Robust Hedging in Incomplete Markets

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Motivation

- Incomplete markets Not all risks are traded in the financial market.
- Model Misspecification The expected asset/ liability return is difficult to estimate based on the historical data. Expose to estimation error.
  - Expected stock return: $\frac{\ln(S_T)-\ln(S_0)}{T} + \frac{1}{2}\sigma^2$, only price at time 0 and T matters.
  - If market volatility is 16%, then the standard error of the equity premium is $\left(\frac{16\%}{\sqrt{T}}\right)$, shrinking with square root of time. Hard to keep the same DGP during entire period.
- Application Pension fund investment with inflation risk and longevity risk.

Contribution and Conclusion

- Contribution We provide a robust optimal hedging strategy in an incomplete market that protects the investor from the parameter uncertainty.
- Conclusion
  1. The agent can benefit from a robust policy in two aspects. 1: Expected loss from estimation error is less sensitive to the estimated parameters. 2: Cost of hedging is reduced if the expected stock return is over estimated.
  2. The robust policy is more conservative than the naive policy when the financial institution is facing solvency risk.
  3. Robustness effect depends heavily on the correlation between the asset risk and the liability risk.

Model: Market Incompleteness

- Two uncorrelated risk factors One hedgeable risk W_1 and one non-hedgeable risk W_2. Both are univariate standard Brownian motions.
- ALM model The investor puts wA_t amount in the stock market at time t. The remaining part of the assets $(1-w)A_t$ is put into the money-market account.

$$dA_t = \big(r + w(\mu - r)\big)\, A_t\, dt + w\sigma\, A_t\, dW_1,$$

- The liability market is incomplete, exposing to both hedgeable risk W_1 and non-hedgeable risk W_2,

$$dL_t = a\, L_t\, dt + b\, L_t \left(\rho\, dW_1 + \sqrt{1-\rho^2}\, dW_2\right),$$

with $\rho \in [-1, 1]$, the correlation between asset risk and liability risk.

Model Misspecification

- Employ Hansen and Sargent (2007) framework. Perturbed evolution:

$$
\begin{aligned}
dA_t &= \big(r + w(\mu - r)\big)\, A_t\, dt + w\sigma\, A_t\, \left(dW_1 + \lambda_1 dt\right), \\
dL_t &= a\, L_t\, dt + b\, L_t\, \Big(\rho\, \left(dW_1 + \lambda_1 dt\right) \\
&\quad + \sqrt{1-\rho^2}\, \left(dW_2 + \lambda_2 dt\right)\Big),
\end{aligned}
$$

The two drift terms λ_1 and λ_2 are defined as two perturbation time series process.

Uncertainty Set

- The values of λ_1 and λ_2 are constrained by an uncertainty set which is derived via distribution theory.
- Estimated drift terms: $\delta = \begin{pmatrix} \mu \\ a \end{pmatrix}$
- Estimation error: $\delta_1 = \begin{pmatrix} \sigma\lambda_1 \\ b\rho\lambda_1 + b\sqrt{1-\rho^2}\lambda_2 \end{pmatrix}$
- Variance covariance matrix: $\Gamma = \begin{pmatrix} \sigma & 0 \\ b\rho & b\sqrt{1-\rho^2} \end{pmatrix}$
- Assume $\delta_1 \sim N(0, \Gamma)$

$$\delta_1' \Sigma^{-1} \delta_1 \leq \kappa^2$$

- **Uncertainty Set**

$$\mathbb{S} = \left\{ \lambda_1, \lambda_2 | \lambda_1^2 + \lambda_2^2 \leq \kappa^2 \right\}$$

Optimization Problem

- Naive Optimization Optimal hedging strategy that minimizes the expected shortfall at time T

$$\min_{w} \mathbb{E}[(L_T - A_T)^+ \mid \mathcal{F}_t]$$

with $w \in [0, 1]$. We call the hedging strategy which does not consider model misspecification a naive policy denoting w_{na}.

Robust Optimization Problem

- Robust Optimization If the agent is afraid of the model misspecification, he seeks a robust policy defined as:

$$\min_{w} \max_{\lambda_1,\lambda_2} \mathbb{E}[(L_T - A_T)^+ \mid \mathcal{F}_t]$$

Two player's game: Player 1, the robust agent makes an instantaneous investment decision w_{rob} to maximize the expected shortfall. Player 2, the (imaginary) Mother Nature. Given the decision from player one, she attempts to minimize the expected shortfall by controlling λ_1 and λ_2.

