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**Title: Change of Measures for Frequency and Severity**

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## Change of Measures for Frequency and Severity

**1. Introduction.** The components of premium for an insurance product are made up Lost Cost-- pure premium-- Expense, and Risk Load. Casualty and Property actuaries apply their knowledge of ratemaking to calculate Loss Cost based on the relationship Loss Cost = Frequency  $\times$  Severity. In many instances, one is interested to find out how the changes in Frequency and/or Severity affect the changes in Loss Cost.

In a world without risk, one may be tempted to use multivariable calculus to explain changes based on Taylor's formula

$$dLC(F, S) = \frac{\partial LC}{\partial F} dF + \frac{\partial LC}{\partial S} dS \quad (1)$$

where  $LC = LC(F, S)$  denotes Loss Cost, and  $F$  and  $S$  represent Frequency and Severity respectively.

In our uncertain world, one cannot rely on rules of ordinary calculus, i.e. equation (1) above to assess the impact of changes on Loss Cost contingent upon changes in Frequency and Severity. What is needed is a specification of suitable probabilistic models for Frequency and Severity. Then, a probabilistic model for Loss Cost can be developed based on how the sub-model components for Frequency and Severity interact together. A probabilistic model for Loss Cost would provide for a description of possible Loss Cost values with associated probabilities of attaining those values. Interest may lie in a probability distribution for Loss Cost at a specified time or in the behavior of Loss Cost as it evolves over time, i.e., a stochastic process governing the Loss Cost.

## 2. Probabilistic models for Frequency, Severity and Loss Cost.

To create a model for Frequency, we begin with the notion of a Point Process, PP. A PP is characterized by a sequence of non-decreasing random variables  $\{T_n, n \geq 1\}$ .  $T_n$  presents the time of occurrence of the  $n$ th event of interest, e.g.,  $T_n$  may be the accident date of the  $n$ th claim to an insurer during an exposure period. When the inter-arrival times,  $T_n - T_{n-1}, n \geq 1$ , with  $T_0 = 0$ , are independent and identically distributed random variables having an exponential distribution with parameter  $\lambda$ , then the PP is a homogeneous Poisson process with intensity  $\lambda$ . With a PP is associated a *counting process*  $\{N(t), t \geq 0\}$  defined as

$$N(t) = \sum_{n=1}^{\infty} I(T_n \leq t), \quad (2)$$

where  $I(A)$  denotes the indicator random variable which takes value one if the event  $A$  occurs and zero otherwise. Thus,  $N(t)$  counts the number of times the event of interest occurs in the interval  $(0, t]$ . For the Poisson process we have

$$\Pr(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, k = 0, 1, \dots \quad (3)$$

The Frequency at a point of time is expressed as a rate based on the formula  $\frac{E[N(t)]}{\text{exposure}}$  where the "exposure" base depends upon the Line of Business considered.

Next, we consider a model for Severity. The cost of an insurance product is not solely dependent upon knowing whether a claim has occurred. We are also interested in knowing the impact of a claim on our book of business. Hence, for the  $n$ th claim occurring at time  $T_n$ , we are interested in the size of that claim denoted by  $X_n$ . A Marked Point Process, MPP, is a bivariate sequence of random variables  $\{(T_n, X_n), n \geq 1\}$  where  $T_n$  is time of the occurrence of the  $n$ th claim, and  $X_n$  is the corresponding amount of loss for the  $n$ th claim. It is assumed that  $X_n$ 's are independent and identically distributed random variables distributed as  $X$  with a probability measure  $P$ . We use the term probability measure and probability distribution interchangeably in this paper. So, the stochastic behavior of the claim size—Severity—is governed by  $P$ . In this paper, we assume  $X$  has a discrete probability distribution.

Finally, we want to create a stochastic model for Loss Cost based on a counting process  $N(t)$  and a probability measure  $P$  for the Severity. The compound Poisson process may be used to present the behavior of Loss Cost over time. A compound Poisson process,  $Y(t)$ , is defined as

$$Y(t) = X_1 + X_2 + \cdots + X_{N(t)} \quad (4)$$

In (4) above, it is understood that  $Y(t)=0$  whenever  $N(t) = 0$ .

