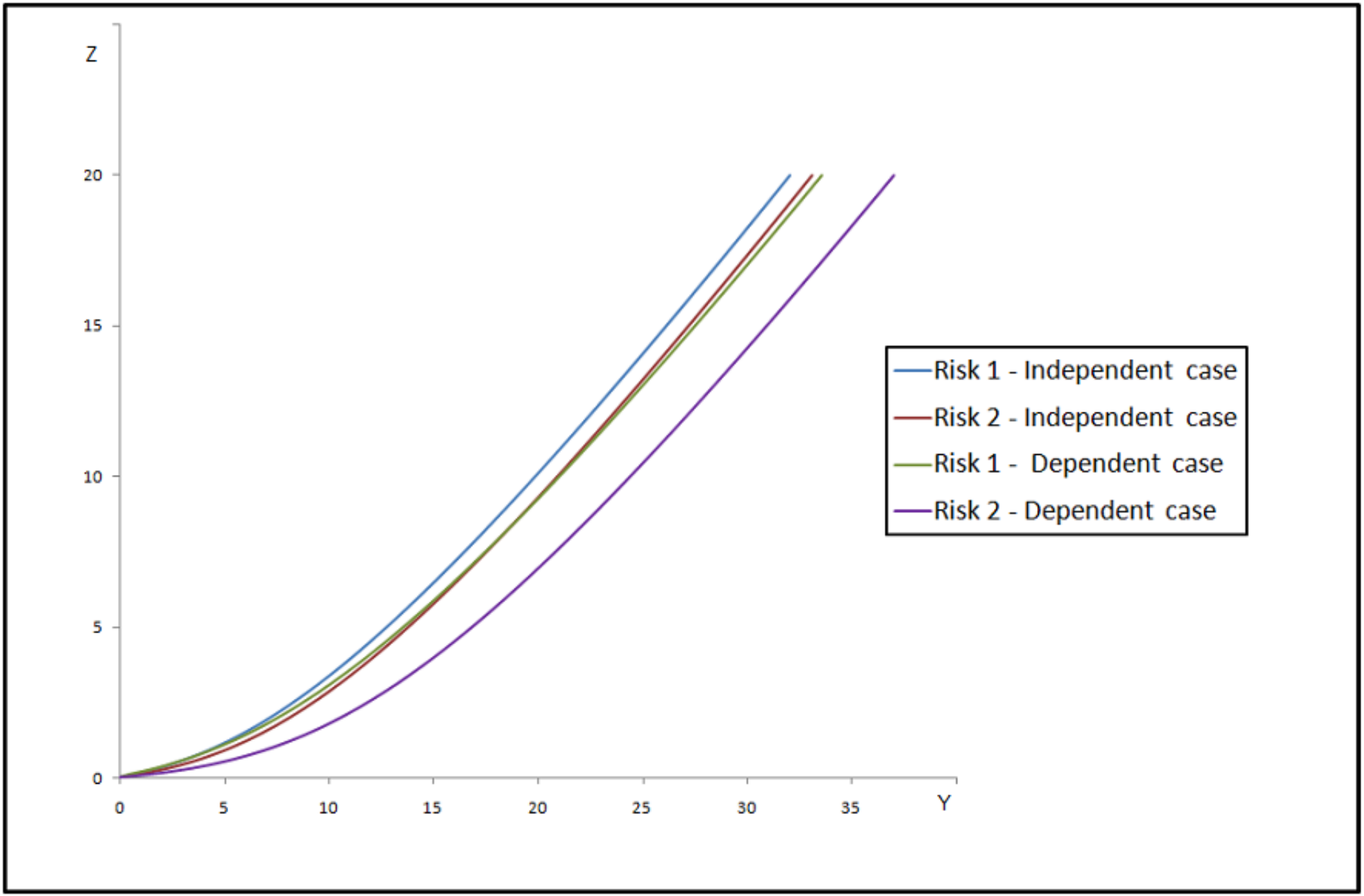
Table 1: **Parameters of optimal treaties** (adjustment coefficient criterion)

Independent N_1, N_2		$\mathrm{Corr}(N_1, N_2) = 0.5$	
best treaties	best XL	best treaties	best XL
$\alpha_{1,1} = 0.487313$ $\alpha_{1,2} = -0.487313$ $\alpha_{2,1} = 0.342036$ $\alpha_{2,2} = -0.342036$	$M_1 = 8.94428$ $M_2 = 15.8155$	$\alpha_{1,1} = 0.580562$ $\alpha_{1,2} = -0.540189$ $\alpha_{2,1} = 0.231556$ $\alpha_{2,2} = -0.229717$	$M_1 = 11.7585$ $M_2 = 21.0894$

Table 2: **Optimal treaties in the independent case vs the dependent case** (adjustment coefficient criterion)

