Pricing of CDO's Based on the Multivariate Wang Transform*

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Plan of my Talk

- ✓ Pricing Principles for Insurance Risk

  Esscher transform, Wang Transform

- ✓ Extension to the Multivariate Setting
- ✓ The Pricing of CDO’s

  Image of CDO, Synthetic CDO, Standard Model

- ✓ Proposed Models for the Pricing of CDO’s
- ✓ Numerical Examples

Premium Principles for Insurance Risk

In the actuarial literature, a popular method for the pricing of financial and insurance risks, among others, is the Esscher transform: $\pi(X) = E[X\mathrm{e}^{-\delta X}] / E[\mathrm{e}^{-\delta X}]$ where $\delta > 0$ stands for risk adjustment.

Recently, Wang (2002, ASTIN) proposed a pricing method based on the following transformation:

$$F^Q(x) = \Phi[\Phi^{-1}(F(x)) + \delta], \quad \delta > 0$$

The distortion $F \to F^Q$ describes risk adjustment.

Note: Insurance markets are *incomplete* and exhibit fat-tailed distributions, as for CDO markets.

Recall that

Esscher: $$\pi(X) = \frac{E[X\mathrm{e}^{-\delta X}]}{E[\mathrm{e}^{-\delta X}]}, \quad \delta > 0$$

Wang: $$\pi(X) = \int x \mathrm{d}F^{Q}(x); \ F^{Q}(x) = \Phi[\Phi^{-1}(F(x)) + \delta]$$

Main drawback of the Esscher transform for the practical use (in the discrete-time setting) is that the MGF $E[\mathrm{e}^{-\delta X}]$ must exist (counterpart of Novikov's condition).

The Wang transform has a merit in this aspect.

Also, Wang (2002, ASTIN) showed that the transform is the only transform among the family of distortions that can recover CAPM and the Black-Scholes formula for options.

The Buhlmann's equilibrium pricing model (1980, ASTIN):

- pure risk-exchange economy
- exponential utility with distinct risk aversion index
- agent *j* faces a risk of potential loss X_j

He derived the following equilibrium pricing formula for risk Y

$$\pi(Y) = E[\eta Y], \quad \eta = \frac{e^{-\lambda Z}}{E[e^{-\lambda Z}]}$$

where $Z = \sum_{j=1}^{n} X_j, \quad \lambda^{-1} = \sum_{j=1}^{n} \lambda_j^{-1}$

Z is the aggregate risk and $\lambda > 0$ is the risk aversion index of the representative agent in the market.

The Esscher transform can be reduced from the Buhlmann's formula by assuming $Y << Z$, whence it has a *sound* economic interpretation.

It can be shown that the Wang transform is the same as the Esscher transform for normally distributed risks.

Moreover, Wang (2003, ASTIN) showed that his transform can be derived from Buhlmann's formula even for general risk under some assumptions on *Z*. Hence, the Wang transform also has a *sound* economic interpretation.

However, actuarial pricing formulas (including the Esscher and Wang) are not linear, yielding arbitrage opportunities.

$$\pi(aX + bY) \neq a\pi(X) + b\pi(Y)$$

$$\Downarrow$$

Kijima (2006, ASTIN) developed a multivariate Wang transform based on the Buhlmann's equilibrium pricing formula.

The multivariate Wang transform

$$F^{Q}(x_1,\ldots,x_n)=\Phi_n(y_1,\ldots,y_n) \quad \text{:CDF of } N_n(0,\Sigma_\rho)$$

$$y_j=\Phi^{-1}[F_j(x_j)]+\sum_{k=1}^{n}\lambda_k\rho_{kj}$$

$$\Sigma_\rho=(\rho_{kj}),\quad \lambda_j=\lambda\sigma_Z w_j$$

Risk aversion index of the representative agent

<u>The normal case</u>:

