Decomposing total risk of a portfolio into the contributions of individual assets

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1. Risk Contributions (RCs)

Meaning, Role, and Advantages

Risk Capital, Risk Budgeting

- Risk Capital

  is distributed from the firm to each business unit, and all the activities of each unit are based on its Risk Capital.

- Basic strategy is

  " Risk for the business unit ≤ Risk Capital for the unit "

  - Firm controls the total risk by allocating Risk Capitals to each business unit. ～ Risk Budgeting

- *How to distribute total risk to each unit ?*
- *Where is the most risky part in the portfolio ?*

"Diversification Effect" must be considered in calculation !

Appropriate measure for Risk Capital ?

What is the measure for Risk Capital of each business unit?

× Simple statistics as a single asset.

- *Standard Deviation … Not used in risk management*
- *VaR (Value at Risk), ES (Expected Shortfall)*
  … "Diversification Effect" is Not considered in them.

- There are 2 candidates (proposed about 10 years before).
  - *Marginal Risk*
  - *Risk Contribution … discussed in this speech.*

Risk Contribution

- Risk Contribution （ abbreviated by RC ）

$$RC_j \equiv a_j \frac{\partial R_P}{\partial a_j}$$

a_j : holding amount of asset j

R_P : risk measure of the portfolio

- R_P satisfies some conditions,

  then RC_j satisfies the additivity

$$\boxed{R_P = \sum_{j=1}^{n} RC_j}$$

  - Some Conditions:
    - ✓ *Rp is continuously differentiable w.r.t. a_j → existence*
    - ✓ *Rp is a first-order homogeneous function.*

    Here, a first-order homogeneous function f satisfies

$$f(\lambda x_1, \lambda x_2, \cdots, \lambda x_n) = \lambda f(x_1, x_2, \cdots, x_n)$$

Hereafter, assume that $F(x) = P\{ X \le x \}$ is strictly increasing.

Examples of Risk Contribution

- Setting

X_j : *future price of asset j per a unit amount.*

X : *future price of the portfolio.* $\left(X = \sum_{j=1}^{n} a_j X_j \right)$

$VaR_X(\alpha)$: *Value at Risk of X with confidence level α.*

$ES_X(\alpha)$: *Expected Shortfall of X with confidence level α.*

$Q_X(\alpha) \equiv \left\{ x \,\middle|\, P\{X \le x\} = \alpha \right\}$: *α percentile of X.*

c : *reference value.*

Then, $VaR_X(\alpha) = c - Q_X(1-\alpha)$

$$ES_X(\alpha) = \frac{1}{1-\alpha}\int_0^{1-\alpha} VaR_X(p)dp = c - E[X \mid X \le Q_X(1-\alpha)]$$

and

$$RC_j^{VaR}(\alpha) \equiv a_j \frac{\partial VaR_X(\alpha)}{\partial a_j} = a_j \frac{\partial c}{\partial a_j} - E\left[a_j X_j \,\middle|\, X = Q_X(1-\alpha) \right]$$

$$RC_j^{ES}(\alpha) \equiv a_j \frac{\partial ES_X(\alpha)}{\partial a_j} = a_j \frac{\partial c}{\partial a_j} - E\left[a_j X_j \,\middle|\, X \le Q_X(1-\alpha) \right]$$

Advantages/Disadvantages of RC

- Advantages
  - ✓ "Diversification Effect" of the portfolio is considered.
  - ✓ Additivity of Risk is satisfied if Rp is a 1-st order homogeneous function.
  - ✓ It has a clear meanings. ・・・ (volume)×(sensitivity)

- Disadvantages
  - ✓ It is difficult to calculate reliable estimates.
    - *Especially, very difficult to calculate by Monte Carlo simulation.*
    - *Now, some studies are proceeding.*

      ex: Glasserman (2005/2006) using importance sampling

2. Estimation method of RCs

Analytical estimaion

by Saddlepoint approximation

$$\left(X = \sum_{j=1}^{n} a_j X_j \right)$$

Density and Distribution Functions

Assumption: s.v. X has a density and a MGF.

- MGF : Laplace transform of a density function

$$M(s) \equiv E\left[e^{sX}\right] = \int_{-\infty}^{\infty} e^{su} f(u) \mathrm{d}u$$

cumulant GF

- Density : Inverse Laplace transform of MGF

$$f(u) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} M(s) e^{-us} \mathrm{d}s = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{K(s) - us} \mathrm{d}s$$

- Cumulative Distribution Function (CDF)

$$F(u) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{M(s)}{s} e^{-us} \mathrm{d}s = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{K(s) - us}}{s} \mathrm{d}s$$

Expression of RC for VaR

- RC for VaR (more concisely, percentile $Q_X(\alpha)$)

$$RC_j^{Q_X}(\alpha) \equiv a_j \frac{\partial Q_X(\alpha)}{\partial a_j} = a_j \frac{\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\partial K_X(s|\mathcal{G}_T)}{\partial a_j} \frac{e^{K_X(s|\mathcal{G}_T)-sQ_X(\alpha)}}{s} ds}{\int_{\sigma-i\infty}^{\sigma+i\infty} e^{K_X(s|\mathcal{G}_T)-sQ_X(\alpha)} ds}$$

  - ✓ derived from derivative of CDF w.r.t. a_j.
  - ✓ proposed by Martin et. al (2001) .