It is useful to use the concept of a MPP, as defined above, to provide an alternative presentation for the compound Poisson process. To a MPP, we associate a *counting process*,  $N(t, A)$ , according to

$$N(t, A) = \sum_{n=1}^{\infty} I(T_n \leq t, X_n \in A) \quad (5)$$

In (5) above,  $N(t, A)$  counts the number of times the event of interest occurs in interval  $(0, t]$  subject to the  $X_n$ 's being restricted by the event  $A$ . For a specified  $t$  and  $A$ , (5) is a random variable. For a specified  $A$ ,  $N(t, A)$  is a counting stochastic process, and for specified  $t$ ,  $N(t, A)$  is probability measure. For a compound Poisson process, it is customary to assume that the  $T_n$  and  $X_n$  are independent of each other, and the counting process related to  $\{T_n, n \geq 1\}$  is a Poisson process with probability distribution (3). We make a further assumption that  $X_n$  has a finite probability distribution with values in the set  $\{x_1, x_2, \cdots, x_m\}$  and  $p_j = P(X = x_j)$ ,  $1 \leq j \leq m$ .

Now, we give an alternative formula for  $Y(t)$  as

$$\begin{aligned} Y(t) &= \sum_{i=1}^{\infty} X_i I(T_i \leq t) \\ &= \sum_{i=1}^{\infty} \left\{ \sum_{j=1}^m x_j I(X_i = x_j) \right\} I(T_i \leq t) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^m x_j \sum_{i=1}^{\infty} I(T_i \leq t, X_i \in A_j) \\
&= \sum_{j=1}^m x_j N(t, A_j) \\
&= \sum_{j=1}^m x_j N_j(t) \tag{6}
\end{aligned}$$

where  $A_j$  is the singleton event  $\{x_j\}$ ,  $N(t, A_j)$  is defined by (5), and we define  $N_j(t)$  as  $N(t, A_j)$ . Note that  $N(t, A_j)$  represents the number of claims when the corresponding claim amounts are exactly the value  $x_j$ .

The following remarks are relevant to  $Y(t)$  as given by (6):

Remark 1. The number of terms in the summation (6) is fixed, being equal to  $m$ .

Remark 2. Since the events  $A_1, A_2, \dots, A_m$  are mutually exclusive, then the random variables  $N(t, A_1), N(t, A_2), \dots, N(t, A_m)$  are independent for any given  $t$ , see Last and Brandt (1995). This fact is useful when one tries to determine the distribution of  $Y(t)$ .

Remark 3. The distribution of  $N_j(t) = N(t, A_j)$  is a Poisson with intensity  $\lambda P(A_j)$  where  $\lambda$  is the intensity of the unrestricted Poisson process  $N(t)$ , and  $P(A_j)$  is the probability of the event  $A_j$  based on the probability measure  $P$  representing the distribution of  $X$ , see Last and Brandt.  $N_j(t)$  is the number of claims when the corresponding claim amounts are  $x_j$ .  $N_j(t)$  may also be viewed as a restricted version of the counting process  $N(t)$ . This point of view is referred to as *thinning* of a Poisson process.

Remark 4.  $Y(t)$ , as given by (6), is a weighted sum of Poisson processes with differing intensities.

After this preliminary discussion of a compound Poisson process, let us consider the notion of a change of measure as it relates to Frequency, Severity and Loss Cost. In this paper, there are three stochastic elements of interest which we are interested to know how they change subject to changes in their underlying probability measures. These random elements are  $N(t)$ , a Poisson process related to Frequency;  $X$  representing the Severity; and finally  $Y(t)$ , a compound Poisson process, presenting the Loss Cost.

### 3. Change of measures for Frequency, Severity and Loss Cost.

We begin by considering the change of measure for Severity first. Recall that the random variable  $X$ , representing the Severity may be presented by alternative probability measures. So, if, initially  $X$  has a probability measure  $P$ , and later on an alternative probability measure  $Q$ , then we express symbolically this change of measure as  $P \rightarrow Q$ .

A change of measure relies upon a fundamental theorem in measure theory known as Radon-Nikodym Theorem. Before, we can state the Radon-Nikodym Theorem, we need to define the concept of the *absolute continuity* of two measures. We say that the measure  $Q$  is absolutely

continuous with respect to the measure  $P$ , and write this as  $Q \ll P$ , if  $Q(A) = 0$  whenever  $P(A) = 0$  for the event  $A$ . Thus,  $P$  and  $Q$  agree on events which have probability of zero. Now, we state a version of Radon-Nikodym Theorem suitable for probability measures, see Jacod and Protter (2002).