In this case, we have

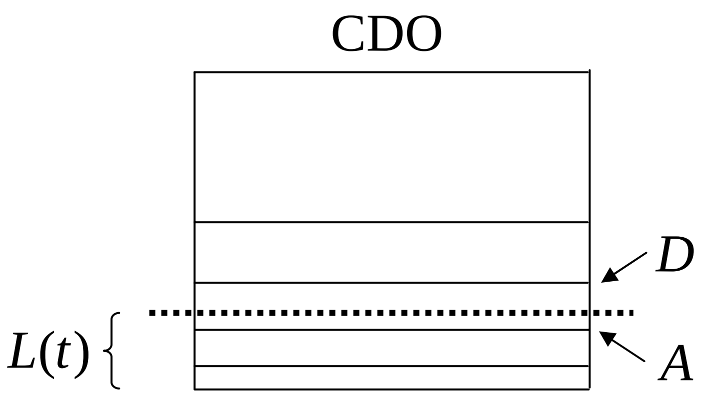
$$w_j=\frac{\sigma_j}{\sigma_Z},\ \lambda_j=\lambda\sigma_j,\ \sum_{k=1}^{n}\lambda_k\rho_{kj}=\frac{\lambda}{\sigma_j}C(X_j,Z)$$

and $\quad F_j(x_j)=\Phi((x_j-\mu_j)/\sigma_j)$

$$\therefore\ F^{Q}(x_1,\ldots,x_n)=\Phi_n(y_1,\ldots,y_n),\ y_j=[x_j-\mu_j+\lambda C(X_j,Z)]/\sigma_j$$

Wang changes $\quad \mu_j \to \mu_j-\overbrace{\lambda C(X_j,Z)}^{\text{risk premium}}$

Pricing of Tranches



Note: We assume that the recovery is zero.

$$M_t = (D-A)1_{\{L(t)<A\}} + (D-L(t))1_{\{A \le L(t) < D\}}$$

Remaining Principal at time t for the tranche

$$B(t) = \mathrm{e}^{\int_0^t r_s \mathrm{d}s}$$: Money Market Account

$$V = E^Q\left[\int_0^T \frac{cM_t}{B(t)}\mathrm{d}t + \frac{M_T}{B(T)}\right]$$

The present value of cash flows arising from the tranche.

Standard Model: One-Factor Gaussian Copula Model

In general,

Assumed to be constant

$$L(t) = \sum_{i=1}^{n} M_i N_i(t), \quad M_i = (1-\delta_i)F_i, \quad N_i(t) = 1_{\{\tau_i \le t\}}$$

We need to model $(\tau_1, \tau_2 \ldots, \tau_n)$ in the *bottom-up* setting.

Consider the *Merton's structural model*:

$$\tau_i \le t \;\Leftrightarrow\; X_i(t) \le x, \quad X_i(t) = \log V_i(t)$$

where V stands for the firm value.

So, instead of modeling the correlated $(\tau_1, \tau_2 \ldots, \tau_n)$

it should be easier to model the correlated $(X_1, X_2, \ldots, X_n)$

Take $(X_1, X_2, \ldots, X_n)$ to be latent variables such that:

$$X_i = \rho_i U + \sqrt{1-\rho_i^2}\, U_i, \quad U \perp U_i \qquad \text{under } Q$$

where U_i has the CDF $H(x)$, U has $G(x)$, and X_i has $K(x)$

By definition, $F_i(t) = Q\{\tau_i \le t\} = Q\{X_i \le x\} = K(x)$

Supposing $\rho_i = \rho$ for all i, (to be calibrated)

Example 1 (The industry standard model): U's follow a normal. Then $K(x)$ is also normal, and $\tau_i = F_i^{-1}(\Phi(X_i))$

Example 2 (Hull and White, 2004): U's follow a t distribution. Then $K(x)$ needs to be evaluated numerically