  - ✓ RC for ES (Expected Shortfall) can be derived similarly. (I will explain the method later.)

$\bar{s}$: saddlepoint

Saddlepoint Approximation

- Approximate "integral in complex domain" by "contribution from the contour integral near the saddlepoint," which is a solution of $dJ_X(\bar{s})/ds = 0$

$$J_X(s) \equiv K_X(s) - us$$

- Approximated Expressions are follows:

$$f_X(u) \simeq \frac{e^{K_X(\bar{s})-\bar{s}u}}{\sqrt{2\pi K_X^{(2)}(\bar{s})}}\left[1+\frac{1}{8}\lambda_{(4)}(\bar{s})-\frac{5}{24}\lambda_{(3)}^2(\bar{s})\right]$$

$$F_X(u) \simeq e^{K_X(\bar{s})-\bar{s}u+\frac{1}{2}\hat{z}^2}\left[\{1-\Phi(\hat{z})\}\left\{1+\frac{1}{6}\lambda_{(3)}(\bar{s})\hat{z}^3+\left(\frac{1}{24}\lambda_{(4)}(\bar{s})\hat{z}^4+\frac{1}{72}\lambda_{(3)}^2(\bar{s})\hat{z}^6\right)\right\}\right.$$
$$\left.+\phi(\hat{z})\left\{-\frac{1}{6}\lambda_{(3)}(\bar{s})(\hat{z}^2-1)-\left(\frac{1}{24}\lambda_{(4)}(\bar{s})(\hat{z}^3-\hat{z})+\frac{1}{72}\lambda_{(3)}^2(\bar{s})(\hat{z}^5-\hat{z}^3+3\hat{z})\right)\right\}\right],$$

$$K_X^{(n)} = \frac{d^n K_X}{ds^n}, \quad \lambda_{(n)} = K_X^{(n)} \Big/ \left(\sqrt{K_X^{(2)}}\right)^n, \quad \hat{z} = \sqrt{\bar{s}^2 K_X^{(2)}(\bar{s})}$$

$$\left(X = \sum_{j=1}^{n} a_j X_j \right)$$

Application to Risk Evaluation (1)

- It is easy to calculate above approximation formulas when s.v. X_js are mutually independent.
- However, independence cannot be assumed in risk evaluation models

  Because the dependences between s.v.s are very important.

So, we will apply the saddlepoint approximation to conditional independent models !

s.v. τ_j, j=1,...,n are (G, P) – conditionally independent,

wher

$$P\{\tau_1 > t_1, \cdots, \tau_n > t_n | \mathcal{G}_T\} = \prod_{j=1}^{n} P\{\tau_j > t_j | \mathcal{G}_T\}$$

Application to Risk Evaluation (2)

Procedure. (We call this "Hybrid Method")

1. Generate scenarios of conditions G_T .
   - ✓ Here, we can use Monte Carlo Simulation.
2. Calculate conditional distribution and RCs numerically by using saddlepoint approximation.
3. Calculate the expectations of above conditionals by Procedure 2, which are the estimates of unconditionals.
   - ✓ Use "chain rule" for conditional expectation.

$$\boxed{F_X(u) = E\left[F_X(u \mid G_T)\right]}$$

Old Estimation Method for ES and its RC

- Application to ES (and CTE, Conditional Tail Expectation)

$$ES_X(\alpha) = c - CTE_X(Q_X(1-\alpha))$$

$$CTE_X(u) \equiv E[X(T)|X(T) \le u] = \frac{1}{P\{X(T) \le u\}} \int_{-\infty}^{u} v f_X(v) dv$$

  - ✓ *Then, written as follows;*

$$CTE_X(u) = \frac{E[X(T)]}{P\{X(T) \le u\}} F_h(u)$$

where $F_h(u)$ is distribution function whose density is $h_X(u) \equiv \frac{u f_X(u)}{E[X(T)]}$.

- *Saddlepoint approximation can be used to evaluate $F_h(u)$, then, ES (CTE) can be estimated.*
- *RCs for ES (or CTE) can be derived similarly as RCs for VaR.*
  - ✓ *Taking derivative of* $CTE_X(u) = \frac{E[X(T)]}{P\{X(T) \le u\}} F_h(u)$ *w.r.t. a_j .*

Problems on ES and its RC

- My preliminary studies show that
  - ✓ *Estimated values of VaR and its RCs are very accurate.*
  - ✓ *But, the estimates of ES (CTE) and its RCs are not accurate.*
  - ✓ *Especially, the additivity of RCs is not satisfied accurately.*
    (Later, I show some numerical examples.)

