The Insurance Risk in the SST and in Solvency II: Modelling and Parameter Estimation

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Abstract:
Both, the Swiss Solvency Test (SST) and solvency II in the EU are the framework of a new, risk based solvency regulation. In this paper we concentrate on the insurance risk. We will compare the two models and discuss what is common and what is different in the two models. Emphasis will be lead on the estimation of the parameters and some new parameter estimators will be presented. In this context we will also address the problem of how to combine individual and industry-wide data by means of credibility. Another special discussion point will be diagonal effects in the reserve risk such as the impact of inflation and super-imposed inflation.

1 Introduction

As early as in 2004 Switzerland developed the first version of the Swiss Solvency Test (SST), a new risk based solvency regulation. The first field test with 10 insurers already took place in 2004. Since 2008 all Swiss companies have to do the SST and from 2011 on the target capital or solvency capital requirement according to the SST will be in force.

In the meantime the EU has developed solvency II, and in 2008 the 4th quantitative impact study QIS has been carried through.

Both, the SST and solvency II have the same aim, namely to install a risk based solvency regulation and to require a solvency capital which is based on the risks of the company, be it on the asset side or be it on the insurance side.

It is now interesting to see the two systems and modelling frameworks, where they are equal and similar and where there are major differences.

In this paper we will concentrate on the non-life insurance risk. In Section 2 we will define the insurance risk, and in Section 3 we will investigate how they are modelled in the two systems. Section 4 will be devoted to parameter estimators, where some new parameter estimators will be presented.

2 The Insurance Risk

By the insurance risk in non-life we denote next year’s technical result \( TR \) defined by

\[
TR = P - K - C^{CY} - C^{PY}
\]  

(1)
where

\[ P = \text{earned premium}, \]
\[ K = \text{administrative costs}, \]
\[ C^{CY} = \text{total claim amount current year}, \]
\[ C^{PY} = \text{total claim amount previous years}. \]

The total claim amount previous years is defined as

\[ C^{PY} = -CDR, \]

where \( CDR \) is the \textit{claims development result}

\[ CDR = R - PA^{PY} - R^{31.12..PY}. \]

where

\[ R = \text{claims reserves per 1.1. (best estimate)}, \]
\[ PA^{PY} = \text{payments for claims of previous years}, \]
\[ R^{31.12..PY.} = \text{claims reserves per 31.12. for claims of previous years}. \]

(1) can also be written as

\[ TR = (P - K - E[C^{CY}]) - (C^{CY} - E[C^{CY}]) - C^{PY} \]
\[ \simeq E[P - K - E[C^{CY}]] - (C^{CY} - E[C^{CY}]) - C^{PY} \] (2)

In (2) it is assumed that the claims reserves are best estimate reserves and that therefore

\[ E[C^{PY}] = 0. \]

Premiums and administrative costs of next year can usually be forecasted with high accuracy and the risk involved in these two components are negligible compared to \( C^{CY} \) and \( C^{PY} \), which are the main risk drivers of the technical result. Replacing \( P \) and \( C \) by their expected values (forecasts at the beginning of the year) leads to (3).

From (3) it is seen that the technical result is composed of three components:

- the expected technical result,
- \( C^{CY} - E[C^{CY}] \), which is the deviation of the total claim amount current year (CY) from its expected value,
- \( C^{PY} \), the total claim amount previous years (PY).

\( C^{CY} - E[C^{CY}] \) is referred to as \textit{premium risk} in solvency II and as the \textit{risk of claim amount current year} in the SST, and \( C^{PY} \) is called \textit{reserve risk} in solvency II and in the SST. In the next two Sections we are going to consider in more detail how these risks are modelled in the SST and in solvency II.
In the SST as well as in solvency II, a *standard model* is in place, but companies might use another *internal model* subject to approval from the supervision authority. The topic of this paper is the modelling of the insurance risk in the standard non-life model.

The SST as well as solvency II distinguishes between different lines of business (lob) $i = 1, 2, \ldots, I$. We will use the index $i$ to denote lob $i$, whereas a dot in the index always indicates summation over the corresponding index. For instance, $C_{i}^{CY}$ will denote the claim amount CY of lob $i$ and $C_{\ast}^{CY}$ the claim amount CY summed up over all lines of business. The segmentation into lob considered in the standard models can be found in Appendix A.1.

Finally, in the SST as well as in solvency II, reserves are discounted with the risk-free yield rate. Thus the reserves have to be interpreted as discounted reserves. The same is the case for related random variables. For instance the claim amount CY has to be interpreted as the discounted claim amount CY.

## 3 Modelling of the Insurance Risk

### 3.1 Modelling of the Insurance Risk in the SST

In the SST standard model the $C^{CY}$ is split into $C^{CY,n}$, the claim amount caused by the bulk of so called "normal" claims and $C^{CY,b}$, the claim amount caused by big claims or big claim events. In a short term notation we will call $C^{CY,n}$ the normal claim amount and $C^{CY,b}$ the big claim amount. All these elements ($C^{CY,n}$, $C^{CY,b}$, $C^{PY}$) are modelled by a stochastic model named the *analytical insurance risk model*. The analytical insurance risk model as well as an analogous analytical model for the market risk are designed to describe adequately the reality except for very seldom extraordinary situations. Therefore, in addition to the analytical model the SST also allows to take into account very seldom and extraordinary situations by means of so called *scenarios* $SC_k$, which are characterised by face amounts $c_k$ and assigned probabilities $p_k$ for their occurrences. There are *financial scenarios* producing a big market loss (e.g. scenario Nikkei drop), *insurance scenarios* causing a big insurance loss (e.g. explosion in an industrial complex) and *mixed scenarios* producing both an insurance and a market loss (e.g. a pandemia). A clear separation between insurance and market risk is not possible for the mixed scenarios. We allocate to the insurance risk all insurance scenarios and those mixed scenarios, where the insurance loss is the dominant one and bigger than the market loss. The insurance scenario risk is then the sum

$$
SC_{\text{ins}}^{\ast} = \sum_{k=1}^{K} SC_k,
$$

where $K$ is the number of scenarios allocated to the insurance risk.

From (3) we obtain

$$
TR = \frac{E \left[ P - K - E \left[ C^{CY} \right] \right]}{\text{expected technical result}} - \left( C_{\ast}^{CY,n} - E \left[ C_{\ast}^{CY,n} \right] \right) - \left( C_{\ast}^{CY,b} - E \left[ C_{\ast}^{CY,b} \right] \right) - C_{\ast}^{PY} - SC_{\text{ins}}^{\ast}.
$$

The SST is distribution based. At the end one wants to determine the distribution for $\Delta = \text{change of the risk bearing capital within a one-year period}$. The risk measure used in
the SST is the 99% mean expected shortfall. Transferred to the insurance risk this means that we want to determine the distribution function of the technical result $TR$, and we define the target capital or solvency capital required for the insurance risk by

$$SCR_{ins} = ES_{99\%}[-TR] = ES_{99\%}[-TR] - E[TR],$$

where, for any random variable $X$, $ES_{99\%}[X] = ES_{99\%}[X - E[X]]$ and where $ES_{99\%}[X]$ denotes the 99% expected shortfall. Note that the solvency capital required as defined above is the necessary capital of the insurance risk on a stand-alone basis (before diversification with the market risks).

3.1.1 Modelling of the normal claim amount current year in the SST

Model Assumptions 3.1 (normal claim amount SST) For each line of business $i$ it is assumed that the current year (=next coming year) is characterised by its risk characteristics $\Theta_i = (\Theta_{1i}, \Theta_{2i})$, where $\Theta_{1i}$ and $\Theta_{2i}$ are independent with $E[\Theta_{1i}] = E[\Theta_{2i}] = 1$, and that conditional on $\Theta_i = (\Theta_{1i}, \Theta_{2i})$

- the normal claim amount $C_{i}^{CY,n}$ is compound Poisson distributed
- with Poisson parameter $\lambda(w_i, \Theta_{1i}) = w_i \lambda_i \Theta_{1i}$, where $w_i$ is a known weight and where $\lambda_i$ is the a priori expected claim frequency,
- and with claim severities $Y_i^{(v)}$ having the same distribution as $\Theta_{2i}Y_i$, where $Y_i$ has a distribution $F_i(y)$ with $E[Y_i] = \mu_i$.

Remarks:

- The underlying risk characteristics $\Theta_i = (\Theta_{1i}, \Theta_{2i})$ reflect the "state of the nature" in the next coming year for lob $i$. Indeed things like weather conditions, economic situation, change in legislation, etc. might have an impact on the claim number and the claim severity. These "conditions" may vary from year to year and the imposed changes increase the risk and affect big companies as much as small companies, i.e. this risk cannot be diversified. The "true claim frequency" in the coming year will be $\lambda_i(\Theta_{1i}) = \lambda_i \cdot \Theta_{1i}$, hence $\Theta_{1i}$ is the random factor by which next year's "true underlying claim frequency" will deviate from the a priori expected claim frequency $\lambda_i$. The interpretation for $\Theta_{2i}$ is quite analogous, but with respect to the "true underlying expected value of the claim severity".
- Note, that no distributional assumption about the claim severities is made in model assumptions 3.1. But it will later be assumed that

$$C_* = C_{*}^{CY,n} + C_{*}^{PY}$$

can be approximated by a lognormal distribution with the corresponding first and second moments (see Section 3.1.3). Another idea would be to assume a lognormal distribution on the level of lines of business. This would be a slightly different model, since the sum of lognormal distributions is not lognormal any more. But
the difference would presumably be rather small without a significant impact on the result, but it would have the disadvantage to be much more complicated and the distribution of $C_*$ could not be expressed by a simple analytical formula any more.

We introduce

$$\tilde{P}_i = E \left[ C_i^{CY,n} \right],$$

which is the pure risk premium for normal claims, and

$$X_i = \frac{C_i^{CY,n}}{\tilde{P}_i}$$

(4)

the corresponding loss ratio.

It follows from appendix A.3, formula (65), that

$$\sigma_i^2 := \text{Var} \left( X_i \right) = \sigma_{i,\text{param}}^2 + \frac{\sigma_{i,\text{fluct}}^2}{\nu_i},$$

(5)

where

$$\sigma_{i,\text{param}}^2 = \text{Var} \left( \Theta_1 i \right) + \text{Var} \left( \Theta_2 i \right) + \text{Var} \left( \Theta_1 i \right) \cdot \text{Var} \left( \Theta_2 i \right)$$

(6)

$$\simeq \text{Var} \left( \Theta_1 i \right) + \text{Var} \left( \Theta_2 i \right),$$

(7)

$$\sigma_{i,\text{fluct}}^2 = \text{CoVa}^2 \left( Y_i^{(v)} \right) + 1.$$  

(8)

and where

$$\text{CoVa} \left( Y_i^{(v)} \right) = \text{the coefficient of variation of the claim severities},$$

$$\nu_i = w_i \lambda (i) = \text{a priori expected number of claims}.$$

Since $\tilde{P}_i = \nu_i \mu_i$ we can also write (5) as

$$\sigma_i^2 := \text{Var} \left( X_i \right) = \sigma_{i,\text{param}}^2 + \frac{\sigma_{i,\text{fluct}}^2}{\tilde{P}_i},$$

(9)

where

$$\tilde{\sigma}_{i,\text{fluct}}^2 = \mu_i \sigma_{i,\text{fluct}}^2.$$

Note that $\sigma_i^2 = \text{CoVa}^2 \left( C_i^{CY,n} \right)$ is composed of two components, the parameter risk and the random fluctuation risk. The parameter risk is independent of the size of the company whereas the fluctuation risk decreases with the size of the company respectively with the number of a priori expected claims.