Change of Measures from P to Q

As for CreditMetrics, we start with assuming that under P

$$X_i = \rho_i U + \sqrt{1-\rho_i^2} U_i, \;\; U \perp U_i : \text{follow } N(0,1)$$

$$Z = \sum_{j=1}^{n} X_j \;\; : \text{Aggregate risk}$$

$$\Rightarrow \; X_i \sim N(0,1), \; C(X_i, X_j) = \rho_i \rho_j, \; C_i \equiv C(X_i, Z) = 1 - \rho_i^2 + \rho_i \sum_{j=1}^{n} \rho_j$$

Apply the multivariate Wang.

$$K^Q(x) = \Phi_{n:\Sigma_\rho}(x_1 + \lambda C_1, \ldots, x_n + \lambda C_n), \;\; \Sigma_\rho = (\rho_i \rho_j)$$

$$\Leftrightarrow \; (X_1^*, \ldots, X_n^*) =^d (X_1 - \lambda C_1, \ldots, X_n - \lambda C_n)$$

risk-adjusted log-firm values

By definition, $F_i^Q(t) = Q\{\tau_i \le t\} = Q\{X_i^* \le x\} = \Phi(x + \lambda C_i)$

$$\therefore F^Q(t) = \Phi_{n:\Sigma_\rho}(\Phi^{-1}(F_1^Q(t_1)), \ldots, \Phi^{-1}(F_n^Q(t_n)))$$

$$F^{Q}(t) = \Phi_{n:\Sigma_\rho}(\Phi^{-1}(F_1^{Q}(t_1)), \ldots, \Phi^{-1}(F_n^{Q}(t_n)))$$

: joint CDF of τ's under Q

$F_i^{Q}(t) = \Phi(x + \lambda C_i)$: marginal CDF of τ_i under Q

risk aversion index

The risk premium is embedded in the marginal default CDF that is calibrated from market quotes for CDS's

Conditional on U in the one-factor model, we have

$$q_i^{Q}(t \mid U) = Q\{\tau_i \le t \mid U\} = Q\{X_i^* \le x \mid U\} = \Phi\left(\frac{\Phi^{-1}(F_i^{Q}(t)) - \rho_i U}{\sqrt{1-\rho_i^2}}\right)$$

By independence, it follows that

$$F_i^{Q}(t) = \int_{-\infty}^{\infty} \left(\prod_{i=1}^{n} q_i^{Q}(t \mid u)\right) \phi(u)\mathrm{d}u$$: the standard model if $\rho_i = \rho$

Derived from Merton's structural model and Buhlmann's principle, whence it has a sound economic interpretation.

Risk-Adjusted Gaussian Copula Model

In the standard model, the correlation $\rho_i = \rho$ is the only free parameter that can be calibrated from market quotes.

The correlation should be thought of as the risk premium.

However, then, what is the default correlation?

Note: In the Gaussian model, the change of measures does not change the variance-covariance structure.

The correlation parameters should be estimated under P and the risk premium is calibrated from market quotes under Q

Fact: The CDO market is segmented into tranches according to investors' preferences against risks.

$F_i^Q(t) = \Phi(x + \underbrace{\lambda C_i}_{})$: marginal CDF of τ_i under Q

risk premium for CDS of name i

Assume: $(X_1^*, \ldots, X_n^*) =^d (X_1 - \lambda C_1 - \underbrace{\lambda_D C_1}_{}, \ldots, X_n - \lambda C_n - \lambda_D C_n)$

The risk premium for tranche with detachment D

When evaluating the tranche, we use

$$F^Q(t) = \Phi_{n:\Sigma_\rho}(\Phi^{-1}(F_1^Q(t_1)) + \lambda_D C_1, \ldots, \Phi^{-1}(F_n^Q(t_n)) + \lambda_D C_n)$$