Therefore, we search other methods for estimating ES and its RCs.

New Proposed Method

- Assume that X is a continuously distributed random variable.
- ES (Expected Shortfall) is written as

$$ES_X(\alpha) = \frac{1}{1-\alpha}\int_0^{1-\alpha} VaR_X(p)dp = c - \frac{1}{1-\alpha}\int_0^{1-\alpha} \boxed{Q_X(p)}dp$$

Accurate!

- Then, RCs for ES can be written as

$$RC_j^{ES}(\alpha) = \frac{\partial\, ES_X(\alpha)}{\partial\, a_j} = \frac{\partial\, c}{\partial\, a_j} - \frac{1}{1-\alpha}\int_0^{1-\alpha} \boxed{\frac{\partial\, Q_X(u)}{\partial\, a_j}}\mathrm{d}u$$

- Since estimates for VaR and its RC are very accurate, the above equations might give us better estimates than before. (Muromachi, 2009)

3. Numerical Examples (preliminary results)

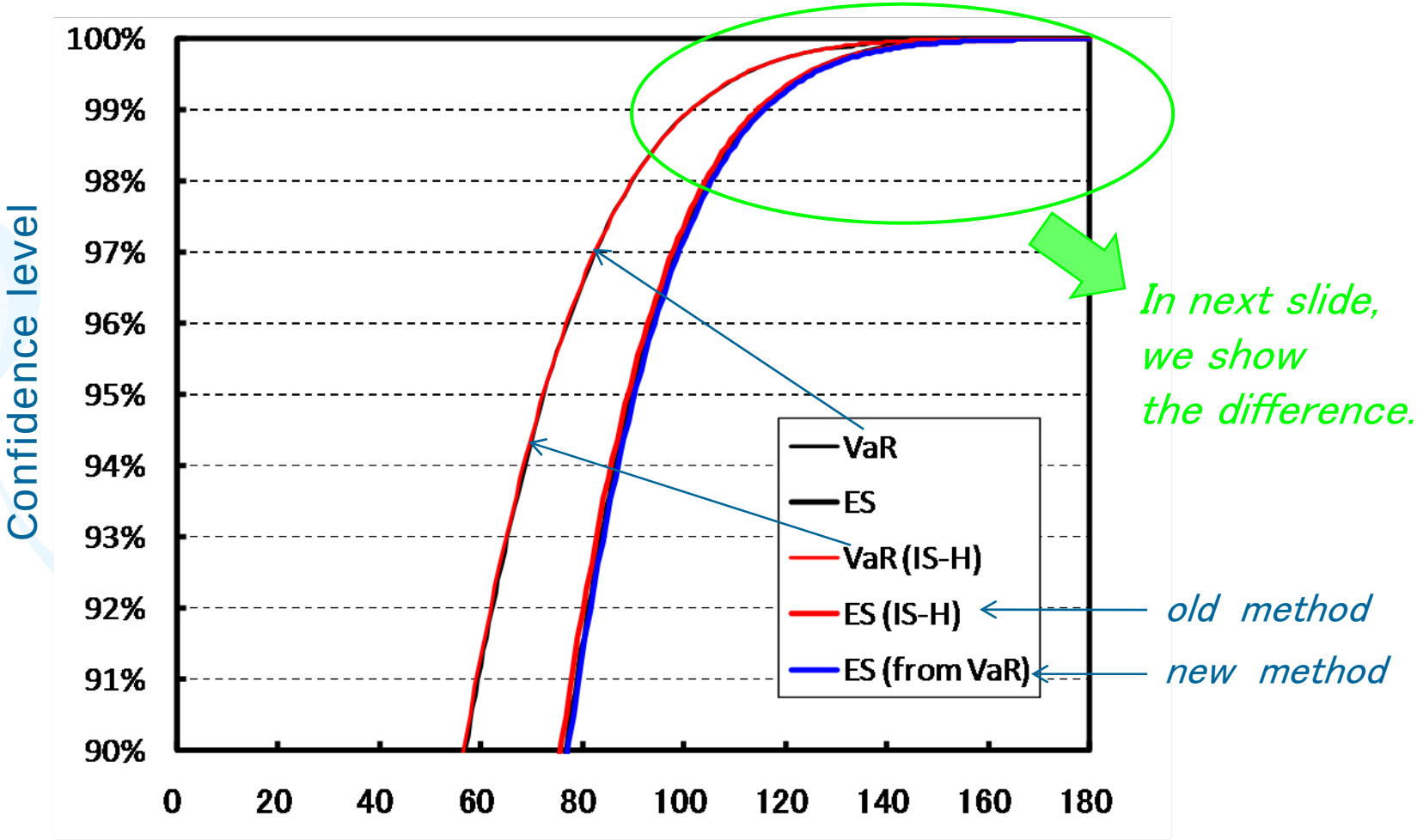
Setting

- Evaluating an interest rate risk and credit risk of a portfolio after 1 year by using a model (Kijima and Muromachi, 2000).
- Portfolio : 100 corporate bonds with maturity 5 years.
- Credit Rating : Aaa(10), Aa(10), A(10), Baa(10), Ba(30), B(30).