The standard values provided by the supervision authority for $\sigma_{i,\text{param}}^2$ and $\sigma_{i,\text{fluct}}^2$ in the SST 2008 (comparable to the market values in solvency II) can be found in Appendix A.2. Companies might deviate from them based on estimates from their own data. Usually there is sufficient data in most of the companies to estimate $\sigma_{i,\text{fluct}}^2$, but there is not yet a sound methodology to estimate $\sigma_{i,\text{param}}^2$ from the own data. Therefore most companies use
the default values for $\sigma^2_{i,\text{param}}$. Estimators for estimating $\sigma^2_{i,\text{param}}$ from the company-own data will be presented in Section 4.2.2.

Model assumptions 3.1 have resulted in the variance structure given by (5). Only this variance structure is then further used in the SST standard model. Hence, alternatively to model assumptions 3.1, we can formulate the model-assumptions directly by the variance condition (9).

Model Assumptions 3.2 (normal claim amount SST, alternative version) It is assumed that

$$
\sigma^2_i := \text{Var} \left( X_i \right) = \sigma^2_{i,\text{param}} + \frac{\sigma^2_{i,\text{fluct}}}{P_i}.
$$

(10)

For calculating the variance of the total normal claim amount (summed up over all lines of business) we have to make assumptions on the correlation of the normal claim amount of different lines of business. Let

$$
\rho^C_{ij} := \text{Corr} \left( C^C_{i,n}, C^C_{j,n} \right),
$$

$$
X_* := \frac{C^C_{i,n}}{P_*}.
$$

Then we obtain

$$
\text{Var} \left( C^C_{i,n} \right) = \sum_{i,j=1}^I P_i \sigma_i \tilde{P}_j \sigma_j \rho^C_{ij}
$$

(11)

and

$$
\sigma^2 := \text{Var} \left( X_* \right) = \frac{1}{P_*^2} \sum_{i,j=1}^I P_i \sigma_i \tilde{P}_j \sigma_j \rho^C_{ij}.
$$

(12)

It is convenient to write (11) in matrix notation. Let

$$
X = (X_1, X_2, \ldots, X_I)^T,
$$

$$
W^C = \left( \tilde{P}_1 \sigma_1, \tilde{P}_2 \sigma_2, \ldots, \tilde{P}_I \sigma_I \right)^T,
$$

$$
R^C = \text{Corr} \left( X, X^T \right).
$$

Note that $R^C$ denotes the correlation matrix of $X$ with the entries

$$
R^C(i,j) = \rho^C_{ij}.
$$

(11) written in matrix notation becomes

$$
\text{Var} \left( C^C_{i,n} \right) = W^C \cdot R^C \cdot W^C.
$$

(13)

The correlation matrix $R^C$ as provided by the supervision authority for the standard model in the SST 2008 can be found in Appendix A.2.
3.1.2 Modelling of the Reserve Risk in the SST

Let

\[ L_i = \text{outstanding claims liabilities at 1.1. for lob i}, \]
\[ R_i = \text{best estimate of } L_i \text{ per 1.1. = best estimate reserve}, \]
\[ \widetilde{R}_i = PA_i^{PY} + R_i^{31.12.PY} = \text{best estimate of } L_i \text{ per 31.12.}, \]

where \( PA_i^{PY} \) are the claim payments for previous years’ claims and \( R_i^{31.12.PY} \) are best estimate reserves per 31.12. of previous years’ claims (see Section 2).

Note that \( \widetilde{R}_i \) is known at the beginning of the year, whereas \( \widetilde{R}_i \) is still a random variable \( (PA_i^{PY} \text{ and } R_i^{31.12.PY} \text{ will only be known at the end of the year}). \)

The claim amount PY becomes

\[ C_i^{PY} = \widetilde{R}_i - R_i. \]

As for the current year risk, we introduce something analogous to the loss ratio, but with the ingoing reserves \( R_i \) (instead of the premiums) as "weight", that is we consider

\[ Y_i = \frac{\widetilde{R}_i}{R_i}. \]

Note that \( E[Y_i] = 0 \) and that \( C_i^{PY} = (Y_i - 1)R_i. \)

**Model Assumptions 3.3 (reserve risk SST)** It is assumed that

\[ \tau_i^2 := \text{Var} (Y_i) = \tau_{i, \text{param}}^2 + \frac{\tau_{i, \text{fluct}}^2}{R_i}. \quad (14) \]

**Remarks:**

- The SST distinguishes between parameter risk and random fluctuation risk, but the model assumptions 3.3 are not written down anywhere in an official document of the SST. As pointed out in [7] the question of how to quantify the reserve is risk is not yet fully answered. Thus the variance structure (14) is my own interpretation and is not yet an integral part of the current SST. I think that from a modelling point of view there are good reasons to assume the above structure, even if this structure is not reflected in the current standard parameters, where the coefficients of variation of the random fluctuation risk do not depend on the size of the reserves. Note that (14) has the same structure as (10).

- As for the current year risk it is again assumed that the variance consists of two components, the parameter risk independent of the weight (=volume of reserves) and the random fluctuation risk which is inversely proportional to this weight. The parameter risk reflects the estimation error which is a risk measure of the deviation of the reserve \( R_i \) from the true expected value \( E[L_i] \) of the outstanding liabilities \( L_i \), whereas the random fluctuation risk encompasses the pure random fluctuation of the ultimate around \( E[L_i] \), which is also called process error in claims reserving.
The table with the standard values for $\tau^2_{i,\text{param}}$ and $\tau^2_{i,\text{fluct}}$ provided by the supervision authority for the SST 2008 can be found in Appendix A.2.

In order to calculate the variance for the reserve risk summed up over all lines of business, we have again to make assumptions about the correlation of the reserve risk of different lob. According to the current SST standard assumption it is assumed that there is no correlation between the reserve risk of different lob.

**Discussion and remarks on the correlation assumption for PY risks in the SST**

- The current standard assumption of no-correlation between the reserve risks of different lines of business is questionable. Calendar year factors may affect the reserves of several lines of business simultaneously and impose a positive correlation. An obvious calendar year factor is certainly inflation, but there might also be others like change in legislation.

- A change of the correlation assumption can be done well within the SST modelling frame-work. Hence the no-correlation assumption does not question the SST-standard-model as such, but is rather a matter of the parameter choice within the SST standard model.

Let

\[
Y_\bullet = \frac{\overline{R}_\bullet}{\overline{R}_\bullet}, \\
Y = (Y_1, Y_2, \ldots, Y_l)^T, \\
W_{PY} = (R_{1\tau_1}, R_{2\tau_2}, \ldots, R_{l\tau_l})^T, \\
R_{PY} = \text{Corr} \left( Y, Y^T \right).
\]

Then we obtain

\[
\text{Var} \left( C_{PY}^\bullet \right) = W_{PY}^T \cdot R_{PY} \cdot W_{PY},
\]

and

\[
\tau^2 := \text{Var} \left( Y_\bullet \right) = \frac{1}{\overline{R}_\bullet^2} \left( W_{PY}^T \cdot R_{PY} \cdot W_{PY} \right).
\]

The no-correlation standard assumption means that $R_{PY}$ is the identity matrix. Then (16) becomes

\[
\tau^2 = \frac{1}{\overline{R}_\bullet^2} \sum_{i=1}^{l} R_{i\tau_i}^2.
\]

**3.1.3 Modelling of the sum of normal claim amount CY and claim amount PY in the SST**

Let

\[
S_i = C_{i,Y}^{CY,n} + \overline{R}_i, \\
Z_i = \frac{C_{i,Y}^{CY,n} + \overline{R}_i}{\overline{P}_i + R_i} = \frac{\overline{P}_i X_i + R_i Y_i}{\overline{P}_i + R_i}.
\]
Whereas $\tilde{P}_i$ and $R_i$ can be considered as the weights attached to $X_i$ and $Y_i$ respectively, the corresponding weight for $Z_i$ is the sum

$$V_i = \tilde{P}_i + R_i.$$  \hfill (20)

From (19) we immediately see that

$$\varphi_i^2 := \text{Var} (Z_i) = \frac{(\tilde{P}_i \sigma_i)^2 + 2\tilde{P}_i \sigma_i R_i \tau_i \text{Corr} (C^C_Y, C^P_Y) + (R_i \tau_i)^2}{V_i^2}. \hfill (21)$$

At the end we are interested in the variance of the total claim amount CY and PY summed up over all lines of business. Denote by

$$R_{CY,PY} = \text{Corr} \left( C^{CY}, C^{PY} \right)$$

the correlation matrix between the claim amounts CY and the claim amounts PY with entries

$$R_{CY,PY} (i, j) = \text{Corr} \left( C^{CY}_i, C^{PY}_j \right).$$

As correlation matrix for the joint random vector of claim amounts CY and claim amounts PY we obtain

$$R = \text{Corr} \left( \begin{pmatrix} C^{CY} \\ C^{PY} \end{pmatrix}, \begin{pmatrix} C^{CY} \\ C^{PY} \end{pmatrix}^T \right)$$

$$= \begin{pmatrix} R_{CY} & R_{CY,PY} \\ R_{CY,PY} & R_{PY} \end{pmatrix}.$$  \hfill (22)

Let

$$S_\bullet = C^{CY}_\bullet + \tilde{R}_\bullet,$$  \hfill (23)

$$Z_\bullet = \frac{C^{CY}_\bullet + \tilde{R}_\bullet}{\tilde{P}_\bullet + R_\bullet}.$$  \hfill (24)

Then we obtain

$$\text{Var} (S_\bullet) = \left( \begin{pmatrix} W^{CY} \\ W^{PY} \end{pmatrix} \right)^T \cdot R \cdot \left( \begin{pmatrix} W^{CY} \\ W^{PY} \end{pmatrix} \right),$$  \hfill (25)

where

$$V_\bullet = \tilde{P}_\bullet + R_\bullet.$$  \hfill (26)

Since

$$Z_\bullet = \sum_{i=1}^I \frac{V_i}{V_\bullet} Z_i,$$  \hfill (27)
we can also express the variance of $Z_\cdot$ in terms of the $Z$-variables, i.e.

$$\text{Var} (Z_\cdot) = \sum_{i,j=1}^I \frac{V_i V_j \varphi_i \varphi_j}{V^2} \text{Corr} (Z_i, Z_j)$$

$$= \frac{1}{V^2} V^T (W^{T}_{CY} \cdot R_{CY} \cdot W_{CY} + 2 \cdot (W^{T}_{CY} \cdot R_{CY,PY} \cdot W_{CY}) + W^{T}_{PY} \cdot R_{PY} \cdot W_{PY}) V,$$

where $V = (V_1, V_2, \ldots, V_I)^T$.

(25) is valid for any correlation matrix $R$. The standard assumption in the current SST is that there is no correlation between current year risks and previous year risks, which means that $R_{CY,PY}$ is a zero-matrix (matrix with only zeros). Together with the assumption that the reserve risks of different lob are not correlated we obtain

$$\text{Var} (Z_\cdot) = \frac{\tilde{P}_\cdot^2 \sigma^2 + R_{\cdot} \tau^2}{(\tilde{P}_\cdot + R_{\cdot})^2}. \quad (27)$$

Discussion and remarks on the correlation assumption between CY and PY risks:

- The current standard assumption in the SST that CY risks and PY risks are not correlated, is questionable. There might well be calendar year factors affecting both, the CY and the PY results. The most obvious of these factors is claims inflation, which has an impact on both, CY claims and PY claims.

**Model Assumptions 3.4 (normal claim amount CY plus claim amount PY)** It is assumed that $S_\cdot$ has a lognormal distribution with

$$E [S_\cdot] = \tilde{P}_\cdot + R_{\cdot},$$

$$\text{Var} (S_\cdot) = (W_{CY}, W_{PY})^T \cdot R \cdot (W_{CY}, W_{PY}).$$

Remark:

- This model assumption has to be interpreted in the way that a lognormal distribution is a sufficiently accurate approximation to the distribution of $S_\cdot$ for solvency purposes.