Static Robust Optimization

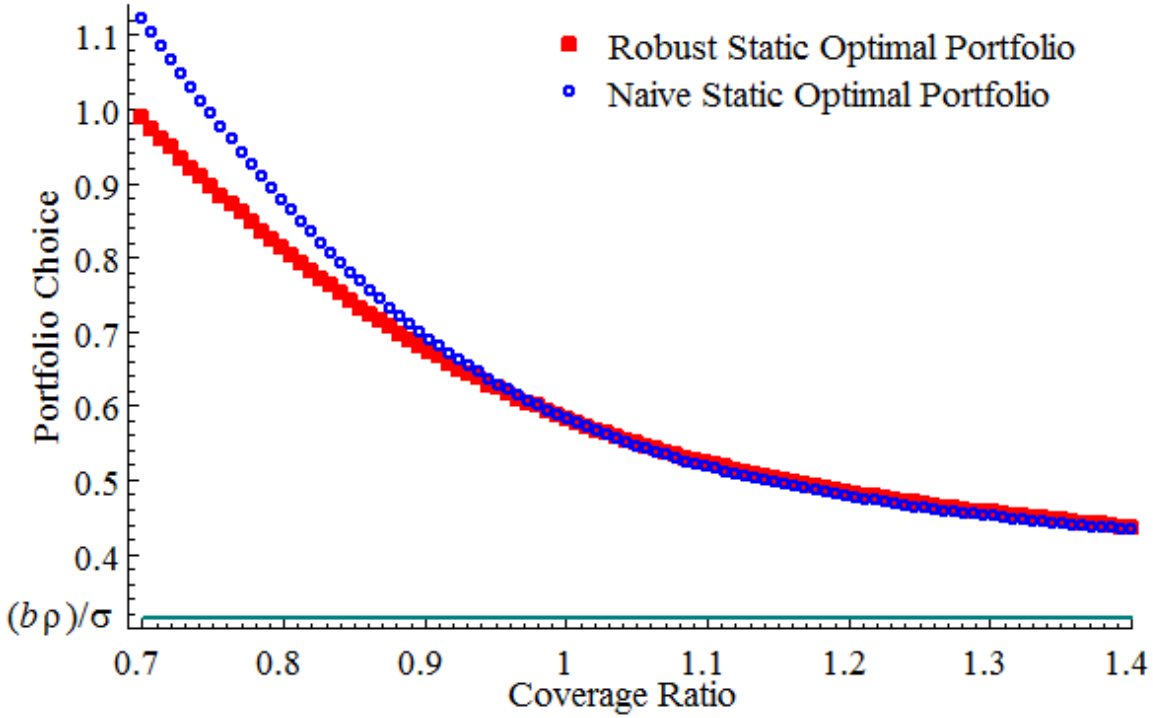
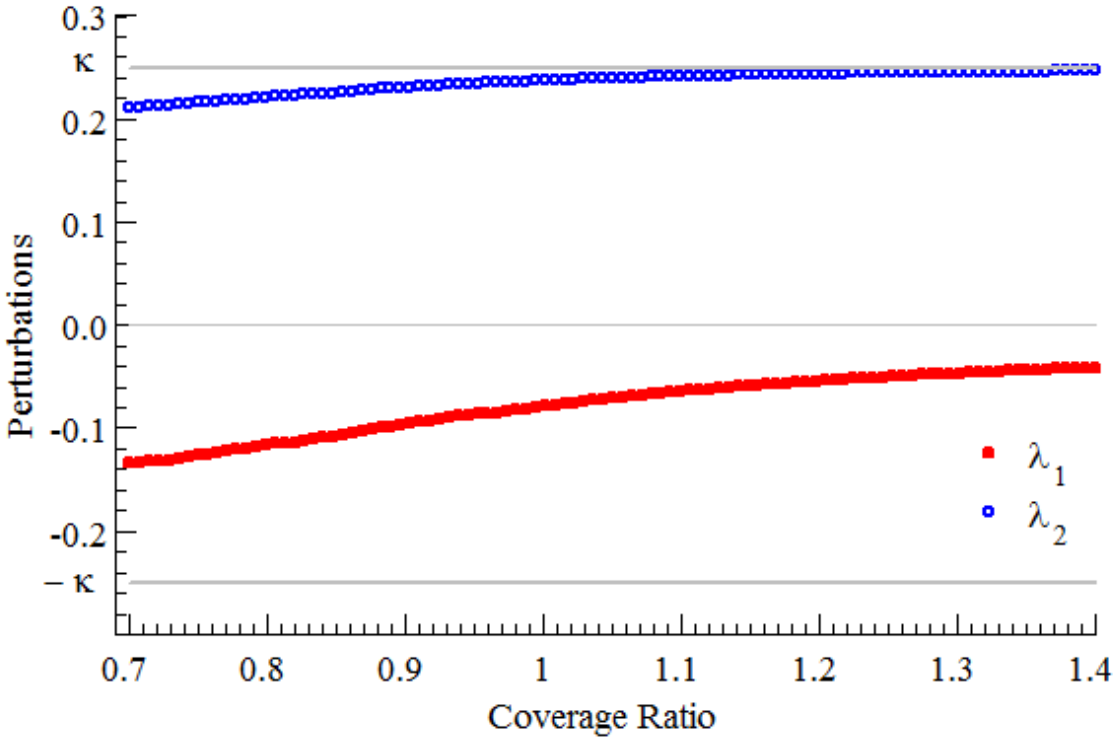