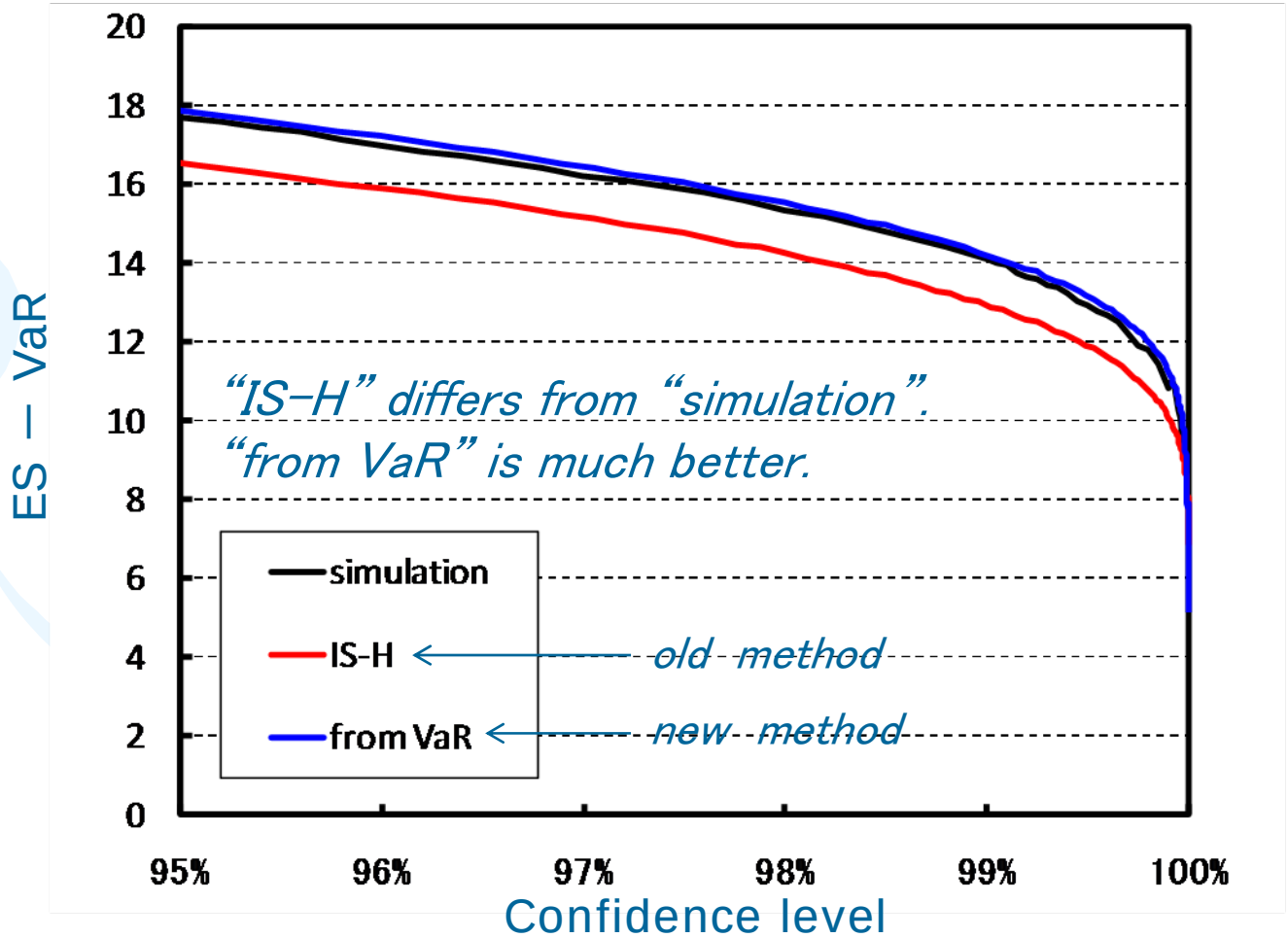
Detailed conditions are omitted here. Please read our article.

- Monte Carlo simulation: 500,000 scenarios. ⇦ use as a reference.
- Hybrid Method with importance sampling technique.
  - ✓ *See Muromachi (2004) in detail.*
- ES and its RCs are estimated by <u>old</u> and <u>new</u> methods.