**3.1.4 Modelling of the Big Claim Amount Current Year in the SST**

Big claims are defined as single individual claims or claim events (natural catastrophes) with a claim amount exceeding a certain threshold (e.g. 1 m CHF or 5 m CHF). For both types of big claims it is assumed that the following model assumptions hold true.
Model Assumptions 3.5 (big claim amount SST)

i) The big claim amount for line of business \( i \) has a compound Poisson distribution

\[
C^{CY,b}_i = \sum_{v=1}^{N^b_i} Y^b_{iv},
\]

where the number of big claims \( N^b_i \) is Poisson-distributed with Poisson parameter \( \lambda^b_i \) and where the claim severities \( Y^b_{iv}, v = 1, 2, \ldots, N^b_i \), are independent and independent from \( N^b_i \) with distribution function \( F_i \).

ii) \( \{ C^{CY,b}_i : i = 1, 2, \ldots, I \} \) are independent.

Remarks:

- The claim severity distributions \( F_i \) are essentially assumed to be Pareto with Pareto parameters \( \alpha_i \). Upper limits can easily be taken into account by truncation of the Pareto distributions. Also xs-reinsurance is no problem to handle by this model.

- Big claim events like hail, wind-storm, flood etc. usually affect several lines of business. However, this is not relevant in this context. Such an event claim might then be associated to the lob with the highest expected claims load. The essential assumption is that all these big claims are independent and compound Poisson.

From these assumptions it follows that

\[
C^{CY,b}_* = \sum_{i=1}^{I} C^{CY,b}_i
\]

is again compound Poisson with parameter

\[
\lambda = \lambda^b = \sum_{i=1}^{I} \lambda^b_i
\]

and claim severity distribution

\[
F = \sum_{i=1}^{n} \frac{\lambda^b_i}{\lambda^b_*} F_i.
\]

The compound Poisson-distribution of \( C^{CY,b}_* \) can for instance be calculated by means of the Panjer-algorithm.

3.1.5 Aggregation

Let

\[
\tilde{T} = C^{CY,n}_* + C^{CY,b}_* + C^{PY}_*
\]

be the total claim amount before scenarios. According to model-assumptions 3.4, the distribution of \( S_* = C^{CY}_* + C^{PY}_* \) is lognormal and according to model-assumptions 3.5, the distribution of \( C^{CY,b}_* \) is compound Poisson. Thus the distribution function \( \tilde{F}(x) \) of \( \tilde{T} \) is obtained by convoluting the lognormal distribution of \( S_* \) with the compound Poisson distribution of \( C^{CY,b}_* \).
3.1.6  Taking the scenarios into account

Insurance scenarios $SC_{k}^{ins}$ are situations or events producing an estimated loss $c_{k}$ (face amount) with probability $p_{k}$. In the SST standard model, beside prescribed scenarios, the company should also think about company specific scenarios. Hence the number $K$ of the scenarios considered by a company is not fixed. However, in the standard SST it is assumed that the scenarios $SC_{k}^{ins}$, $k = 1, 2, \ldots, K$ exclude each other, i.e. that only one of the scenarios can materialize during the next year. This exclusion property was made because of simplicity reasons and is not such a big restriction as it might seem on the first side. If one wants to incorporate the situation, that e.g. $SC_{i}^{ins}$ and $SC_{j}^{ins}$ might occur in the same year, one can add a new scenario $SC_{new}^{ins}$ with face amount $c_{new} = c_{i} + c_{j}$ and a corresponding probability $p_{new}$, for instance $p_{new} = p_{i}p_{j}$ if the the two scenarios are assumed to be independent.

Let

$$T = C_{CY}^{*} + C_{PY}^{*} + C_{PY}^{*} + SC_{ins}^{*},$$

be the total claim amount including scenarios, and denote by $F (x)$ the distribution of $T$. Define

$$p_{0} = 1 - \sum_{k=1}^{K} p_{k} \quad \text{and} \quad c_{0} = 0.$$ 

Then

$$F (x) = \sum_{k=0}^{K} p_{k} \bar{F} (x - c_{k}),$$

where $\bar{F} (.)$ is the distribution before scenarios.

3.2  Modelling of the Insurance Risk in Solvency II

For comparison with the SST we consider a non-life insurance company working in only one region. In the terminology of solvency II the solvency capital required for the insurance risk is denoted by $SCR_{nl}$, where the index $nl$ stands for non-life.

In solvency II, one also distinguishes between current year risk, which is called premium risk, and previous year risk, which is called reserve risk. Contrary to the SST, the claim amount CY is not split into a normal claim amount (caused by the bulk of normal claims) and a big claim amount (caused by big claims or big claim events). However, solvency II considers in addition to the CY- and PY-risk the category of CAT-risks, which are modelled with a face amount (analogous to the scenarios in the SST) indicating the expected loss of natural catastrophes and man-made catastrophes. Thus the CAT-risks cover to some extent the big claim events taken into account by the big-claim modelling in the SST and also allow to consider situations taken into account as scenarios in the SST.

The risk measure in solvency II is the 99.5% value at risk. Solvency II is, however, not distribution based. At the end one comes up with a figure, the solvency capital required (SCR) and not with a distribution from which the SCR is obtained.
Solvency II also distinguishes between different lines of business \( i = 1, 2, \ldots, I \). As in the SST there are provided "standard parameters" called "market parameters", which are used in the standard solvency II calculation.

Solvency II is a framework with formulas how to calculate the SCR. From these formulas we can derive the underlying model assumptions behind solvency II leading to these formulas.

In the next Section we introduce the formulae in solvency II for calculating the SCR and in the following Sections we then discuss the model assumptions behind these formulas.

### 3.2.1 The Calculation of the SCR in Solvency II

As in the whole paper we concentrate on the non-life insurance risk and hence on the solvency capital required named \( SCR_{\text{nd}} \) in solvency II.

We will use the analogous notation as in Section 3.1, that is

\[
X_i = \frac{C_{i}^{\text{CY}}}{P_i} = \text{loss ratio CY},
Y_i = \frac{\widetilde{R}_i}{R_i},
\sigma_i^2 = \text{Var}(X_i),
\tau_i^2 = \text{Var}(Y_i),
\]

where

\[
P_i \quad \text{premium}
\]

\[
R_i \quad \text{reserve at the beginning of the year = best estimate of the outstanding liabilities } L_i, \text{per 1.1.}
\]

\[
\widetilde{R}_i \quad \text{a posteriori best estimate per 31.12. of } L_i.
\]

**Remark:**

- The premium \( P_i \) is further specified in solvency II (earned, written, ...).

The following formula is used for the standard deviation of the premium risk.

\[
\sigma_i = \sqrt{\alpha_i \cdot \sigma_i^2 + (1 - \alpha_i) \cdot \sigma_{i,M}^2},
\]

where \( \alpha_i \) is a credibility weight depending on the lob and on the number of historical years of data available, \( \sigma_{i,M} \) is a given standard parameter (market wide estimate for the premium risk) and \( \sigma_i^2 \) is a company-specific estimate of \( \sigma_i^2 \) based on the company’s own historical data. Hence \( \sigma_i^2 \) is a credibility weighted mean between the company specific estimate \( \sigma_{i,\text{ind}}^2 \) and the market estimate \( \sigma_{i,M}^2 \). For estimating \( \sigma_{i,\text{ind}}^2 \), denote by \( P_{ij} \) and \( X_{ij} \) the premiums and the observed loss ratios of the individual company in lob \( i \) in year \( j \) and by \( n_i \) the number of historical data available and taken into account for lob \( i \). Then the following formula is used in solvency II.

\[
\sigma_{i,\text{ind}}^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} \frac{P_{ij}}{P_i} (X_{ij} - \bar{X}_i)^2,
\]
where 
\[ \overline{X_i} = \sum_{j=1}^{m_i} \frac{P_{ij}}{P_i} X_{ij}. \]

The standard deviations \( \tau_i \) for the reserve risk are given standard parameters which are the same for all companies and which do not depend on the size of the company.

Next the premium plus reserve risk per line of business is considered. As in Section 3.1.3 we introduce 
\[ Z_i = \frac{1}{V_i} (P_i X_i + R_i Y_i), \]
where \( V_i = P_i + R_i. \)

Solvency II assumes, that \( \text{Corr} (X_i, Y_i) = \rho_{CY, PY} = 50\% \) for all \( i. \) Then 
\[ \varphi_i := \sqrt{\text{Var} (Z_i)} = \sqrt{\left( \frac{P_i \sigma_i}{V_i} \right)^2 + 2 \rho_{CY, PY} P_i \sigma_i \tau_i + \left( \frac{R_i \tau_i}{V_i} \right)^2}. \]

(30)

Next it is assumed in solvency II that 
\[ \text{Corr} (Z_i, Z_j) = \rho_{ij} \]

(31)
where the correlations \( \rho_{ij} \) are given standard values equal for all companies. For 
\[ Z_* = \sum_{i=1}^{I} \frac{V_i}{V_*} Z_i \]
we then obtain 
\[ \varphi^2 = \text{Var} (Z_*) = \sum_{i=1}^{I} \frac{V_i V_j \rho_{ij}}{V_*^2} \rho_{ij}, \]

(32)
which is the same as (25).

The formula used in solvency II for calculating the SCR of the combined premium and reserve risk is
\[ \text{SCR}_{pr+res} = V_* \left( \frac{\exp \left( \Phi^{-1} (0.995) \cdot \sqrt{\log (\varphi^2 + 1)} \right)}{\sqrt{\varphi^2 + 1}} - 1 \right) \]
\[ = V_* \text{VAR}_{0.995} (\Psi) \]

(33)

(34)
where 
\[ \Psi = \text{lognormal distributed random variable with } E [\Psi] = 1 \text{ and } \text{Var} (\Psi) = \varphi^2, \]
\[ \text{VAR}_{0.995} (\Psi) = 99.5\% \text{ value at risk of } \Psi - E [\Psi]. \]
\[ V_* = P_* + R_* \]
\[ \Phi (x) = \text{standard normal distribution}. \]
The CAT-risks are similar to the scenarios in the SST. A CAT-risk $CAT_k$ is characterised by a face amount $c_k$ to be interpreted as the expected loss for the company if this catastrophe happened. For natural catastrophes the amounts $c_k$ for a specific company are proportional to the underlying premium ($c_k P_t$ if only lob $t$ is hit by the event resp. a corresponding sum over several lob in the case that several lob are hit). The solvency capital required is for the total of the cat-risks is then calculated by

$$SCR_{CAT} = \sqrt{\sum_{k=1}^{K} c_k^2}.$$  \hspace{1cm} (35)

Finally, the SCR for the non-life insurance risk including the CAT risks is

$$SCR_{nl} = \sqrt{SCR_{pr+res}^2 + SCR_{CAT}^2}.$$ 

3.2.2 Modelling of the Claim Amount CY in Solvency II

In (28) neither $\sigma^2_{i,M}$ nor the credibility weight $\alpha_i$ depends on the size of the company. Therefore we conclude that the following implicit model assumption regarding the CY claim amount is behind solvency II.

Model Assumptions 3.6 (CY claim amount solvency II) It is assumed that

$$\text{Var} (X_i) = \sigma^2_i.$$

Model assumptions 3.6 should be compared with the model assumptions 3.2 in the SST. From this comparison we see that in solvency II the variance of $X_i$ does not depend on the size of the company, or in the terminology of the SST, there is only a parameter risk and no random fluctuation risk.