	Independent N_1, N_2		$\mathrm{Corr}(N_1, N_2) = 0.5$	
	best treaties	best XL	best treaties	best XL
R	0.311772	0.284421	0.262623	0.238882
$E[\hat{Z}_1]$	0.037230	0.000701	0.067591	0.000416
$E[\hat{Z}_2]$	0.015349	3.176×10^{-6}	0.010809	1.368×10^{-6}
$\frac{E[\hat{Z}_1]}{E[\hat{Y}_1]}$	0.148920	0.002803	0.270363	0.001664
$\frac{E[\hat{Z}_2]}{E[\hat{Y}_2]}$	0.020465	4.235×10^{-6}	0.014412	1.824×10^{-6}
$P_1(Z_1)$	0.079324	0.035215	0.113004	0.030710
$P_2(Z_2)$	0.103890	0.004838	0.072157	0.003648

If we compute the adjustment coefficient of the retained risk under treaties Z_1^i, Z_2^i (i for independent) but the risks were dependent the answer is $R_{Z_1^i, Z_2^i} = 0.258863$ (versus $R_{Z_1^d, Z_2^d} = 0.262623$).


Z
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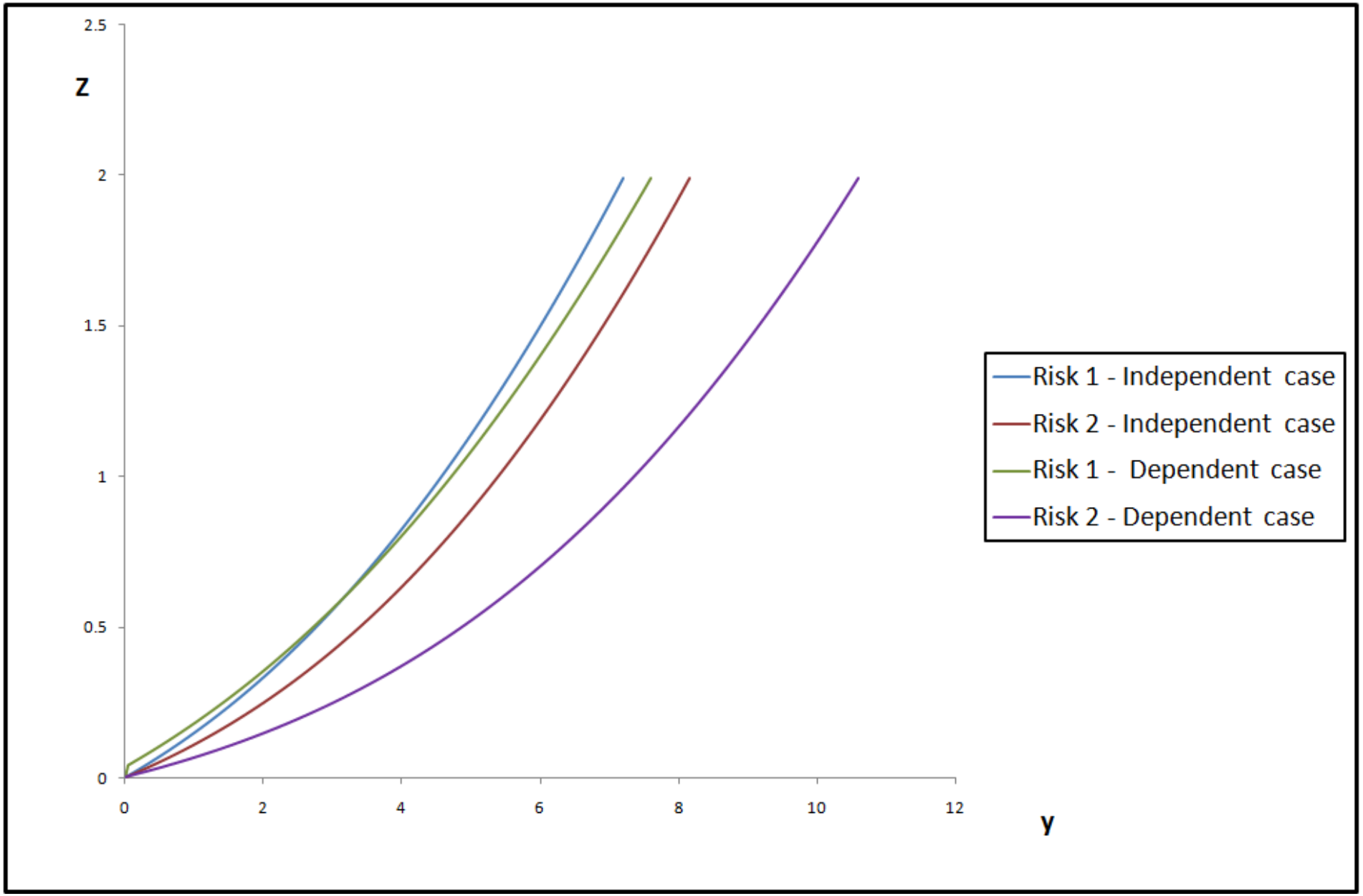
Y
Risk 1 - Independent case
Risk 2 - Independent case
Risk 1 - Dependent case
Risk 2 - Dependent case