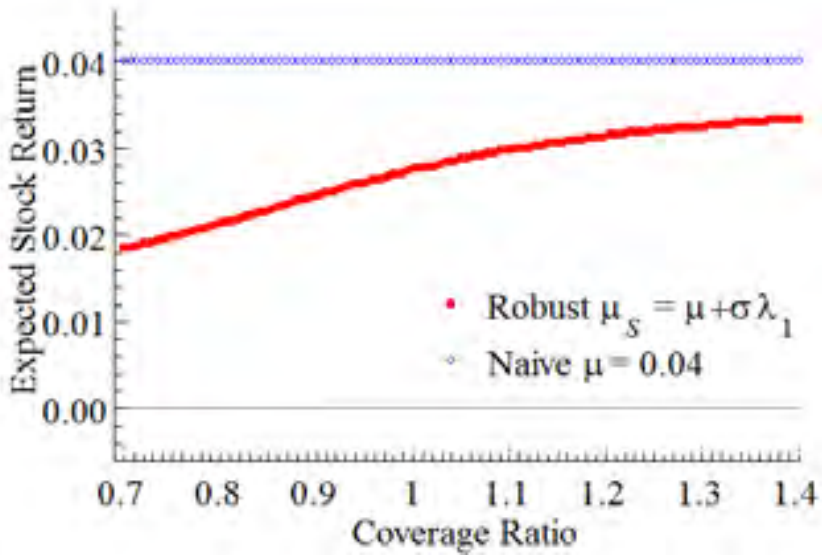
Figure: Static optimal portfolio choice. The results are based on the benchmark estimation parameters $\mu = 0.04$, $\sigma = 0.16$, $r = 0$, $a = 0$, $b = 0.1$, $\rho = 0.5$, $\kappa = 0.25$, and $T = 5$.

Mother Nature's Choice

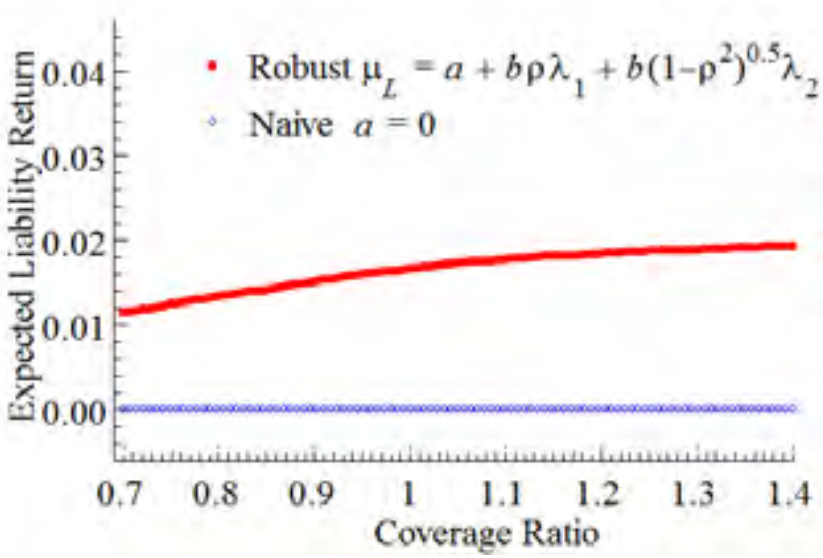


- $\lambda_1 < 0$ representing the agent's fear for an over estimated asset return.
- Low C_0, more risk exposure, more negative λ_1.