Discussion and Remarks:

- The assumption that the variance of the loss ratio is independent of the size of the company is questionable. From a conceptual modelling point of view the assumptions in the SST that there is a parameter risk not depending on the size of the company and a random fluctuation risk being inversely proportional to $P_i$ seems to be more realistic.

- Solvency II resp. formula (28) is from a pure mathematical point of view not fully stringent in the sense that (29) is a best unbiased estimator linear in $(X_{ij} - \overline{X}_i)^2$ under the model assumption that $\text{Var} (X_i) = \sigma^2_i / P_i$. Under the model assumptions 3.6 the non weighted mean

$$\sigma^2_{i,ind} = \frac{1}{n_i-1} \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_i)^2,$$
where
\[
X_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}
\]
would be a better estimator than (29).

### 3.2.3 Modelling of the Reserve Risk in Solvency II

**Model Assumptions 3.7 (PY claim amount solvency II)**  
*It is assumed that*
\[
\text{Var} (Y_i) = \tau_i^2.
\]

Model assumptions 3.7 should again be compared with the model assumptions 3.3 in the SST. From this comparison we see that in solvency II the variance of \(Y_i\) does not depend on the size of the company, or in the terminology of the SST, there is only a *parameter risk* and no random fluctuation risk.

**Discussion and Remarks:**

- We can add here the same remark as for the CY risk. The assumption that the variance of the reserve risk is independent of the size of the company is rather questionable.

### 3.2.4 Modelling of the Sum of CY Risk and PY risk in Solvency II

We use the same notation as in Section 3.1.3 and introduce
\[
S_i = C_i^{\text{CY}} + \bar{R}_i, \\
Z_i = \frac{C_i^{\text{CY},n} + \bar{R}_i}{P_i + R_i} = \frac{P_i X_i + R_i Y_i}{P_i + R_i}, \\
V_i = P_i + R_i.
\]

As described in Section 3.2.1, it is assumed that
\[
\text{Corr} (X_i, Y_i) = \rho_{\text{CY,PY}} = 50\% \text{ for all } i, \quad (39) \\
\text{Corr} (Z_i, Z_j) = \rho_{ij}, \quad (40)
\]

where the \(\rho_{ij}\) are given standard parameters (market values). From (39) and (40) follows that
\[
\varphi^2 = \text{Var} (Z_\star) = \sum_{i,j=1}^{I} \frac{V_i V_j \varphi_i \varphi_j}{V_\star^2} \rho_{ij}, \quad (41)
\]
Discussion and remarks on the correlation assumption in solvency II:

- (40) is assumed to be true for any company. Hence it should be fulfilled for a company which has just started and which has no reserve risk as well as for a company in the run-off and having no CY risk and only a reserve risk. Hence we can conclude that

\[
\text{Corr}(X_i, X_j) = \text{Corr}(Y_i, Y_j) = \text{Corr}(Z_i, Z_j) = \rho_{ij},
\]

which means that the correlation matrices of \(X, Y\) and \(Z\) are all the same. To assume the same correlation matrix for the CY-risks and the PY-risks is, however, questionable. The reason that there are correlations between lines of business results from the fact that there might be factors affecting different lines of business simultaneously. However, the calendar year factors for CY- and PY-risks are not necessarily the same and if they are the same they might impact CY- and PY-risks differently. Hence the correlation matrix of \(X, Y\) and \(Z\) are hardly the same in reality. For instance, calendar year factors such as weather conditions which might have an impact on the claim frequency affect only the CY risks but not the reserve risks.

- Using the notation of Section 3.1.3 we can conclude that in solvency II \(R_{CY} = R_{PY}\) with entries \(R_{CY}(i, j) = \rho_{ij}\) and that the diagonal elements of \(R_{CY,PY}\) are equal to \(\rho_{CY,PY} = 50\%\). However the elements of \(R_{CY,PY}\) outside the diagonal depend on the volumes \(P_i\) and on \(R_i\), which is a bit strange. It is very difficult to interprete and to see the structure of \(R_{CY,PY}\) resulting from the solvency II correlation assumptions.

Finally the following model assumption is behind formula (33).

**Model Assumptions 3.8 (claim amount CY + PY solvency II)** It is assumed that \(S_\bullet - E[S_\bullet]\) has the same distribution as \(V_\bullet (\Psi - 1)\), where \(\Psi\) has a lognormal distribution with

\[
E[\Psi] = 1,
\]

\[
\text{Var}(\Psi) = \frac{1}{V_\bullet^2} \varphi^2
\]

with \(\varphi^2 = \text{Var}(Z_\bullet)\) as given in formula (41).

**Remarks and discussion:**

- Model assumptions 3.8 are a direct consequence of formula (33) and the fact that the risk measure in solvency II is the 99.5% value at risk

- Model assumptions 3.8 should be compared with model assumptions 3.4 in the SST. At first glance it seems as if they were identical. As in the SST it holds that

\[
S_\bullet - E[S_\bullet] = V_\bullet (Z_\bullet - E[Z_\bullet]).
\]
However, $E[Z_{\Psi}] = 1$ in the SST, whereas in solvency II $E[Z_{\Psi}]$ is normally smaller than one, because the premiums used for the loss ratios are not the pure risk premiums and hence $E[X_{\Psi}]$ is the expected loss ratio for CY claims, which is usually smaller than one. Thus in solvency II the random variable $Z_{\Psi} - E[Z_{\Psi}]$ is approximated by the random variable $\Psi - 1$, or in other words, $S_{\Psi}$ is modelled with a lognormal distribution with mean $E[S_{\Psi}]$, but with a variance which is slightly different from $\text{Var}(S_{\Psi})$.

If $S_{\Psi}$ was modelled with a lognormal distribution with $E[S_{\Psi}]$ and $\text{Var}(S_{\Psi})$ as in the SST, one would obtain instead of (33)

$$SCR_{pr+res} = \mu_Z V_{\Psi} \left( \frac{\exp \left( \Phi^{-1}(0.995) \cdot \sqrt{\log \left( \frac{\sigma^2}{\mu_Z} + 1 \right)} \right) - 1}{\sqrt{\frac{\sigma^2}{\mu_Z} + 1}} \right) = V_{\Psi} V_{\text{VaR0.995}}(Z_{\Psi}),$$

where

$$\mu_Z = E[Z_{\Psi}].$$

The following table compares $V_{\text{VaR0.995}}(\Psi)$ with $V_{\text{VaR0.995}}(Z_{\Psi})$ for different values of $\varphi$ with $\alpha = 99.5\%$ and $\mu_Z = 85\%$.

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>$V_{\text{VaR0.995}}(\Psi)$</th>
<th>$V_{\text{VaR0.995}}(Z_{\Psi})$</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>0.090</td>
<td>0.091</td>
<td>1.008</td>
</tr>
<tr>
<td>2%</td>
<td>0.189</td>
<td>0.192</td>
<td>1.016</td>
</tr>
<tr>
<td>3%</td>
<td>0.286</td>
<td>0.303</td>
<td>1.024</td>
</tr>
<tr>
<td>4%</td>
<td>0.413</td>
<td>0.426</td>
<td>1.032</td>
</tr>
<tr>
<td>5%</td>
<td>0.540</td>
<td>0.562</td>
<td>1.041</td>
</tr>
<tr>
<td>6%</td>
<td>0.678</td>
<td>0.711</td>
<td>1.050</td>
</tr>
<tr>
<td>7%</td>
<td>0.828</td>
<td>0.877</td>
<td>1.059</td>
</tr>
<tr>
<td>8%</td>
<td>0.991</td>
<td>1.059</td>
<td>1.068</td>
</tr>
<tr>
<td>9%</td>
<td>1.168</td>
<td>1.259</td>
<td>1.078</td>
</tr>
<tr>
<td>10%</td>
<td>1.361</td>
<td>1.480</td>
<td>1.088</td>
</tr>
<tr>
<td>11%</td>
<td>1.570</td>
<td>1.724</td>
<td>1.098</td>
</tr>
<tr>
<td>12%</td>
<td>1.797</td>
<td>1.991</td>
<td>1.108</td>
</tr>
<tr>
<td>13%</td>
<td>2.043</td>
<td>2.286</td>
<td>1.119</td>
</tr>
<tr>
<td>14%</td>
<td>2.310</td>
<td>2.609</td>
<td>1.130</td>
</tr>
<tr>
<td>15%</td>
<td>2.599</td>
<td>2.965</td>
<td>1.140</td>
</tr>
<tr>
<td>16%</td>
<td>2.913</td>
<td>3.355</td>
<td>1.152</td>
</tr>
<tr>
<td>17%</td>
<td>3.253</td>
<td>3.782</td>
<td>1.163</td>
</tr>
<tr>
<td>18%</td>
<td>3.621</td>
<td>4.251</td>
<td>1.174</td>
</tr>
<tr>
<td>19%</td>
<td>4.019</td>
<td>4.765</td>
<td>1.186</td>
</tr>
<tr>
<td>20%</td>
<td>4.450</td>
<td>5.328</td>
<td>1.197</td>
</tr>
</tbody>
</table>

### 3.2.5 Modelling and Aggregation of CAT-Risks

The assumptions for the cat-risks leading to formula (35) are as follows:

**Model Assumptions 3.9 (CAT risks solvency II)** *The cat-risks $CAT_k; k = 1, 2, \ldots, I$, are independent and normally distributed with $V_{\text{VaR0.995}}(CAT_k) = c_k$.**
Discussion and Remark:
The assumption that the cat-risks are normally distributed follows from the quadratic
sum in (35). Of course, this assumption is not adequate for cat-risks, or in other words,
the aggregation formula (35) is a practical, but simplified way to calculate the necessary
capital for cat-risks.

3.3 Summary and Comparison of the modelling in the SST and
in Solvency II

In this Section we shortly summarise the modelling of the insurance risk in the SST and
in solvency II and work out, what is common in the two modelling frameworks and where
there are major differences.

First we should note that both, the SST and solvency II are parameter-based models.
Whereas solvency II is essentially a factor model, the SST is distribution based and the
parameters enter in the calculation of the resulting distribution. From a mathematical
point of view, the distribution based procedure is not a big complication and does not
need more than techniques which are nowadays well known by people having passed the
actuarial exams to become a full actuary.

A first difference is the risk measure. In the SST the 99% expected shortfall is used,
whereas in solvency II it is the 99.5% value at risk. As shown in [6], the difference is
rather small in the case of a lognormal distribution within the relevant parameter range.

A major difference is the modelling of the variance of the CY (premium) and the PY
(reserve) risk. In the SST it is assumed, that the variance is the sum of a parameter risk
and a random fluctuation risk, where the first is independent of the size of the company
and the latter becomes smaller the bigger the company is. In solvency II this variance is
assumed to be independent of the size of the company.

A further difference between the SST and solvency II is that for the CY risks the SST
models the claims load caused by the bulk of normal claims and the one caused by the
big claims separately, whereas solvency II does not make such a distinction.

Major differences also exist in the assumptions about the correlation matrices. The
current SST assumptions are such that there are no correlations for the reserve risks
between lines of business and that there is also no correlation between CY- and PY- risks,
whereas solvency II assumes a correlation of 50% between the premium and reserve risk
of the same line of business and that the correlations between lob are the same for the
premium risk and the reserve risk. Neither the assumptions in the SST nor the ones in
solvency II are fully satisfactory from a conceptual point of view.

Finally, the scenarios in the SST and the modelling of the CAT-risks in solvency II are
somewhat similar. However, they are aggregated in a fully different manner. In solvency II
the corresponding SCR are just added by quadratic summation. In the SST they are fully
integrated in the calculation of the resulting distribution. Here the fundamental difference
between the distribution based approach in the SST and the factor-model approach and
aggregation of the resulting SCR with a correlation matrix becomes most obvious.
4 Parameter Estimation

The standard parameter values in the STT and in solvency II differ substantially (see appendices A and B). This fact by itself reveals that the estimation of the parameters is a problem on its own. In the following we present estimators of the parameters based on the observations of the individual data of the company. For some of the parameters we also show how the individual estimators can be combined with the standard or market values by credibility techniques.