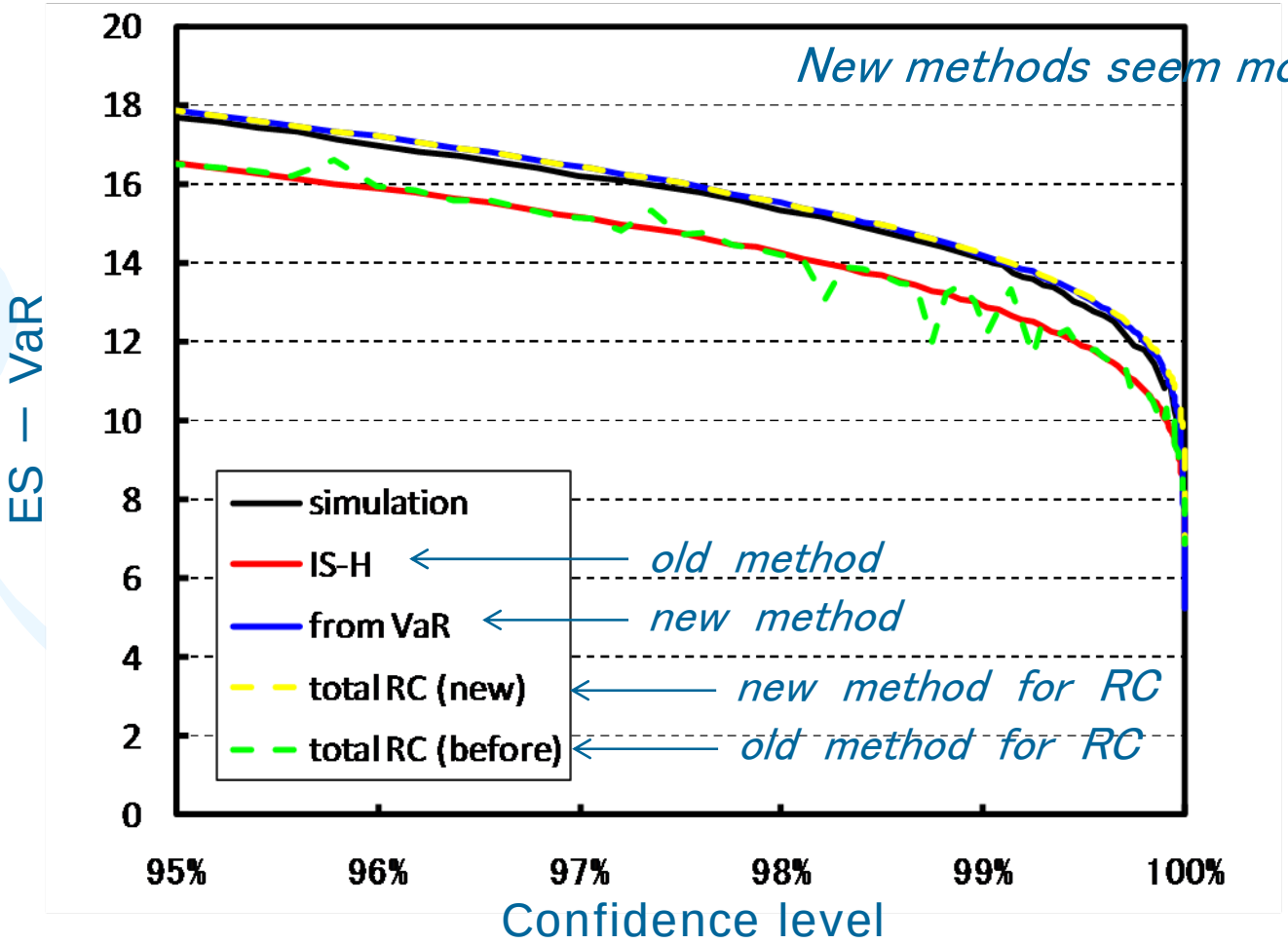
Estimated VaR and ES



Estimated Difference : ES − VaR

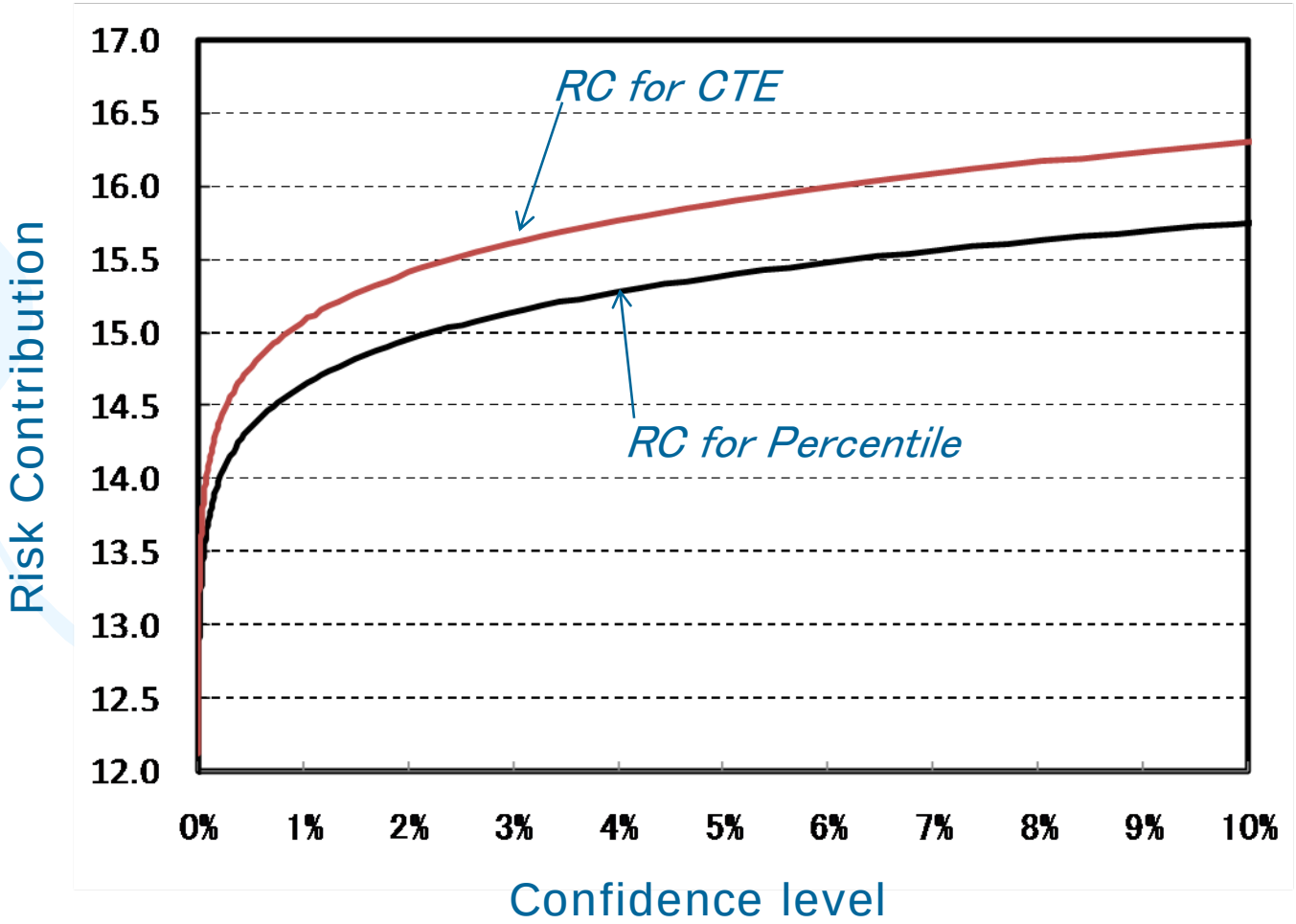


ES − VaR : Sum of RCs



New methods seem more accurate and reliable.

RC for Percentile and CTE (Baa, 30)



Conclusion

- Our method by using saddlepoint approximation gives much more reliable estimates than ordinary Monte Carlo simulation.
- Especially, our original method gives very good estimates of (1) VaR and (2) RC for VaR, but not so good of (3) ES and (4) RC for ES.
- However, our new proposed method gives good estimates of (3) ES and (4) RC for ES in wide ranges.
  - ✓ *This is our first preliminary results. Please wait for more detailed results.*

References

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Kijima, M. and Y. Muromachi, “Evaluation of credit risk of a portfolio with stochastic interest rate and default processes,” Journal of Risk, 3(1), 2000.