**Radon-Nikodym Theorem:** Let  $P$  and  $Q$  be two probability measures defined on the same measure space. If  $Q \ll P$ , then there exists a nonnegative random variable  $Z$  such that

$Q(A) = \int_A Z dP = E_P\{Z I(A)\}$  for event  $A$ . Moreover,  $Z$  is unique  $P$ -a.s. We write  $Z = \frac{dQ}{dP}$ , and  $Z$  is referred to as Radon-Nikodym derivative.

The following two comments are relevant with respect to the Radon-Nikodym Theorem. First, if we start with a probability measure  $P$  and a nonnegative measurable function,  $Z$ , then it is not hard to show that  $Q(A)$  as define by  $Q(A) = \int_A Z dP$  is a set function and is indeed a measure. The significance of Radon-Nikodym theorem is that converse result is also valid. That is, given two measures  $Q$  and  $P$ , with  $Q \ll P$ , then there **exists** a measurable function  $Z$  such that  $Q$  and  $P$  are related according to  $Q(A) = \int_A Z dP$ . The Radon-Nikodym states the **existence** of  $Z$  without providing an explicit expression for  $Z$ .

Second, when  $P$  is a discrete probability distribution (measure), corresponding to  $X$  taking values in a denumerable set, then it is easy to give an **explicit** expression for  $Z$ , the Radon-Nikodym derivative. For  $X$  having a discrete distribution, we have

$$Z_j = \frac{\Pr_Q(X = x_j)}{\Pr_P(X = x_j)} = \frac{Q(X = x_j)}{P(X = x_j)}, j = 1, 2, \dots \quad (7)$$

The above  $Z_j$ 's may be viewed as likelihood ratios.

In this paper, the change of measure for Severity,  $X$ , as given by (7) above may be stated by

$$Z_j = \frac{q_j}{p_j}, j = 1, 2, \dots, m \quad (8)$$

where  $p_j = \Pr_P(X = x_j)$ , subject to  $\sum_{j=1}^m p_j = 1$  and  $q_j$  is similarly defined as  $q_j = \Pr_Q(X = x_j)$ .

Next, we consider the change of measure for Frequency. Since, Frequency is mainly determined by a Poisson process, we consider next the change of measure for a Poisson random variable. The Poisson random variable or process is characterized by intensity parameter  $\lambda$ , so we indicate symbolically a change of measure for the Poisson process by  $\lambda \rightarrow \lambda'$ . Poisson random variable is a discrete random variable and we can apply formula (7) to derive a change of measure for a Poisson process  $N(t)$  by using (3)

$$\begin{aligned}\frac{\Pr_{\lambda'}(N(t) = k)}{\Pr_{\lambda}(N(t) = k)} &= e^{-\lambda't} \frac{(\lambda't)^k}{k!} \left[ 1 / e^{-\lambda t} \frac{(\lambda t)^k}{k!} \right] \\ &= e^{(\lambda-\lambda')t} \left( \frac{\lambda'}{\lambda} \right)^k\end{aligned}$$

So the Radon-Nikodym derivative (process) is in this case is

$$Z(t) = e^{(\lambda-\lambda')t} \left( \frac{\lambda'}{\lambda} \right)^{N(t)} \quad (9)$$

Finally, we want so derive a change of measure for a compound Poisson process,  $Y(t)$ , by considering change in measures for Frequency and Severity combined. We write symbolically the change of measure for  $Y(t)$  by  $(\lambda, P) \rightarrow (\lambda', Q)$ .

A compound Poisson Process is an example of a stochastic process with jumps. Understanding the notion of change of measure for stochastic processes with jumps requires the knowledge of stochastic calculus and the application of the Girsanov's Theorem. The interested reader may consult Shreve (2004) for further information on this subject. In this exposition, we discuss the notion of a change of measure for a compound Poisson process in an informal and intuitive fashion.

We use the formula (6) for  $Y(t)$ . The values of  $\Pr(Y(t) = y)$  will be non-zero only if  $y$  is of the form

$$y = x_1 N_1(t) + x_2 N_2(t) + \dots + x_m N_m(t),$$

and

$$N(t) = N_1(t) + N_2(t) + \dots + N_m(t)$$

Given  $N(t) = k$ ,  $Y(t)$  has a multinomial probability distribution according to

$$\binom{k}{k_1 \quad k_2 \quad \dots \quad k_m} p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$$

with  $k_j \geq 0$ ,  $k_1 + k_2 + \dots + k_m = k$

Based on Remark 3,  $N_j(t)$  is a Poisson process with intensity  $\lambda p_j$ . (10)

Using Remark 2,  $N_1(t), N_2(t), \dots, N_m(t)$  are independent random variables. (11)