4.1 Estimation of Parameter in Solvency II

The following straightforward estimator is part of the official solvency II guideline and was already encountered in Section 3.2.1, formula (29).

\[
\hat{\sigma}_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} \frac{P_{ij}}{P_i} (X_{ij} - \bar{X}_i)^2,
\]

where

\[
X_{ij} = \frac{C_{ij}}{P_{ij}} = \text{observed loss ratio of lob } i \text{ in year } j,
\]

\[
\bar{X}_i = \sum_{j=1}^{n_i} \frac{P_{ij}}{P_i} X_{ij},
\]

\[
n_i = \text{number of historical years taken into account}.
\]

An analogous estimator was suggested in [6] for the reserve risk, namely

\[
\hat{\tau}_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} \frac{R_{ij}}{R_i} (Y_{ij} - \bar{Y}_i)^2,
\]

where

\[
Y_{ij} = \frac{\hat{R}_{ij}^{PY}}{R_{ij}},
\]

\[
\bar{Y}_i = \sum_{j=1}^{n_i} \frac{R_{ij}}{R_i} Y_{ij}.
\]

Remarks:

- (43) and (44) are a best unbiased estimate of \( \sigma_i^2 \) and \( \tau_i^2 \) linear in \((X_{ij} - \bar{X}_i)^2\) and \((Y_{ij} - \bar{Y}_i)^2\), respectively, under the assumption that \( \text{Var}(X_{ij}) = \sigma_i^2 / P_{ij} \) and \( \text{E}[X_{ij}] = \mu_X \) and \( \text{Var}(Y_{ij}) = \tau_i^2 / R_{ij} \).

- In practice (44) usually underestimates the reserve risk since despite the reserves \( R_{ij} \) being best estimate reserves they often contain some smoothing elements.

The above estimators are simple straightforward estimators. In the next Section we will consider estimators of the parameters used in the SST. However, these parameter estimators could also be used in an internal model in solvency II. At the end, in solvency II as well as in the SST, the aim is to estimate the coefficients of variation of the premium and the reserve risk.
4.2 Estimation of the Parameters in the SST

The SST framework does not yet provide formulas for estimating the parameters from individual data. In the following we are going to suggest such estimators.

The SST distinguishes for the premium risk between normal claims CY and big claims CY. However, if one does not want to make this distinction, the estimators developed in Sections 4.2.1 and 4.2.2 are also valid when applied to the total of CY-claims.

4.2.1 Estimation of the random fluctuation risk normal claims CY

In Section 3.1.1 we have seen that

\[ \sigma^2_{i, \text{fluct}} = \text{CoVa} a^2 \left( Y_i^{(v)} \right) + 1. \]

Hence we have to estimate the coefficient of variation of the claim sizes. For this purpose we calculate first the observed coefficient of variations for different accident years \( j \), that is

\[ \text{CoVa} a_{ij} = \frac{1}{N_{ij} - 1} \sum_{v=1}^{N_{ij}} \left( Y_{ij}^{(v)} - \bar{Y}_i \right)^2 \frac{1}{\bar{Y}_i^2}, \]  

where \( N_{ij} \) is the number of normal claims of lob \( i \) in accident year \( j \), \( Y_{ij}^{(v)}, v = 1, 2, \ldots, N_{ij} \), the individual claim amounts and \( \bar{Y}_i \) the mean of the \( Y_{ij}^{(v)} \). However, one should have in mind that the individual claim amounts \( Y_{ij}^{(v)} \) in recent accident years \( j \) consist to a great part of case estimates. It is a known fact that, especially in long tail lines of business, (45) underestimates in recent accident years the true ultimate coefficient of variation. Therefore, the values resulting from (45) should first be extrapolated to their ultimate value. The following table shows the development triangle of the coefficient of variation in motor liability of a big Swiss insurance company.

| Development triangle CoVa claim amounts in motor liability |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| AY/DY           | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
| 1998            | 6.1 | 7.1 | 8.1 | 8.5 | 9.1 | 9.7 | 10.1| 10.2| 10.2| 10.9|
| 1999            | 6.1 | 7.6 | 8.1 | 9.0 | 10.0| 10.3| 10.6| 10.6| 10.7| 10.7|
| 2000            | 5.8 | 7.4 | 8.9 | 9.5 | 9.6 | 9.6 | 10.4| 10.4| 10.7| 10.7|
| 2001            | 5.7 | 7.8 | 9.0 | 9.0 | 9.2 | 10.3| 10.6| 10.6| 10.6| 10.6|
| 2002            | 5.6 | 8.2 | 8.8 | 9.1 | 10.0| 10.2| 10.6| 10.9| 10.9| 10.9|
| 2003            | 6.6 | 8.5 | 9.2 | 10.1| 10.7| 11.1| 11.5| 11.9| 11.9| 11.9|
| 2004            | 6.7 | 8.5 | 9.2 | 10.1| 10.7| 11.1| 11.5| 11.9| 11.9| 11.9|
| 2005            | 5.2 | 7.1 | 8.7 | 9.2 | 9.7 | 10.1| 10.5| 10.8| 10.8| 10.8|
| 2006            | 5.4 | 7.2 | 8.7 | 9.3 | 9.0 | 9.5 | 9.9 | 10.3| 10.6| 10.6|
| 2007            | 5.7 | 7.4 | 8.4 | 8.8 | 9.4 | 9.7 | 10.1| 10.4| 10.4| 10.4|

The values above the diagonals are the observed empirical coefficients of variation resulting from (45) of accident years \( i \) calculated at different development years. The values below the diagonals are simple chain ladder forecasts. We can see from this table that for instance in the development years 0 or 1 the coefficients of variation are substantially...
underestimated compared to the ultimate values. We can also see that the observed coefficients of variation are rather stable and that the mean of the extrapolated coefficients of variation (column 9) over the different accident years is an accurate estimator for the coefficient of variation of the claim sizes. Indeed, this parameter can usually be estimated with great accuracy from the own data.

### 4.2.2 Estimation of the parameter risk of the normal claim amount CY

For the parameter risk, most companies in Switzerland use the default standard values provided by the supervision authority. One of the reasons is that there has not yet been developed an estimator for the parameter risk for the own data which is based on a sound actuarial basis. In the following we try to fill this gap and we will present such estimators.

We assume that we have observed historical data on the accident years $j = 1, 2, \ldots, J$. We consider a specific lob $i$. To simplify notation we drop in the following the index $i$ and we write for instance $X_j$ for the observation of the specific lob considered in the year $j$. It is assumed,

i) that each year is characterised by its characteristics $\Theta_j = (\Theta_{j1}, \Theta_{j2})^T$ and that for each year the model-assumptions 3.1 are fulfilled with underlying claim frequency parameter $\lambda_j$ and claim severity parameter $\mu_j$,

ii) that the coefficient of variation for the claim sizes $CoVa \left( Y_j^{(v)} \right)$ is the same for all years,

iii) that random variables belonging to different years are independent and $\Theta_1, \Theta_1, \ldots, \Theta_J$ are independent and identically distributed.

From the above assumptions and from (5), (7), (62), (63), we see that we have the following situation:

The random variables $X_j$ are independent with

\[
\text{Var} \left( X_j \right) = \sigma_{\text{param}}^2 + \frac{\sigma_{\text{fluct}}^2}{\nu_j} \approx \sigma_{\text{param}}^2 + \frac{\sigma_{\text{fluct}}^2}{\bar{P}_j},
\]

where

\[
\begin{align*}
\nu_j &= \text{number of a priori expected claims}, \\
\bar{P}_j &= E \left[ C_j^{CY,n} \right], \\
\sigma_{\text{fluct}}^2 &= CoVa^2 \left( Y^{(v)} \right) + 1,
\end{align*}
\]

and with

\[
\begin{align*}
E \left[ X \mid \Theta \right] &= \Theta_1 \Theta_2, \\
\text{Var} \left( X \mid \Theta \right) &= \frac{1}{w^2} \cdot \Theta_1 \Theta_2^2 \left( CoVa^2 \left( Y \right) + 1 \right).
\end{align*}
\]
Hence the observations $X_j, j = 1, 2, \ldots, J$, fulfill the conditions of the Bühlmann-Straub model (see for instance [1], Chapter 4). However, we do not have the "standard situation" of the Bühlmann-Straub model, where we have a collective of risks $i = 1, 2, \ldots, I$, and where for each of these risks observations $X_{ij}, j = 1, 2, \ldots, n$, over several years are available. Here the risks are the different years $j = 1, 2, \ldots, J$ and for each of these "risks", one has only one observation $X_j$. In the standard situation of the Bühlmann-Straub model, the observations over several years for each risk are used to estimate the within risk variance component. But here the within risk variance is $\sigma^2_{\text{fluct}}$, which can be estimated otherwise as described in Section 4.2.1.

Therefore we can use the standard estimators in the Bühlmann-Straub model (see e.g. [1], Chapter 4.8) and obtain

$$
\hat{\sigma}^2_{\text{param}} = c \cdot \left\{ \frac{J}{J-1} \sum_{j=1}^{J} \frac{w_j}{w_\ast} \frac{(X_j - \overline{X})^2}{J\hat{\sigma}^2_{\text{fluct}}/n_\ast} \right\}, \tag{48}
$$

where

$$
c = \frac{I - 1}{I} \left\{ \sum_{i=1}^{I} \frac{w_i}{w_\ast} \left( 1 - \frac{w_i}{w_\ast} \right) \right\}^{-1},\nonumber$$

$$
\hat{\sigma}^2_{\text{fluct}} = \frac{\text{CoVa}^2(Y^{(v)})}{\overline{X}} \text{ estimator of the coefficient of variation of the claim sizes},
$$

$$
n_\ast = \text{observed number of claims}.
$$

Whereas the total claim amounts $C_j^{\text{CY,n}}$ are known figures and recorded in the files of a company, this is not the case for the corresponding pure risk premiums $\bar{P}_j = E\left[ C_j^{\text{CY,n}} \right]$. Hence, before being able to apply (48) one has to determine $\bar{P}_j$, which are used for calculating the $X_j$.

Denote by

$$
LR_j = \frac{C_j^{\text{CY,n}}}{P_j}
$$

the observed claims ratio for normal claims in year $j$, where $P_j$ is the earned premium. Under the assumption that

$$
E\left[ LR_j \right] = \mu_{LR} \tag{49}
$$

is the same for all years, it is suggested to use

$$
\bar{P}_j = \overline{LR} \cdot P_j
$$

where

$$
\overline{LR} = \sum_{j=1}^{J} \frac{P_j}{\bar{P}_j} LR_j.
$$

23
Often (49) is not fulfilled because of things like business cycles and premium policies. Then the premiums have first have to be adjusted by such effects, such that the loss ratios calculated with the adjusted premiums should then fulfil (49).

Finally, we have to decide which weights \( w_j \) should be used in (48). We suggest to take the \( \tilde{P}_j \) as weights, which is justified by the variance property (47). A numerical example will be given in the next Section 4.2.3.

The estimator (48) is based on the properties and the variance structure of \( X_j \). It does, however not make use of the fact that according to (7)

\[
\sigma_{\text{param}}^2 \simeq \text{Var}(\Theta_1) + \text{Var}(\Theta_2).
\]

An alternative approach to (48) is therefore to estimate \( \text{Var}(\Theta_1) \) and \( \text{Var}(\Theta_2) \) directly, which then involves the claim frequencies and the claim averages. Hence the following estimators are recommended for lines of business where there is a "natural" volume measure, which is necessary that the claim frequencies are comparable over time. This is the case for personal lines like motor liability, household insurance etc., but not for corporate property, corporate liability or industrial fire insurance.