Z
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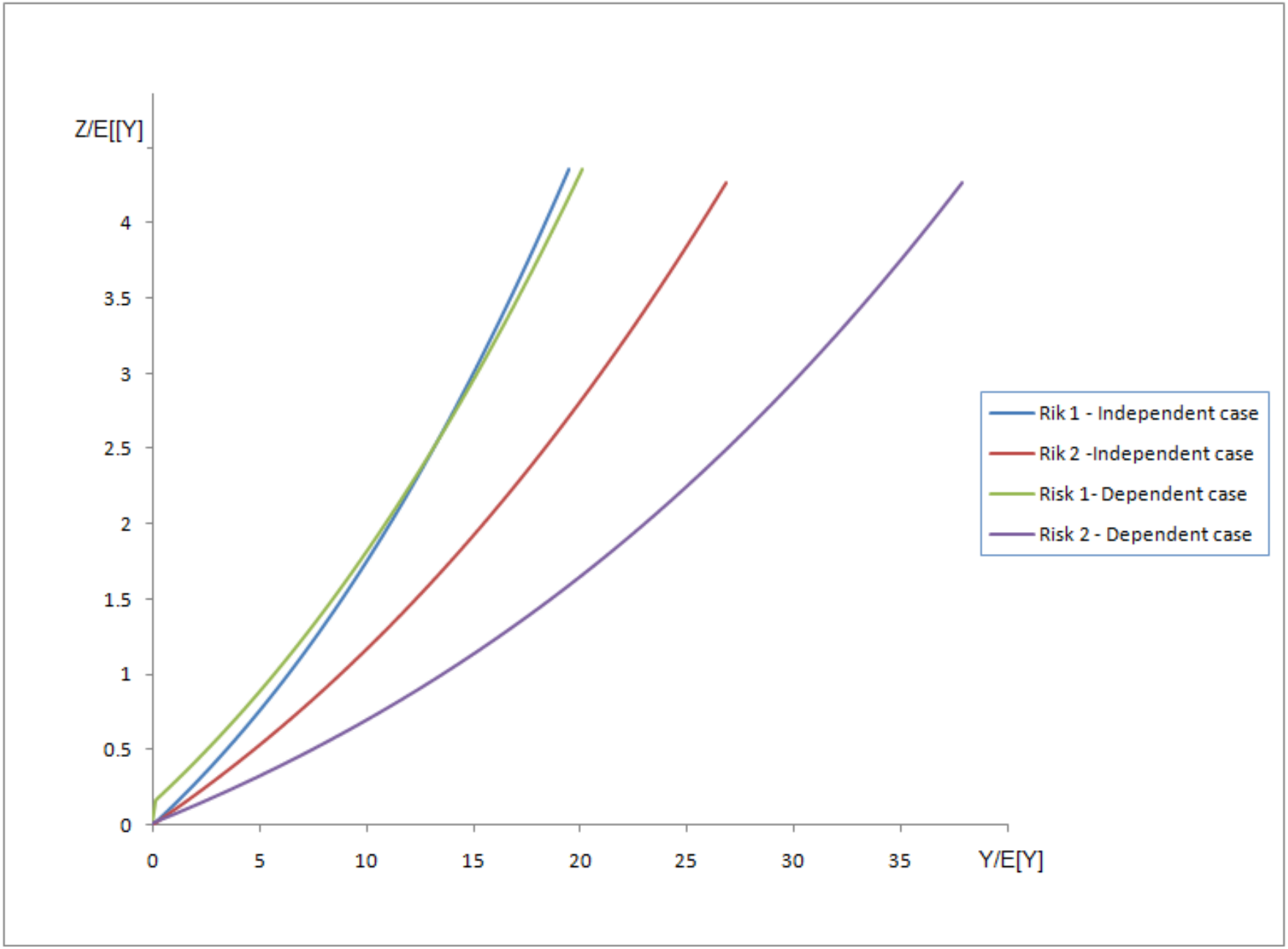
Risk 1 - Independent case
Risk 2 - Independent case
Risk 1 - Dependent case
Risk 2 - Dependent case


Z/E[[Y]
4
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35
Y/E[Y]
Rik 1 - Independent case
Rik 2 -Independent case
Risk 1- Dependent case
Risk 2 - Dependent case