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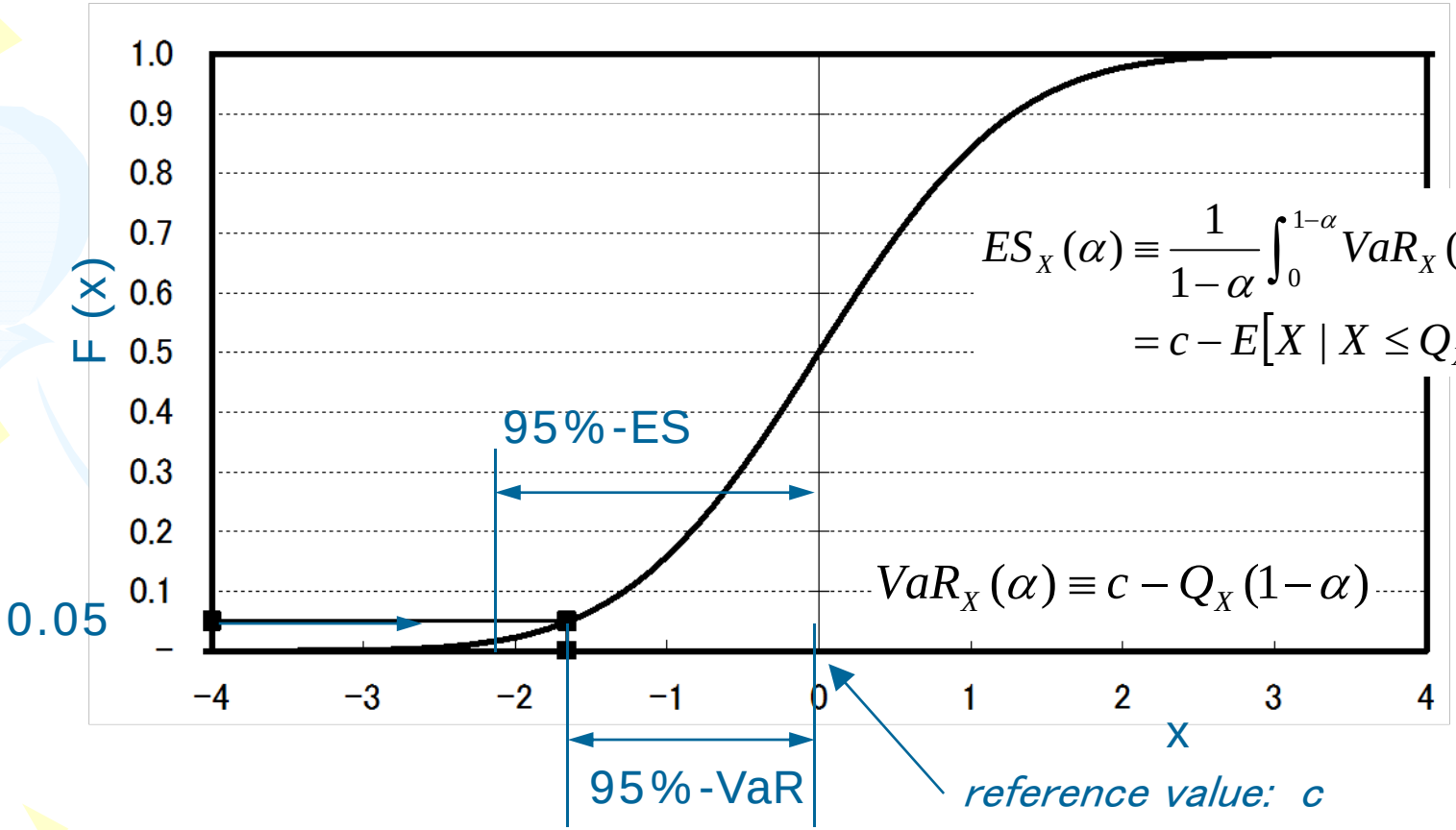
Thank you for your attention.

VaR (Value at Risk), ES (Expected Shortfall)

X is future value, F(x) is its distribution function.

✓ *For simplicity, assume that F(x) is continuous and strictly increasing.*



$$ES_X(\alpha) \equiv \frac{1}{1-\alpha}\int_0^{1-\alpha} VaR_X(p)dp$$
$$= c - E\left[X \mid X \le Q_X(1-\alpha)\right]$$

$$VaR_X(\alpha) \equiv c - Q_X(1-\alpha)$$

Marginal Risk

- Definition (in this context, not universally used)

  Marginal Risk of "Asset j"

  = Risk of a Portfolio − Risk of a Portfolio excluding Asset j

- Advantages
  - *"Diversification effect" of the portfolio is considered.*
    (Marginal Risk depends on the portfolio which includes the asset.)
  - *The values can be estimated easily.*

- Disadvantages
  - *"Additivity of Risk" is Not satisfied.*

  Here, "Additivity of Risk" means the following equation.

  Sum of Marginal Risks of All Assets = Total Risk of the Portfolio

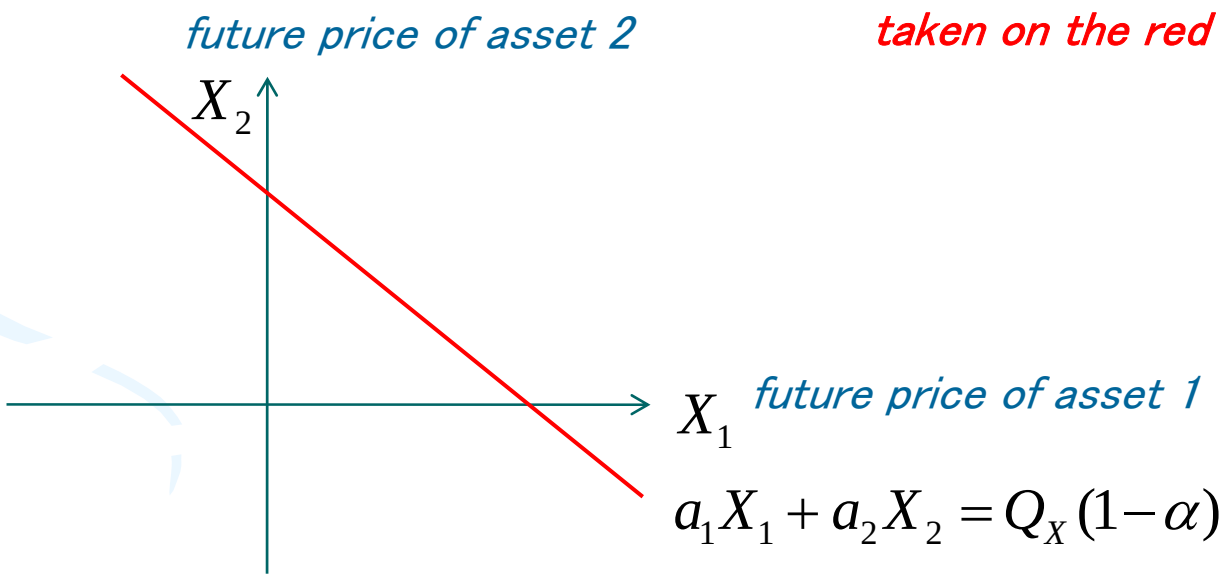
  - *Many risk measures do not satisfy the above equality.*

Why difficult to calculate RC?