Let

\[
F_j = \frac{N_j}{\nu_j},
\]

where \( N_j \) = number of claims in year \( j \),

\[
\nu_j = \text{a priori expected number of claims in year } j.
\]

Note that \( F_j \) is a "standardised" frequency with \( E[F_j] = 1 \). If there are no trends in the claim frequency, that is if

\[
E[F_j] = \lambda \text{ for all } j,
\]

then one can put

\[
\nu_j = \hat{\lambda} \cdot JR_j,
\]

where \( JR_j \) = number of risks in year \( j \),

\[
\hat{\lambda} = \frac{N_*}{JR_*}.
\]

If there are trends, then use \( \nu_j = \hat{\lambda}_j \cdot JR_j \), where \( \hat{\lambda}_j \) is the trend adjusted estimate of the a priori claim frequency in year \( j \).

Under model-assumptions 3.1, where \( N_j \) is assumed to be conditionally Poisson, it holds that

\[
\text{Var}(F_j) = \text{Var}(\Theta_1) + \frac{\lambda}{\nu_j}
\]

and one can show that

\[
\sigma_{\Theta_1}^2 = \left( c \cdot \frac{\nu_*}{J} \right)^{-1} \left( \frac{V_F}{F} - 1 \right)
\]
where

\[ V_F = \frac{1}{J-1} \sum_{j=1}^{J} \nu_j (F_j - \bar{F})^2, \]

\[ \bar{F} = \sum_{j=1}^{J} \frac{\nu_j}{\nu_0} F_j, \]

\[ c = \sum_{j=1}^{J} \frac{\nu_j}{\nu_0} \left( 1 - \frac{\nu_j}{\nu_0} \right). \]

is an unbiased estimator of \( \text{Var} (\Theta_1) \).

For estimating \( \text{Var} (\Theta_2) \), one has to look at the observed claims averages

\[ \bar{Y}_j = \frac{1}{N_j} \sum_{t=1}^{N_j} Y_{j}^{(t)} \]

in different years. Because of inflation or possible other trends, one has first to adjust the claim sizes in a given year \( j \) by these factors to bring them on the same level, which can be done for instance by linear regression. After these adjustments, the claim sizes and claim averages fulfil the Bühlman-Straub credibility model for claim sizes as presented in Chapter 4.11 of [1] and one can use this theory to estimate \( \text{Var} (\Theta_2) \).

### 4.2.3 Numerical Example

The following table shows the figures in motor liability of a big insurance company in Switzerland. For confidentiality reasons the figures have been multiplied by a constant factor.

<table>
<thead>
<tr>
<th>year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>earned premiums</td>
<td>444'256</td>
<td>442'133</td>
<td>439'773</td>
<td>454'701</td>
<td>473'404</td>
<td>504'481</td>
<td>523'446</td>
<td>544'809</td>
<td>557'566</td>
<td>557'291</td>
</tr>
<tr>
<td>adjusted for business cycle</td>
<td>470'610</td>
<td>492'902</td>
<td>499'742</td>
<td>528'722</td>
<td>530'722</td>
<td>543'037</td>
<td>541'870</td>
<td>544'265</td>
<td>553'141</td>
<td>557'291</td>
</tr>
<tr>
<td>number of risks</td>
<td>606'599</td>
<td>602'911</td>
<td>592'008</td>
<td>605'140</td>
<td>609'140</td>
<td>603'345</td>
<td>600'003</td>
<td>598'497</td>
<td>595'263</td>
<td></td>
</tr>
<tr>
<td>number of claims</td>
<td>464'37</td>
<td>668'14</td>
<td>660'11</td>
<td>668'94</td>
<td>655'83</td>
<td>655'82</td>
<td>633'28</td>
<td>630'12</td>
<td>609'55</td>
<td>613'97</td>
</tr>
<tr>
<td>claim frequency</td>
<td>10.6%</td>
<td>11.6%</td>
<td>11.2%</td>
<td>11.2%</td>
<td>10.8%</td>
<td>10.9%</td>
<td>10.6%</td>
<td>10.3%</td>
<td>10.3%</td>
<td>10.3%</td>
</tr>
<tr>
<td>total claim amount</td>
<td>443'614</td>
<td>492'726</td>
<td>485'613</td>
<td>489'085</td>
<td>480'175</td>
<td>511'006</td>
<td>478'947</td>
<td>452'545</td>
<td>476'601</td>
<td>473'176</td>
</tr>
<tr>
<td>claims average</td>
<td>695'7</td>
<td>706'1</td>
<td>735'7</td>
<td>731'1</td>
<td>732'2</td>
<td>779'2</td>
<td>756'3</td>
<td>7'182</td>
<td>7'819</td>
<td>7'707</td>
</tr>
<tr>
<td>adjusted for trends</td>
<td>799'6</td>
<td>801'3</td>
<td>821'0</td>
<td>802'7</td>
<td>790'9</td>
<td>828'4</td>
<td>7'916</td>
<td>7'402</td>
<td>7'937</td>
<td>7'707</td>
</tr>
<tr>
<td>loss ratio</td>
<td>443'614</td>
<td>492'726</td>
<td>485'613</td>
<td>489'085</td>
<td>480'175</td>
<td>511'006</td>
<td>478'947</td>
<td>452'545</td>
<td>476'601</td>
<td>473'176</td>
</tr>
<tr>
<td>observed</td>
<td>100.1%</td>
<td>99.7%</td>
<td>99.1%</td>
<td>102.5%</td>
<td>106.7%</td>
<td>113.7%</td>
<td>118.0%</td>
<td>122.8%</td>
<td>125.7%</td>
<td>125.6%</td>
</tr>
<tr>
<td>with adjusted premiums</td>
<td>106.1%</td>
<td>111.1%</td>
<td>112.7%</td>
<td>118.2%</td>
<td>119.6%</td>
<td>122.4%</td>
<td>122.1%</td>
<td>122.7%</td>
<td>124.7%</td>
<td>125.6%</td>
</tr>
</tbody>
</table>

This data set was one of the many data sets which were evaluated in [5]. In the table below the results for the different estimators are listed, where it was assumed that the number of the a priori expected claims for the CY is 62'000.

Estimator 1 shows the result of the straightforward estimator (43), with and without the business cycle adjustments in the premiums. We see that the difference is quite big between the two results. Indeed, there were quite remarkable business cycles during the
last 10 years, such that the risk of the CY claims is overestimated when just using the formula from solvency II and not taking them into account.

Estimator 2 is the resulting estimator when estimating the coefficient of variation with the procedure described in Section 4.2.1 and where the parameter risk is estimated with formula (48). Again the parameter risk is substantially overestimated when not taking into account the business cycles.

Estimator 3 differs from estimator 2 only in the estimation of the parameter risk. Here the variance of \( \Theta_1 \) and of \( \Theta_2 \) are estimated by using (52) and the estimator described at the end of Section 4.2. The estimate for \( \text{Var}(\Theta_2) \) turned out to be zero. This was the case in nearly all lob investigated in [5], which indicates that the main driver for the parameter risk are random fluctuations from year to year in the claim frequency. Thus it might be a good idea just to concentrate on the claim frequency and to estimate the parameter risk from it. This estimator does not depend on the premiums and hence the results with and without premium adjustments are the same.

The same holds true for estimator 4 where we have just carried through the SST calculations with the SST standard parameters.

In this example, the results in the column "with adjusted premiums" are fairly near to each other, whereas for estimators 1 and 2 there are big differences between the first and the second column.

<table>
<thead>
<tr>
<th>estimator</th>
<th>with earned premiums</th>
<th>with adjusted premiums</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10.5%</td>
<td>5.8%</td>
</tr>
<tr>
<td>( CoV_a(Y^{(\nu)}) )</td>
<td>10.8</td>
<td>10.8</td>
</tr>
<tr>
<td>( \sigma_{fluct} )</td>
<td>4.4%</td>
<td>4.4%</td>
</tr>
<tr>
<td>( \sigma_{param} )</td>
<td>10.3%</td>
<td>4.2%</td>
</tr>
<tr>
<td>estimator 2</td>
<td>11.2%</td>
<td>6.1%</td>
</tr>
<tr>
<td>( \sigma_{fluct} )</td>
<td>4.4%</td>
<td>4.4%</td>
</tr>
<tr>
<td>( \sqrt{\text{Var}(\Theta_1)} )</td>
<td>3.8%</td>
<td>3.8%</td>
</tr>
<tr>
<td>( \sqrt{\text{Var}(\Theta_2)} )</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>( \sigma_{param} )</td>
<td>3.8%</td>
<td>3.8%</td>
</tr>
<tr>
<td>estimator 3</td>
<td>5.3%</td>
<td>5.3%</td>
</tr>
<tr>
<td>( CoV_a(Y^{(\nu)}) ) ( \sqrt{(CoV_a(Y^{(\nu)})^2 + 1)}/\nu )</td>
<td>4.0%</td>
<td>4.0%</td>
</tr>
<tr>
<td>( \sigma_{param} )</td>
<td>3.5%</td>
<td>3.5%</td>
</tr>
<tr>
<td>estimator 4 (with standard parameters SST)</td>
<td>5.8%</td>
<td>5.8%</td>
</tr>
</tbody>
</table>

### 4.2.4 Estimation of the Pareto Parameters for Big Claims

We consider again a specific lob \( i \) and drop the index \( i \) to simplify notation. Assume that \( Y_1, Y_2, \ldots \) are the big claims above a certain threshold \( c \) and we assume that they are
Pareto-distributed with Pareto-parameter \( \hat{\vartheta} \).

Assume you have observed \( n \) such claims. It is well known that

\[
\hat{\vartheta} = \left( \frac{1}{n-1} \sum_{\nu=1}^{n} \ln \left( \frac{Y_{\nu}}{c} \right) \right)^{-1}
\]  

(53)

is an unbiased estimator of \( \vartheta \) with

\[
E \left[ \hat{\vartheta} \right] = \vartheta, \\
CoVa \left( \hat{\vartheta} \right) = \frac{1}{\sqrt{n-2}}.
\]  

(54)

The number of observed big claims of an individual company in the considered lob might be rather small. From (54) it is seen that the uncertainty of this estimate then becomes fairly big. Hence, it is desirable to take also into account in some way the standard value of the SST, which can be considered as an estimate gained from industry-wide data.

Indeed, the estimator \( \hat{\vartheta} \) fulfills the Bühmann-Straub credibility model and we can use credibility techniques to combine the individual experience with the a priori estimate given by the standard value of the SST. The credibility estimator is given by (see [10] or [1], Chapter 4.14)

\[
\hat{\vartheta}^{\text{cred}} = \alpha \hat{\vartheta} + (1 - \alpha) \vartheta_0
\]  

(55)

where \( \hat{\vartheta} \) is as in (53) and where

\[
\alpha = \frac{n-2}{n-1 + \kappa}, \quad \vartheta_0 = \text{standard value from the SST}, \quad \kappa = CoVa \left( \Theta \right)^{-2}.
\]

Here \( CoVa \left( \Theta \right) \) denotes the coefficient of variation of the Pareto parameter within the different companies. It could be estimated for instance by the supervision authority from the data of the different companies, or one can assess it in a pure Bayesian way, let’s say to assume that it is 25%, which would result in a value of 16 for \( \kappa \). For instance if your individual estimator is based on 10 observed big claims, then you would give a credibility weight of 32% to your own Pareto-estimate and if your estimator is based on 30 observed big claims, then the credibility weight given to your own estimate would be 62%.