<u>Parameter estimation and calibration</u>

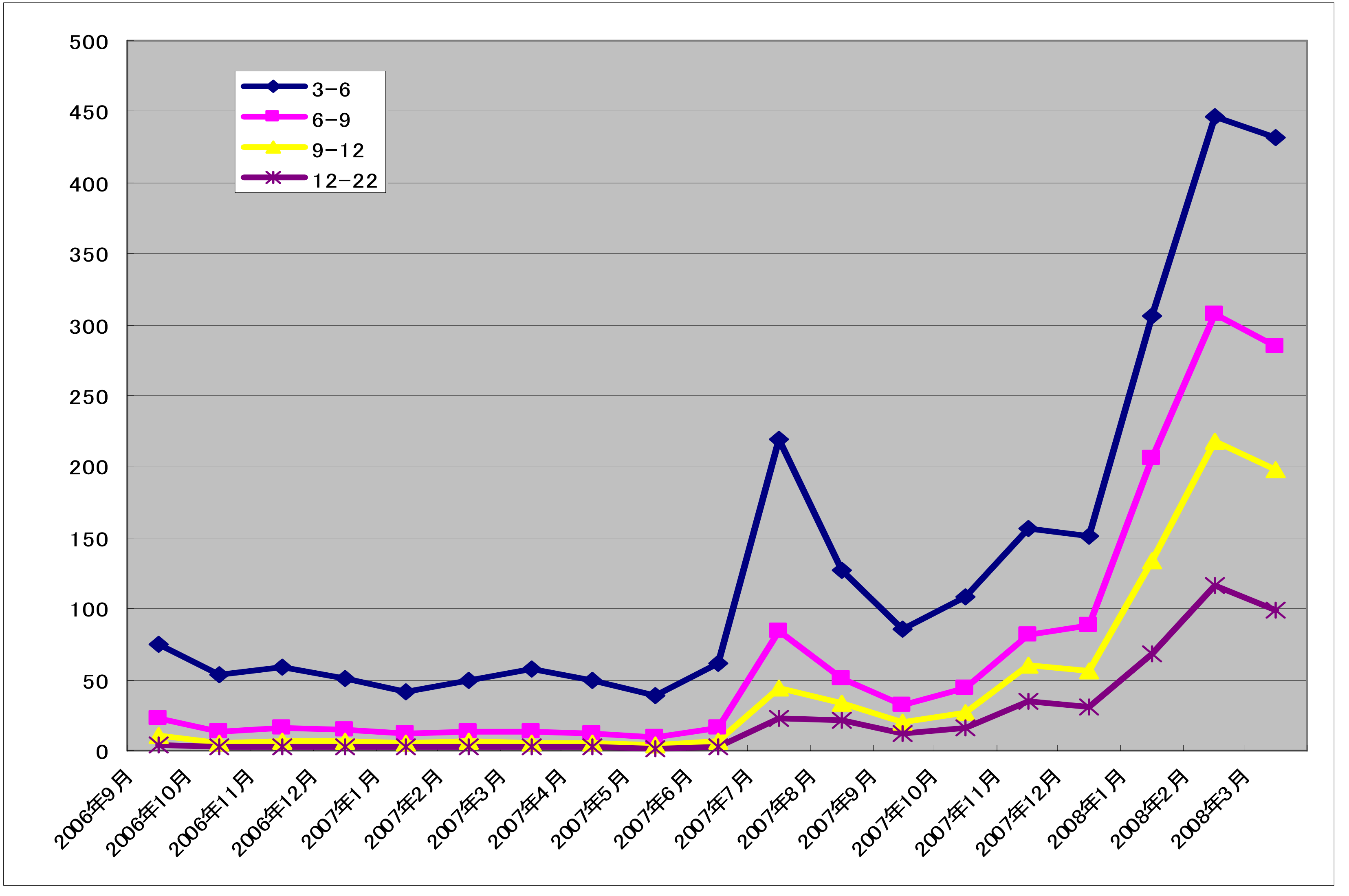
$\Sigma_\rho = (\rho_{ij});\ \rho_{ij} = \rho_i \rho_j$ where $X_i = \rho_i U + \sqrt{1 - \rho_i^2}\, U_i = \rho_i U + \varepsilon_i$