Applying (10), and (9), we have the Radon-Nikodym derivative  $Z_j(t)$  for  $N_j(t)$  as

$$Z_j(t) = \exp(\lambda p_j - \lambda' q_j) \left( \frac{\lambda' q_j}{\lambda p_j} \right)^{N_j(t)} \quad (12)$$

Furthermore, due to independence result given by (11), we have

$$\Pr_{(\lambda', Q)}(N_1(t) = k_1, N_2(t) = k_2, \dots, N_m(t) = k_m)$$

$$= \prod_{j=1}^m \Pr_{(\lambda', Q)}(N_j(t) = k_j) \quad (13.a)$$

$$= \prod_{j=1}^m Z_j(t) \Pr_{(\lambda, P)}(N_j(t) = k_j) \quad (13.b)$$

$$= e^{(\lambda - \lambda')t} \prod_{j=1}^m \left( \frac{\lambda' q_j}{\lambda p_j} \right)^{k_j} \Pr_{(\lambda, P)}(N_j(t) = k_j) \quad (13.c)$$

$$= \left\{ e^{(\lambda - \lambda')t} \prod_{j=1}^m \left( \frac{\lambda' q_j}{\lambda p_j} \right)^{k_j} \right\} \Pr_{(\lambda, P)}(N_1(t) = k_1, N_2(t) = k_2, \dots, N_m(t) = k_m) \quad (13.d)$$

In (13.a), we used the fact that  $N_j(t)$ 's are independent based on the “new” measure  $\Pr_{(\lambda', Q)}$ . A change of measure is used for the Poisson variable  $N_j(t)$  in (13.b). In (13.c), we used the fact that  $\sum_{j=1}^m p_j = 1 = \sum_{j=1}^m q_j$  to simplify our expression. And, finally in (13.d) we used the notion of independence for  $N_j(t)$ 's under the “original” measure  $\Pr_{(\lambda, P)}$ .

We need to make a final adjustment to the product term,  $\prod_{j=1}^m \left( \frac{\lambda' q_j}{\lambda p_j} \right)^{k_j}$ , appearing in right-hand-side of (13.d). It would be instructive to illustrate this point with reference to an example. Suppose  $m = 3$  and  $N(t) = 4$  with  $k_1 = 1, k_2 = 2, k_3 = 1$ . Then, among  $X_1, X_2, X_3$ , and  $X_4$ , we have one instance of  $X_i$  being  $x_1$  with probability  $p_1$ ; two instances of  $X_i$  being  $x_2$  with probability  $p_2^2$ ; and one instance  $X_i$  being  $x_3$  with probability  $p_3$ . Note, we have assigned probabilities according to  $P$ . A similar assignment of probabilities according to the measure  $Q$  is also valid. Let us define  $P(X_i)$  to be  $p_j$  when the realized value of  $X_i$  is  $x_j$  for  $i = 1, 2, 3, 4$ . Similarly, we define  $Q(X_i)$ . Again, we note that  $X_i, i = 1, 2, 3, 4$  are **identically** distributed under either measure  $P$  or measure  $Q$ . Then, for our example, we can write

$$\prod_{j=1}^m \left( \frac{\lambda' q_j}{\lambda p_j} \right)^{k_j} = \left( \frac{\lambda' q_1}{\lambda p_1} \right) \left( \frac{\lambda' q_2}{\lambda p_2} \right)^2 \left( \frac{\lambda' q_3}{\lambda p_3} \right) = \prod_{i=1}^{k_1 + k_2 + k_3} \left( \frac{\lambda' Q(X_i)}{\lambda P(X_i)} \right)$$

With this refinement, then

$$\prod_{j=1}^m \left( \frac{\lambda' q_j}{\lambda p_j} \right)^{k_j} = \prod_{i=1}^{k_1+k_2+\dots+k_m} \left( \frac{\lambda' q(X_i)}{\lambda p(X_i)} \right) \quad (14)$$

Combining (13.d) with (14), gives the Radon-Nikodym derivative  $Z(t)$  for the compound Poisson process as

$$Z(t) = e^{(\lambda-\lambda')t} \prod_{i=1}^{N(t)} \left( \frac{\lambda' q(X_i)}{\lambda p(X_i)} \right) \quad (15)$$

To summarize, the change of measure, as given by Radom-Nikodym derivative is (9) for the Frequency only; is (8) for the Severity only; and is (15) for Frequency and Severity combined.

## References

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