### 4.3 Estimation of the Reserve Risk

The reserve risk is the same in the SST and in solvency II. Estimator (44) is straightforward, but as already mentioned in Section 4.1, this estimator usually underestimates the reserve risk.

We believe that the reserve risk should be determined based on the techniques used for estimating these reserves. The most known formula for estimating the variance or mean squared error of the reserves is the famous formula of Mack for the chain ladder reserving
method (see [8]). However, Mack’s formula measures the ultimate reserve risk whereas for solvency purposes we need the one-year reserve risk.

The one-year reserve risk is best understood when we look at the recursive formula for the chain ladder reserving method. This has been done in [2]. The formula derived there coincides with the result found in [9].

In the following, we briefly summarize these results.

Assume that at time $I$, there is given a development triangle or trapezoid

\[ D_I = \{ C_{i,j} : 0 \leq i \leq I, \ 0 \leq j \leq J, \ i + j \leq I \}, \]

where $C_{i,j}$ denote cumulative claims payments or cumulative incurred claims of accident year $i$ in development year $j$ and where $C_{i,j}$ is the ultimate claim. Let

\[
\widehat{C}_{i,J} = C_{i,I-i} \prod_{j=I-i}^{J-1} \widehat{f}_j,
\]

\[
\widehat{R}_I = \widehat{C}_{i,J} - C_{i,J}^{\text{paid}},
\]

\[
\widehat{\sigma}_j^2 = \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} C_{i,j} \left( F_{i,j} - \widehat{f}_j \right)^2.
\]

\[
F_{ij} = \frac{C_{i,j+1}}{C_{i,j}},
\]

\[
S_j^{[k]} = \sum_{i=0}^{k} C_{i,j},
\]

\[
\widehat{f}_j = \frac{S_j^{[I-j-1]}}{S_j^{[I-j-1]}},
\]

(56)

Note that the $\widehat{f}_j$ are the well known chain ladder factors, $\widehat{C}_{i,J}$ is the chain ladder forecast for the ultimate claim and $\widehat{R}_I$ is the chain ladder reserve for accident year $i$.

Then in [4] and [9] the following result has been derived for the one-year reserve risk.

**Theorem 4.1 (one-year mse)** The one-year mean square error can be estimated as follows:

i) single accident year $i$

\[
mse \left( \widehat{R}_i \right) = C_{i,I-i} \Gamma_{I-i} + C_{i,I-i}^2 \Phi_{I-i}
\]

(58)

where

\[
\Gamma_{I-i} = \widehat{\sigma}_{I-i}^2 \left( 1 + \frac{C_{I,I-i}}{S_j^{[I-i]}} \right) \prod_{j=I-i+1}^{J-1} \widehat{f}_j,
\]

\[
\Phi_{I-i} = \frac{\widehat{\sigma}_{I-i}^2}{S_j^{[i]} \prod_{j=I-i+1}^{J-1} \widehat{f}_j^2},
\]
ii) all accident years

\[
mse \left( \hat{R}_t \right) = \sum_{i=I-J+1}^{I} mse \left( \hat{R}_i \right) + 2 \sum_{I-J+1 \leq i < k \leq I} C_{i,I-i} \widehat{C}_{k,I-k} (\Gamma_{I-i} + \Phi_{I-i}), \quad (59)
\]

where \( \widehat{C}_{k,I-k} \) is the chain ladder forecast of \( C_{k,I-k} \).

Remarks:

- Since the reserves \( \hat{R}_t \) are proportional to \( C_{i,I-i} \), model assumptions 3.1 (or (14), respectively) are consistent with model assumptions 3.3 (or (58) respectively). In (58) the term \( C_{i,I-i} \Gamma_{I-i} \) corresponds to the random fluctuation risk and the term \( C_{i,I-i}^2 \Phi_{I-i} \) to the parameter risk.

- We believe that the parameter risk for the chain ladder reserving method can be treated in a rigorous mathematical way only in a Bayesian set-up (see[2], [3], [4]). With the Bayesian set-up and taking a non-informative prior the resulting estimates are slightly different from Mack’s formula for the ultimate risk or the above formula for the one-year risk. Mack’s formula and the above formula are then obtained by a first order Taylor approximation from the corresponding results with the Bayesian approach.

When looking at the reserve risk from a solvency point of view, we are interested in the 100- or 200-year adverse event. What are such events that first come into our mind? There are things like a high inflation or a change in legislation like a decrease of the technical interest rate to calculate the lump sums in motor liability insurance, hence situations with a high adverse calendar year effect. But such events are usually not observed in the triangles and not captured by chain-ladder or similar reserving methods. Hence they are also not reflected by the above estimator of the reserve risk. Thus the question arises whether we do the right thing when looking at the reserve risk from a solvency point of view. The answer is that the above method is adequate for the reserve risk except in extraordinary situations with a huge adverse calendar year effect. Hence, it seems absolutely necessary to complement the reserve risk calculation with a scenario of a hyper-inflation or another adverse situation which might happen. Indeed, nobody knows whether the actual finance crisis will not lead to high inflation in some years.

The following example is taken from [5]. First you find below a development triangle in private-liability of a major Swiss insurance company. For confidentiality reasons the figures are multiplied by a constant factor. The figures above the diagonal are the observed cumulative payments, whereas the figures below the diagonal show the chain ladder forecasts.
The next table shows the resulting chain ladder reserves and the corresponding standard-deviation for each accident year and for the total. The ultimate reserve risk calculated with Mack’s formula amounts to 13.3%, whereas the one year reserve risk according to formula (59) reduces to 6.6%.

<table>
<thead>
<tr>
<th>acc. year</th>
<th>CI-Reserve</th>
<th>ultimate (Mack)</th>
<th>one year</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CoVa</td>
<td>sqrt mse</td>
<td>Vko_CDR</td>
</tr>
<tr>
<td>1979-1987</td>
<td>84</td>
<td>206.4%</td>
<td>174</td>
</tr>
<tr>
<td>1988</td>
<td>86</td>
<td>198.0%</td>
<td>170</td>
</tr>
<tr>
<td>1989</td>
<td>199</td>
<td>129.2%</td>
<td>257</td>
</tr>
<tr>
<td>1990</td>
<td>214</td>
<td>111.7%</td>
<td>275</td>
</tr>
<tr>
<td>1991</td>
<td>326</td>
<td>92.1%</td>
<td>301</td>
</tr>
<tr>
<td>1992</td>
<td>415</td>
<td>78.9%</td>
<td>328</td>
</tr>
<tr>
<td>1993</td>
<td>475</td>
<td>70.8%</td>
<td>336</td>
</tr>
<tr>
<td>1994</td>
<td>687</td>
<td>73.6%</td>
<td>506</td>
</tr>
<tr>
<td>1995</td>
<td>708</td>
<td>73.6%</td>
<td>506</td>
</tr>
<tr>
<td>1996</td>
<td>971</td>
<td>70.0%</td>
<td>475</td>
</tr>
<tr>
<td>1997</td>
<td>1'015</td>
<td>59.7%</td>
<td>605</td>
</tr>
<tr>
<td>1998</td>
<td>1'259</td>
<td>51.4%</td>
<td>647</td>
</tr>
<tr>
<td>1999</td>
<td>1'413</td>
<td>46.8%</td>
<td>662</td>
</tr>
<tr>
<td>2000</td>
<td>1'627</td>
<td>42.5%</td>
<td>696</td>
</tr>
<tr>
<td>2001</td>
<td>1'701</td>
<td>48.4%</td>
<td>823</td>
</tr>
<tr>
<td>2002</td>
<td>2'015</td>
<td>43.0%</td>
<td>867</td>
</tr>
<tr>
<td>2003</td>
<td>2'343</td>
<td>41.2%</td>
<td>964</td>
</tr>
<tr>
<td>2004</td>
<td>2'714</td>
<td>34.4%</td>
<td>933</td>
</tr>
<tr>
<td>2005</td>
<td>3'809</td>
<td>28.6%</td>
<td>1'088</td>
</tr>
<tr>
<td>2006</td>
<td>10'157</td>
<td>13.1%</td>
<td>1'328</td>
</tr>
<tr>
<td>Total</td>
<td>32'250</td>
<td>13.1%</td>
<td>4'294</td>
</tr>
</tbody>
</table>

We have also looked at the claims development results over the last 10 years of the same lob and then applied the estimator (44). The resulting reserve risk was only 3.7%, which
confirms our belief mentioned in Section 4.1, that the true reserve risk is underestimated by (44).

Finally, we should consider an adverse extraordinary scenario and also take it into account when considering the reserve risk. If we assume that claims inflation increases by 3 \% points (additional inflation to the one already existent in the development triangle) and stays on this level for 10 years, then this would create a reserve loss of 4’401 or 13.6\% of the reserves. Inflation scenarios would also affect the reserves of other lob, and its impact should be taken into account as a reserve scenario for the total business of a company.

The observations in the development triangles might vary quite a lot due to random fluctuations, in particular for small and medium sized companies. It would therefore be helpful to know an industry wide payment pattern and a technique how to combine the individual pattern with the industry wide pattern to obtain an optimal estimate. Here we just want to mention that such a technique was presented in [4]. The main idea is to use credibility to obtain credibility estimates of the chain ladder factors, which are then a weighted mean between the industry wide factors and the factors obtained from the company’s own triangle.

4.4 On the Estimation of the Correlation Matrices

We believe that we should first think about the reasons why risks of different lob or the premium and reserve-risk of the same lob are correlated. The main reasons for such correlations are calendar year effects affecting the different risks simultaneously.

To understand the impact of diagonal factors let us have a closer look at the reserve risk. However, the same considerations could also be applied to the CY-risk.

Using the same notation as in Section 3.1.2 we find

\[ \widetilde{R}_i = Y_i C_i \]  

(60)

with

\[ E [Y_i] = 1, \]
\[ \text{Var} (Y_i) = \tau_i^2. \]

Assume that (60) reflects the situation before calendar year effect and that the reserves risk of different lob are independent, that is

\[ \text{Cov} (Y_i, Y_j) = 0 \text{ for } i \neq j. \]

Add now a diagonal effect \( \Delta \) independent of \( \widetilde{R}_i \) with \( E [\Delta] = 1 \) affecting different lob simultaneously (e.g. claims inflation) and denote by

\[ \widetilde{R}_i^* = R_i C_i \Delta, \]
\[ C_i^{PY*} = \widetilde{R}_i^* - R_i, \]
the a posteriori reserve, respectively the a posteriori claim amount PY after calendar year effect. Then

\[
\text{Var}\left( \tilde{R}_i^* \right) = R_i^2 \left( E\left[ Y_i^2 \right] E\left[ \Delta^2 \right] - 1 \right) \\
= R_i^2 \left\{ \left( E\left[ Y_i^2 \right] - 1 \right) (1 + E\left[ \Delta^2 \right] - 1) + E\left[ \Delta^2 \right] - 1 \right\} \\
= R_i^2 \left\{ \tau_i^2 + \sigma_\Delta^2 + \tau_i^2 \sigma_\Delta^2 \right\} \\
\simeq R_i^2 \left\{ \tau_i^2 + \sigma_\Delta^2 \right\},
\]

where in the last equation we have assumed that \( \tau_i^2 \sigma_\Delta^2 \ll \tau_i^2 + \sigma_\Delta^2 \) and where

\( \sigma_\Delta^2 = \text{Var}\left( \Delta \right). \)