Use the simple regression to estimate ρ_i

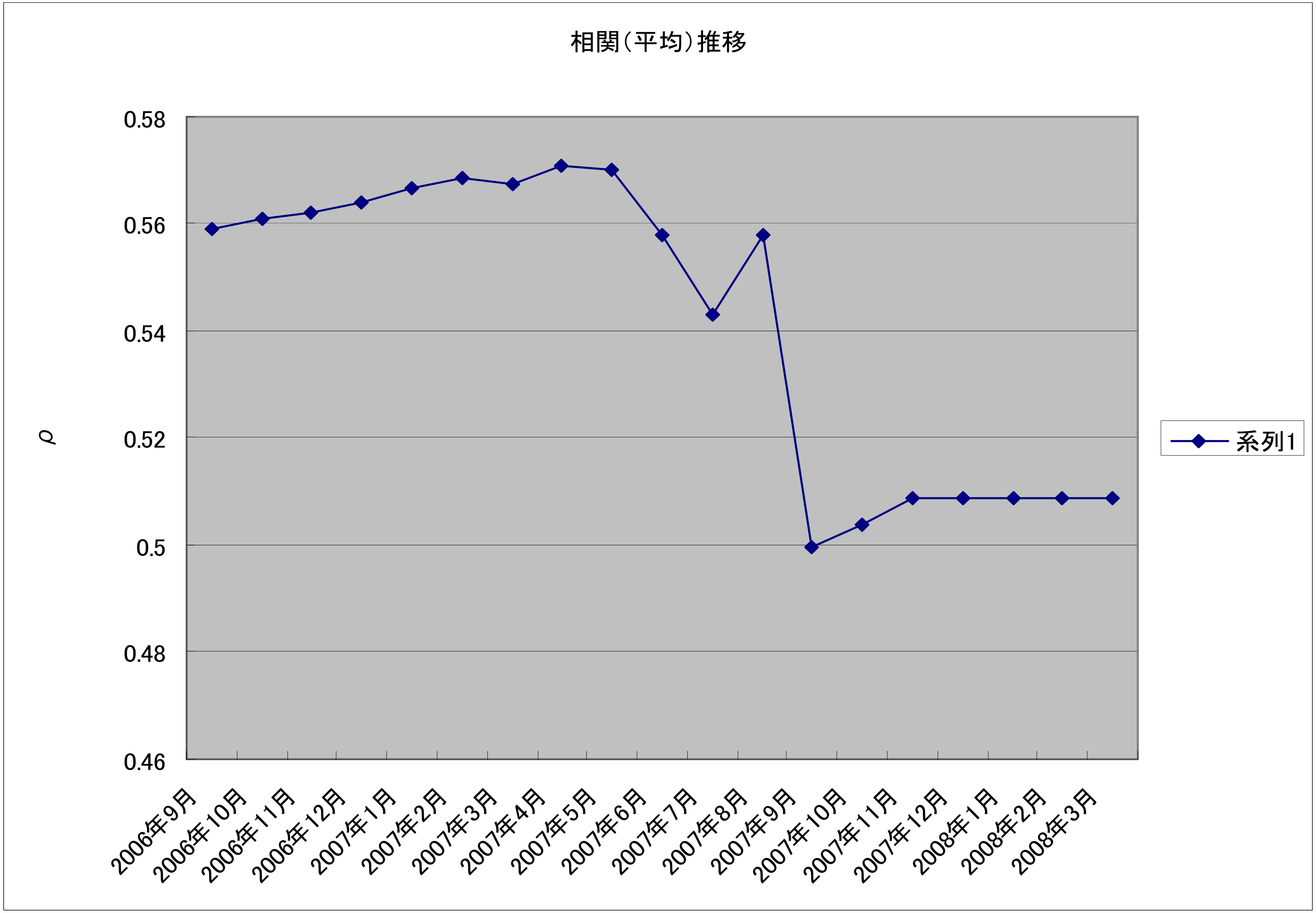
$F_i^Q(t)$: calibrated from market quotes for CDS's