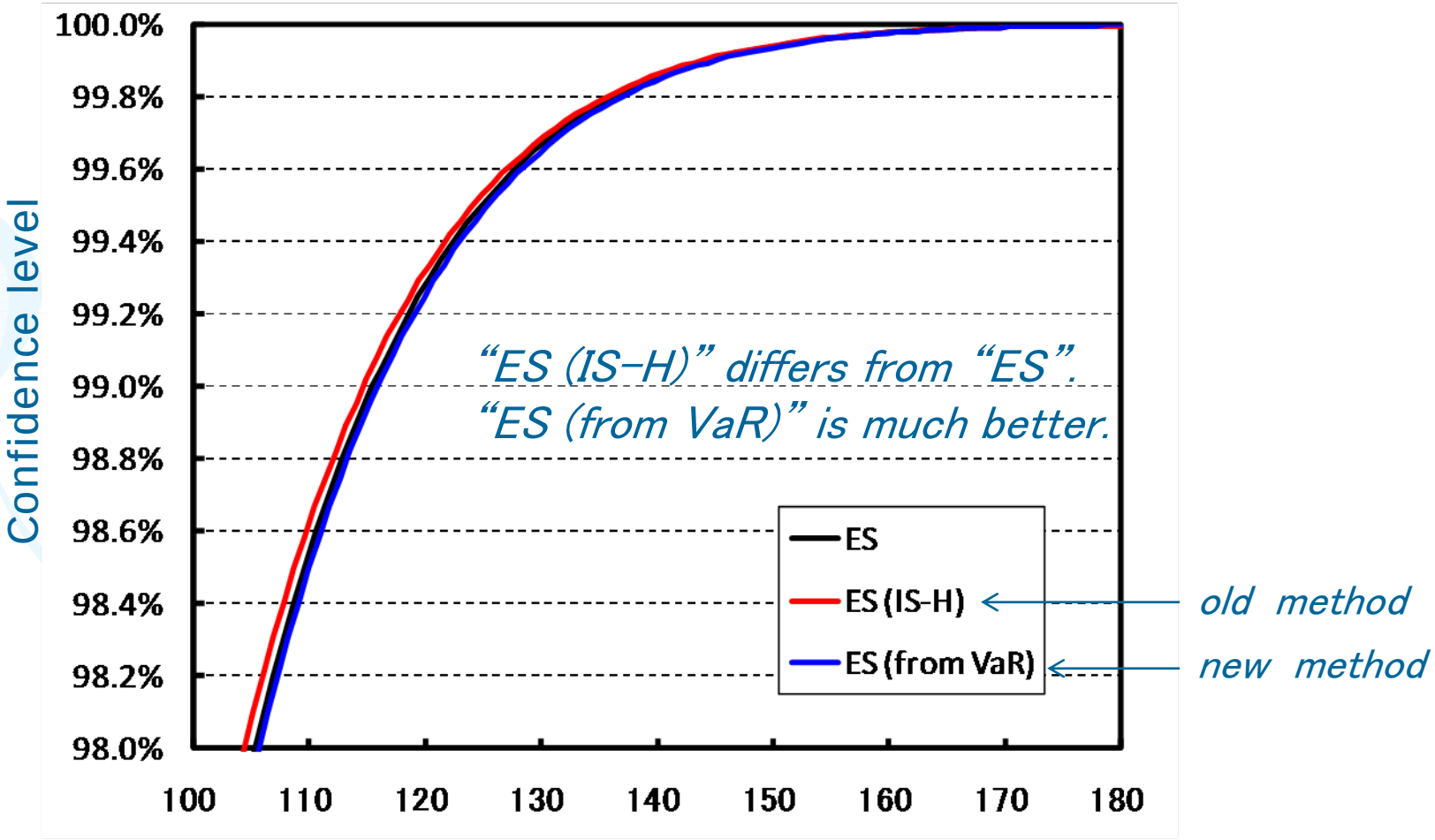
- Consider a portfolio consisting of two assets

$$RC_j^{VaR}(\alpha) = a_j \frac{\partial\, VaR_X(\alpha)}{\partial\, a_j} = a_j \frac{\partial\, c}{\partial\, a_j} - \underline{\underline{E\left[a_j X_j \,\middle|\, X = Q_X(1-\alpha) \right]}}$$

This conditional expectation is taken on the red line below.



Estimated ES by 3 Methods



ES − VaR : Sum of RCs (2)