Under the assumption that the calendar year effect is also effective for lob \( j \) we obtain

\[
\text{Cov}\left( \tilde{R}_i^*, \tilde{R}_j^* \right) = \text{Cov}\left( C_i^{PY*}, C_j^{PY*} \right) \\
= R_i R_j \text{Var}\left( \Delta \right),
\]

\[
\text{Corr}\left( \tilde{R}_i^*, \tilde{R}_j^* \right) = \text{Corr}\left( C_i^{PY*}, C_j^{PY*} \right) \\
= \frac{\sigma_\Delta^2}{\sqrt{\tau_i^2 + \sigma_\Delta^2} \sqrt{\tau_j^2 + \sigma_\Delta^2}}. \tag{61}
\]

(61) is an intuitive and handy formula, which gives quite a good qualitative insight: correlation induced by a calendar year effect becomes smaller the smaller \( \sigma_\Delta^2 \) is compared to \( \tau_i^2 \) and \( \tau_j^2 \). As an example consider claims inflation. Assume that the yearly standard deviation for claims inflation is 1% and that the reserve risks before the calendar year effect "inflation" for lob \( i \) and \( j \) are 3%. Then we obtain from (61) that

\[
\text{Corr}\left( C_i^{PY*}, C_j^{PY*} \right) = 11\%.
\]

If we increase the calendar year inflation to 3%, the correlation would be 50%. This shows for instance, that the correlation induced by varying inflation is bigger in countries with high inflation than in countries with low inflation. The same considerations can also be applied to other calendar year effects and might help experts to assess the correlation matrices.
Appendices

A  Details on the SST

A.1  Lines of Business

The standard SST models splits the business into the following lines of business:

<table>
<thead>
<tr>
<th>lob</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>motor liability</td>
</tr>
<tr>
<td>2</td>
<td>motor hull</td>
</tr>
<tr>
<td>3</td>
<td>property</td>
</tr>
<tr>
<td>4</td>
<td>general liability</td>
</tr>
<tr>
<td>5</td>
<td>workers compensation (UVG)</td>
</tr>
<tr>
<td>6</td>
<td>corporate accident without UVG</td>
</tr>
<tr>
<td>7</td>
<td>corporate health</td>
</tr>
<tr>
<td>8</td>
<td>individual health</td>
</tr>
<tr>
<td>9</td>
<td>marine</td>
</tr>
<tr>
<td>10</td>
<td>aviation</td>
</tr>
<tr>
<td>11</td>
<td>credit and surety</td>
</tr>
<tr>
<td>12</td>
<td>legal protection</td>
</tr>
<tr>
<td>13</td>
<td>others</td>
</tr>
</tbody>
</table>

A.2  Standard Parameters

The supervision authority provided the following standard default parameters for the SST 2008, which could be used by the companies, if they did not have accurate estimates from their own data.

<table>
<thead>
<tr>
<th>lob</th>
<th>standard default parameters for CY-risks</th>
<th>standard default parameters for PY-risks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>σpar</td>
<td>CoVa (claim size)</td>
</tr>
<tr>
<td>1</td>
<td>3.50%</td>
<td>7.00</td>
</tr>
<tr>
<td>2</td>
<td>3.50%</td>
<td>2.50</td>
</tr>
<tr>
<td>3</td>
<td>5.00%</td>
<td>5.00</td>
</tr>
<tr>
<td>4</td>
<td>3.50%</td>
<td>8.00</td>
</tr>
<tr>
<td>5</td>
<td>3.50%</td>
<td>7.50</td>
</tr>
<tr>
<td>6</td>
<td>4.75%</td>
<td>4.50</td>
</tr>
<tr>
<td>7</td>
<td>5.75%</td>
<td>2.50</td>
</tr>
<tr>
<td>8</td>
<td>5.75%</td>
<td>2.25</td>
</tr>
<tr>
<td>9</td>
<td>5.00%</td>
<td>6.50</td>
</tr>
<tr>
<td>10</td>
<td>5.00%</td>
<td>2.50</td>
</tr>
<tr>
<td>11</td>
<td>5.00%</td>
<td>5.00</td>
</tr>
<tr>
<td>12</td>
<td>4.50%</td>
<td>2.25</td>
</tr>
<tr>
<td>13</td>
<td>4.50%</td>
<td>5.00</td>
</tr>
</tbody>
</table>
A.3 Derivation of the Variance of the Normal Claim Amount

We consider a specific lob $i$. For simplicity reasons we drop the index $i$ and we write $S$ instead of $C_{\text{CY},i}$. As in Section 3.1.1 we also consider

$$X = \frac{S}{\bar{P}},$$

where $\bar{P} = E[S]$ is the pure risk premium. For any random variable $Z$, the coefficient of variation is defined by

$$CoVa(Z) = \sqrt{\frac{\text{Var}(Z)}{E[Z]}}.$$  

Note that $\text{Var}(X) = CoVa(S)$.

From model assumptions 3.1 follows that, on the condition given $\Theta$,

$$S = \sum_{\nu=1}^{N} Y^{(\nu)},$$

where

$$N \sim \text{Poi}(w \lambda \Theta_1),$$

$$\Theta_1 = \text{random variable with } E[\Theta_1] = 1,$$

and where the claim severities $Y^{(\nu)}, \nu = 1, 2, \ldots, N$, are independent and have the same distribution as $\Theta_2 Y$ with $E[Y] = \mu$. Hence we have

$$E\left[ Y^{(\nu)} \middle| \Theta \right] = \Theta_2 \mu,$$

$$\text{Var} \left( Y^{(\nu)} \middle| \Theta \right) = \Theta_2^2 \text{Var}(Y),$$

$$\Theta_2 = \text{random variable with } E[\Theta_2] = 1.$$  

We denote by $\nu = w \cdot \lambda$ the a priori expected number of claims. Since $\Theta_1$ and $\Theta_2$ are independent we have

$$E\left[ S \middle| \Theta \right] = w \lambda \mu \Theta_1 \Theta_2,$$

$$\text{Var} \left( S \middle| \Theta \right) = w \lambda \Theta_1 \left[ \Theta_2^2 \cdot (\text{Var}(Y) + \mu^2) \right],$$

$$E\left[ X \middle| \Theta \right] = \Theta_1 \Theta_2$$  \hspace{1cm} (62)

$$\text{Var} \left( X \middle| \Theta \right) = \frac{1}{w \lambda} \cdot \Theta_1 \Theta_2^2 \left( CoVa^2(Y) + 1 \right),$$  \hspace{1cm} (63)
\[ E[S] = w \cdot \lambda \cdot \mu, \]
\[ \text{Var} \left( Y^{(v)} \right) = E \left[ \Theta_2^2 \right] \text{Var} \left( Y \right) + \text{Var} \left( \Theta_2^2 \right) \mu^2 \]
\[ = E \left[ \Theta_2^2 \right] \left( \text{Var} \left( Y \right) + \mu^2 \right) - \mu^2, \]
\[ \text{Var} \left[ S \right] = E \left[ \text{Var} \left( S \left| \Theta \right. \right) \right] + \text{Var} \left( E \left[ S \left| \Theta \right. \right] \right) \]
\[ = w \cdot \lambda \cdot E \left[ \Theta_2^2 \right] \left( \text{Var} \left( Y \right) + \mu^2 \right) + \left( w \lambda \mu \right)^2 \cdot \text{Var} \left[ \Theta_1 \cdot \Theta_2 \right] \]
\[ = w \cdot \lambda \cdot \left( \text{Var} \left( Y^{(v)} \right) + \mu^2 \right) + \left( w \lambda \mu \right)^2 \cdot \text{Var} \left[ \Theta_1 \cdot \Theta_2 \right]. \]

(64)

Analogously we obtain by conditioning on \( \Theta_1 \)
\[ \text{Var} \left[ \Theta_1 \cdot \Theta_2 \right] = E \left[ \Theta_1^2 \right] \cdot \text{Var} \left( \Theta_2 \right) + \text{Var} \left( \Theta_1 \right) \]
\[ = \left( \text{Var} \left( \Theta_1 \right) + 1 \right) \text{Var} \left( \Theta_2 \right) + \text{Var} \left( \Theta_1 \right), \]

which, inserted into (64), yields
\[ \text{Var} \left[ S \right] = \left( w \lambda \mu \right)^2 \left( \text{Var} \left( \Theta_1 \right) + \text{Var} \left[ \Theta_2 \right] + \text{Var} \left( \Theta_1 \right) \cdot \text{Var} \left[ \Theta_2 \right] \right) + w \cdot \lambda \cdot \left[ \sigma_Y^2 + \mu^2 \right], \]
\[ \text{Var} \left[ X \right] = \frac{\text{Var} \left[ \Theta_1 \right] + \text{Var} \left[ \Theta_2 \right] + \text{Var} \left( \Theta_1 \right) \cdot \text{Var} \left[ \Theta_2 \right]}{\text{parameter risk}}, \]
\[ \text{Cov} \alpha^2 \left( S \right) = \frac{\text{Var} \left[ \Theta_1 \right] + \text{Var} \left[ \Theta_2 \right] + \text{Var} \left( \Theta_1 \right) \cdot \text{Var} \left[ \Theta_2 \right]}{\text{parameter risk}} \]
\[ + \frac{1}{w \lambda} \left( \text{Cov} \alpha^2 \left( Y^{(v)} \right) + 1 \right) \]
\[ \approx \frac{\text{Var} \left[ \Theta_1 \right] + \text{Var} \left[ \Theta_2 \right] + \frac{1}{w \lambda_0} \left( \text{Cov} \alpha^2 \left( Y^{(v)} \right) + 1 \right)}{\text{random risk}}. \]

(66)

(66) follows from the fact that usually \( \text{Var} \left[ \Theta_1 \right] \cdot \text{Var} \left[ \Theta_2 \right] \ll \text{Var} \left[ \Theta_1 \right] + \text{Var} \left[ \Theta_2 \right]. \)

B Details on Solvency II

B.1 Lines of Business

Currently solvency II distinguishes between the following lines of business:

<table>
<thead>
<tr>
<th>lob</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>motor, third party liability</td>
</tr>
<tr>
<td>2</td>
<td>motor, other classes</td>
</tr>
<tr>
<td>3</td>
<td>marine, aviation &amp; transport (MAT)</td>
</tr>
<tr>
<td>4</td>
<td>fire and other damage to property</td>
</tr>
<tr>
<td>5</td>
<td>third-party liability</td>
</tr>
<tr>
<td>6</td>
<td>credit and suretyship</td>
</tr>
<tr>
<td>7</td>
<td>legal expenses</td>
</tr>
<tr>
<td>8</td>
<td>assistance</td>
</tr>
<tr>
<td>9</td>
<td>miscellaneous non-life insurance</td>
</tr>
<tr>
<td>10</td>
<td>NP reins property</td>
</tr>
<tr>
<td>11</td>
<td>NP reins casualty</td>
</tr>
<tr>
<td>12</td>
<td>NP reins MAT</td>
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</table>
B.2 Parameters

The following parameters were used in QUIS IV for solvency II.

<table>
<thead>
<tr>
<th>lob</th>
<th>CY-risk (premium) σ</th>
<th>CY-risk (reserve) τ</th>
<th>max historical years m_n</th>
</tr>
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<tr>
<td>1</td>
<td>9%</td>
<td>12%</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>9%</td>
<td>7%</td>
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</tr>
<tr>
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<td>10</td>
</tr>
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<td>10%</td>
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</tr>
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<td>15%</td>
<td>15%</td>
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</tr>
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<td>15%</td>
<td>10</td>
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</table>

<table>
<thead>
<tr>
<th>credibility weights α_i for σ^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>historical years available</td>
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<tr>
<td>m_n</td>
</tr>
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<td>10</td>
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<td>15</td>
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</table>

<table>
<thead>
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<th>correlation matrix for Z-variables (combined premium and reserve risk)</th>
</tr>
</thead>
<tbody>
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<tr>
<td>-----</td>
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</table>
References


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General Guisan-Strasse 40

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Switzerland