λ_D : calibrated from market quotes for the CDO tranche

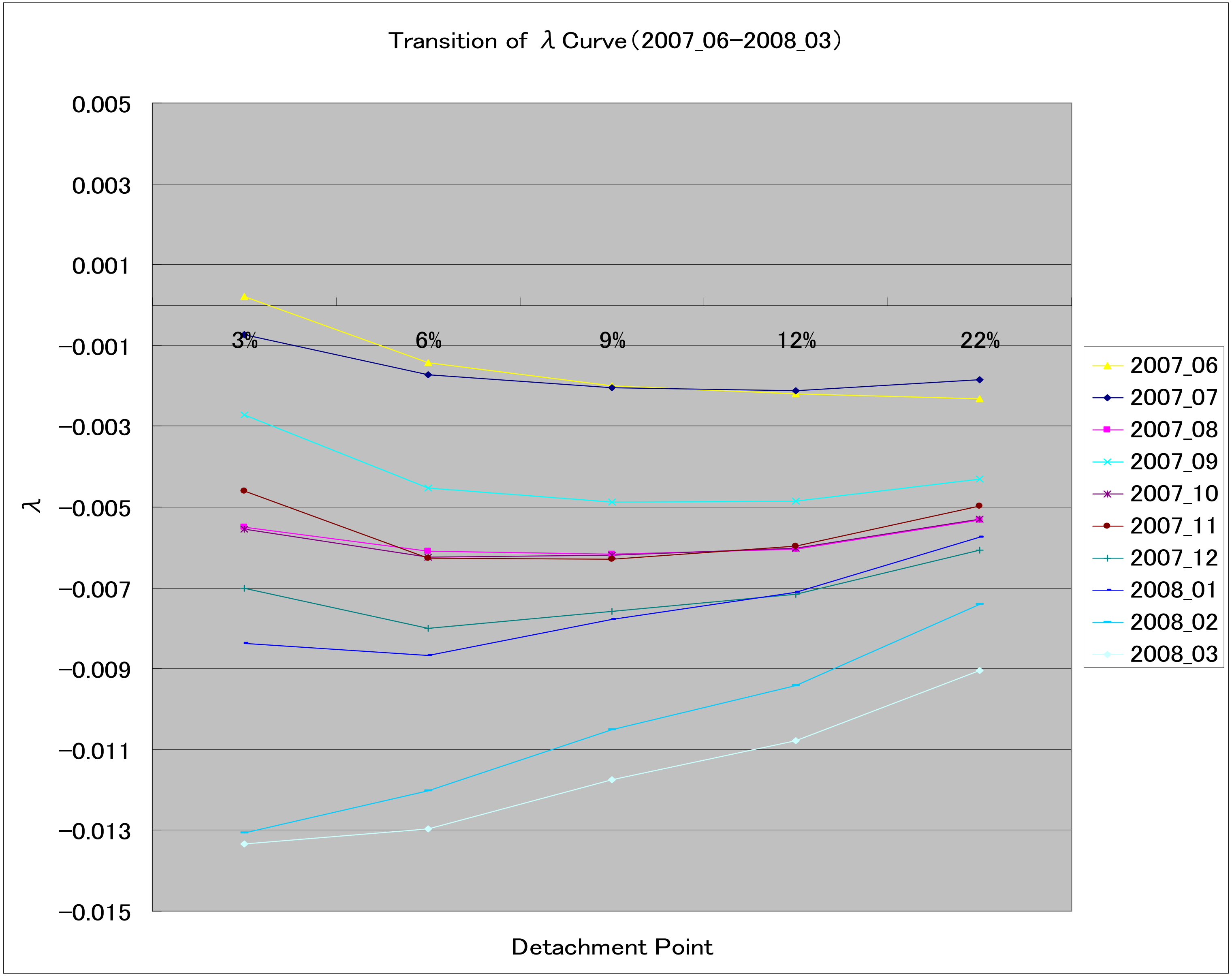
Prices of iTraxx Tranches (since 2006/9)



Averaged Correlation under P (since 2006/9)



Calibrated Risk Premium Curves since 2007/6/30



Risk-Adjusted *t* Copula Model

The risk-adjusted Gaussian model can fit perfectly market quotes for all tranches of standard CDO's by calibrating the risk adjustment parameters λ_D

Note: We have interpreted $\lambda + \lambda_D$ as the risk aversion index for tranche *D* of the representative agent in the market.

⇩

It may be more plausible to assume that λ_D is increasing in *D*

We consider, e.g., the case $\lambda_D = a + b \log D$

Also, financial markets often exhibit fat-tailed distributions.

⇩

✓ Introduce the *t* copula framework

✓ Easy to calculate

Extension by Kijima and Muromachi (2008, IME)

$$F^Q(x) = E_Y[\Phi(\alpha(x)Y + \theta)],\ \theta > 0 \quad \text{: risk premium}$$

where $Y > 0$ and $\alpha(x) = G^{-1}(F(x))$ for some $G(x)$.

In particular, $Y = 1$ implies the Wang transform.

Also, when $Y = \sqrt{\chi_\nu^2 / \nu}$, we have

$$F^Q(x) = E_Y[\Phi(G^{-1}(F(x))Y + \theta)] = P_{\nu:-\theta}[t_\nu^{-1}(F(x))]$$

where, $P_{\nu:\delta}(x)$ denotes the CDF of a non-central t distribution.

However, against our expectation, we can prove

$$\Phi(\Phi^{-1}(F(x)) + \theta) > E_Y[\Phi(G^{-1}(F(x))Y + \theta)],\ \theta \geq 0$$

Therefore, following the Wang's idea, we propose

$$F^Q(x_1, \ldots, x_n) = t_\nu(y_1, \ldots, y_n)$$

$$y_j = \Phi^{-1}[F_j(x_j)] + \sum_{k=1}^{n} \lambda_k \rho_{kj},\ \Sigma_\rho = (\rho_{kj}),\ \lambda_j = \lambda \sigma_Z w_j$$

Pricing of CDO's based on the 2-parameter Wang Transform

The Proposed Model: For $(\tau_1, \tau_2, \ldots, \tau_n)$

$$F^Q(t) = t_{n:\nu,\Sigma_\rho}(\beta), \quad \beta_j = \Phi^{-1}(F_j^Q(t_j)) + \lambda_D C_j \quad \text{under } Q$$

However, this formula involves a double integral.

⇓ Adopt the following approximation

Assume

$$X_i^* = \frac{X_i}{Y_i(\nu)} - (\lambda + \lambda_D) C_i, \; Y_i(\nu) = \sqrt{\chi_{i:\nu}^2 / \nu}, \; X_i = \rho_i U + \sqrt{1 - \rho_i^2}\, U_i$$

Define $\xi_i(u) = \dfrac{U_i + \delta_i(u)}{Y_i(\nu)}, \quad \delta_i(u) = \dfrac{\rho_i u}{\sqrt{1 - \rho_i^2}}$

⇓

$$q_i^Q(t \mid U) = Q\{\tau_i \le t \mid U\} = Q\{X_i^* \le x \mid U\} = P_{\nu:\delta_i(U)}\left(\frac{\Phi^{-1}(F_i^Q(t)) + \lambda_D C_i}{\sqrt{1 - \rho_i^2}} \right)$$

Calibration Results

A-1.DJ iTraxx 5-year Index Tranches(EUR) 2004/8/23

Tranches	0-3%	3-6%	6-9%	9-12%	12-22%	RMSE
Market mid Price	25.5%	146.0	60.3	36.3	7.7	
Bid/ask spread	1.3%	10.0	5.5	5.5	3.5	
Jump-diffusion intensities	25.0%	145.0	58.6	38.1	17.7	0.34
Pure diffusion intensities	30.0%	187.1	27.4	3.5	0.1	5.11
Gaussian copula	27.4%	222.3	52.5	13.8	1.6	4.58
RFL Gaussian copula	25.3%	148.9	52.4	43.4	17.9	0.90
Double-t copula	24.0%	153.4	56.5	32.4	17.4	0.84
Risk-Adjusted t Copula	**25.8%**	**145.2**	**49.1**	**28.4**	**17.1**	**1.15**

Fitted parameters :

$\mu = 3$

$\lambda = 0.0045 \times \ln(\text{Detachment Point}) - 0.0372$

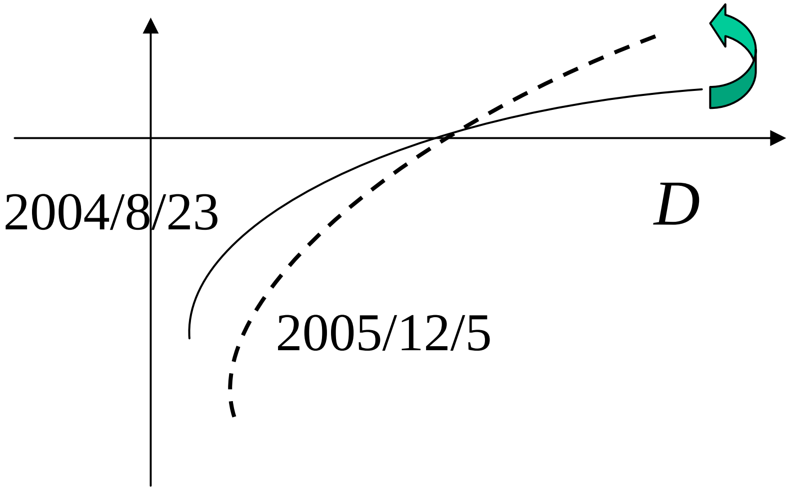
A-2.DJ iTraxx 5-year Index Tranches(EUR) 2005/12/5

Tranches	0–3%	3–6%	6–9%	9–12%	12–22%	RMSE
Market mid Price	26.3%	80.6	23.1	10.3	5.8	
Bid/ask spread	0.6%	3.3	2.6	2.0	1.3	
Jump-diffusion intensities	28.7%	86.3	18.7	14.4	10.4	2.88
Pure diffusion intensities	32.5%	104.3	8.9	0.8	0.0	6.99
Gaussian copula	34.6%	99.9	2.9	0.1	0.0	8.44
RFL Gaussian copula	27.0%	83.2	9.4	7.4	7.3	2.54
Double-t copula	29.8%	101.1	24.4	13.2	6.6	3.99
Risk-Adjusted *t* Copula	**26.5%**	**77.2**	**18.5**	**12.6**	**8.3**	**1.37**

Fitted parameters :

$\mu = 1$

$\lambda = 0.024 \times \ln(\text{Detachment Point}) - 0.3675$



Concluding Remarks

1. We showed that, contrary to the criticism, the one-factor Gaussian copula model is consistent with Buhlmann's economic premium principle, whence it has a sound economic interpretation.

2. Based on this finding, we developed an alternative within the Buhlmann's framework. Namely,
   - we introduce the risk aversion index for each tranche to be calibrated, while keeping the correlation structure as given under the actual probability measure;
   - we also apply the Student t copula.

3. Numerical experiments reveal that our model provide a better fit than the existing models in the literature.

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