Estimation Error of Drift Terms



(a) Robust vs. Naive μ_S



(b) Robust vs. Naive μ_L

- Perturbed expected stock return is reduced by $|\sigma\lambda_1|$ amount.
- Liability drift is pushed up by $|b\rho\lambda_1 + b\sqrt{1-\rho^2}\lambda_2|$ amount.

Policy Evaluation

- Minimum hedging cost: $q(\delta) = \min_{w} Q(w, \delta)$ for given δ, estimated expected return.
- Minimum cost is q_0 by following optimal hedge w_0 if the true return δ_0 is known.
- Any other alternative hedging policies w_a $w_a \neq w_0$ requires more hedging expense than q_0.
- Loss function $K(w_a|\delta_0)$ as the difference between the cost of hedging following a suboptimal policy $Q(w_a, \delta_0)$ and the true minimum cost:

$$K(w_a|\delta_0) = Q(w_a, \delta_0) - q_0$$

- When does robust policy beats the naive policy?

Policy Evaluation

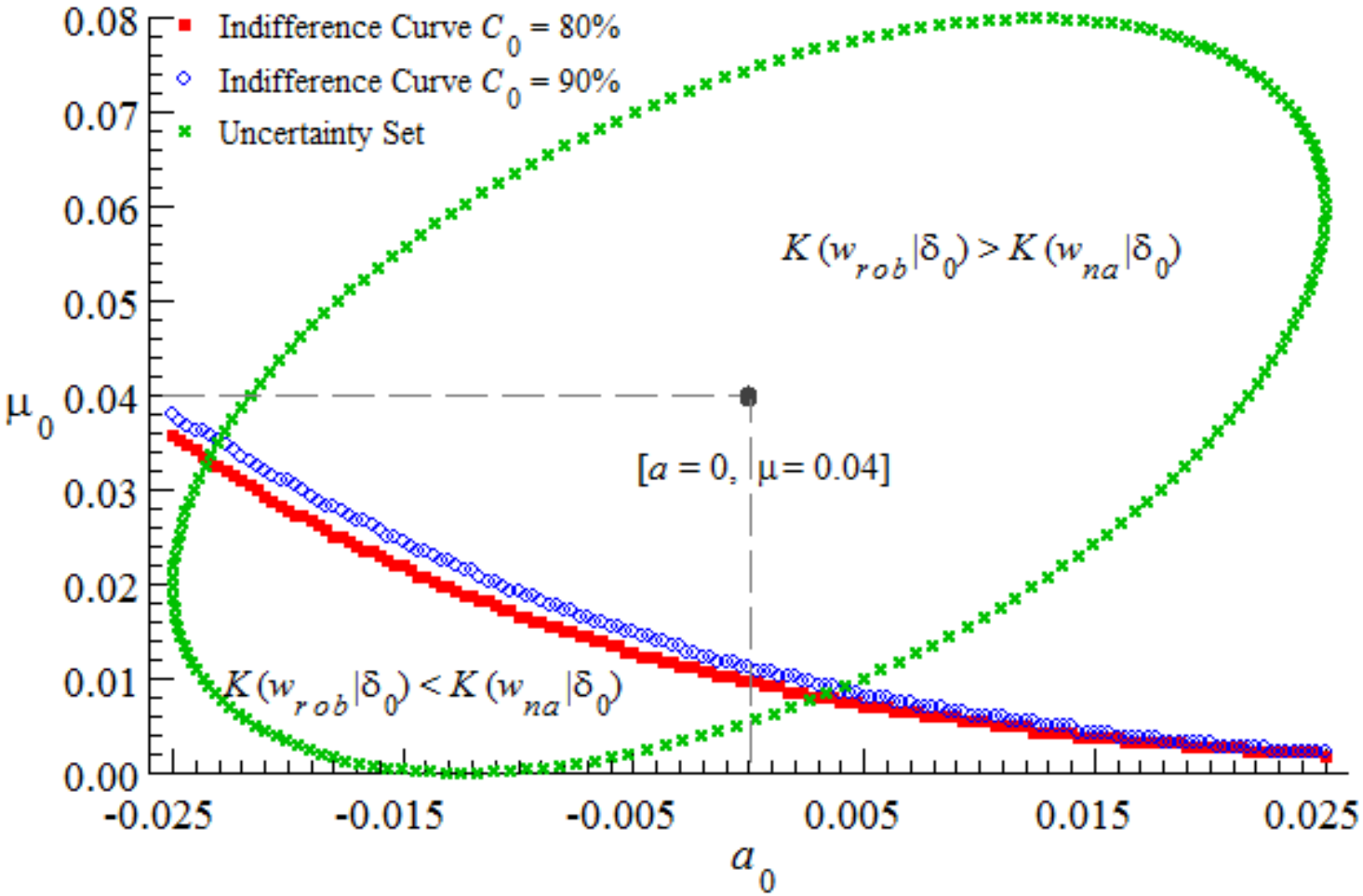
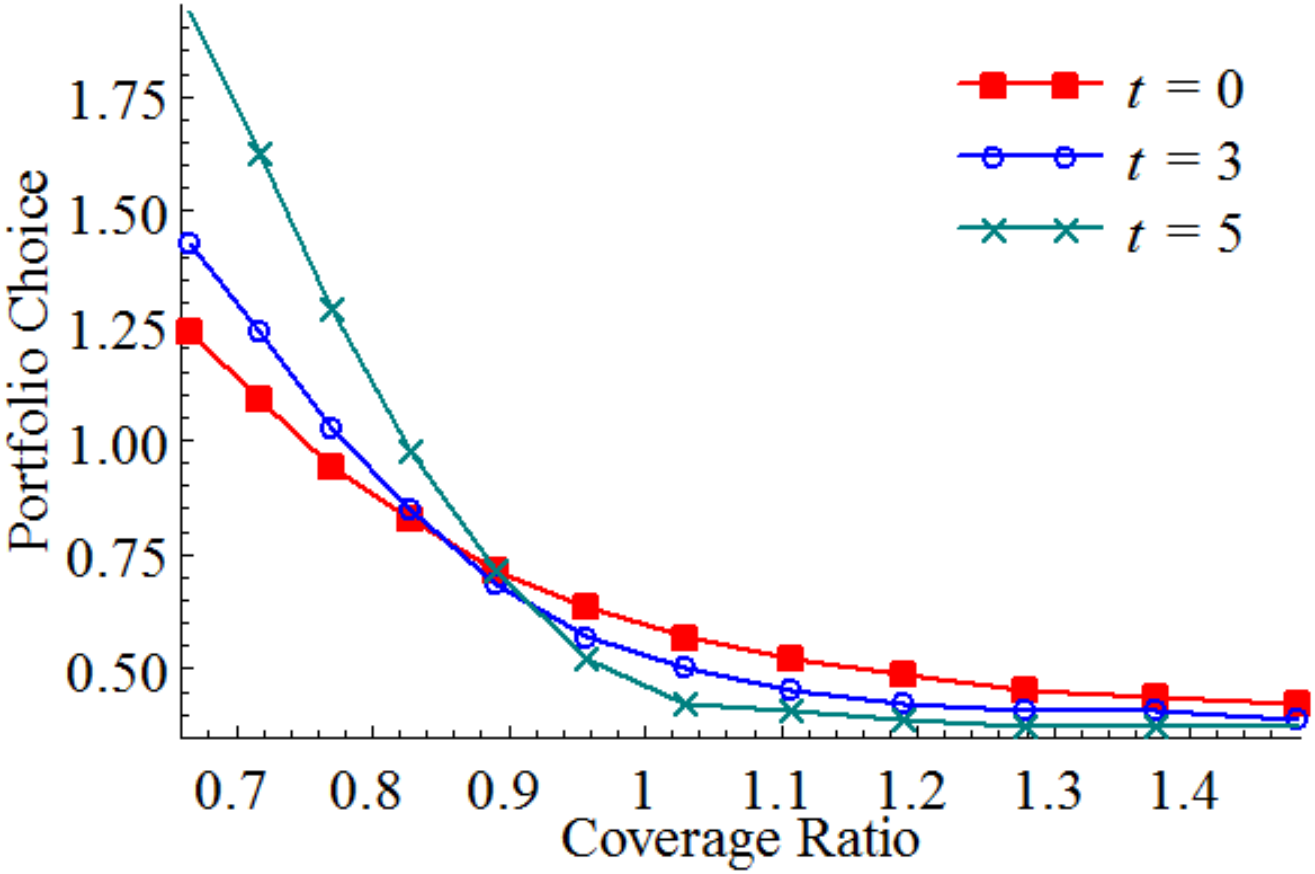


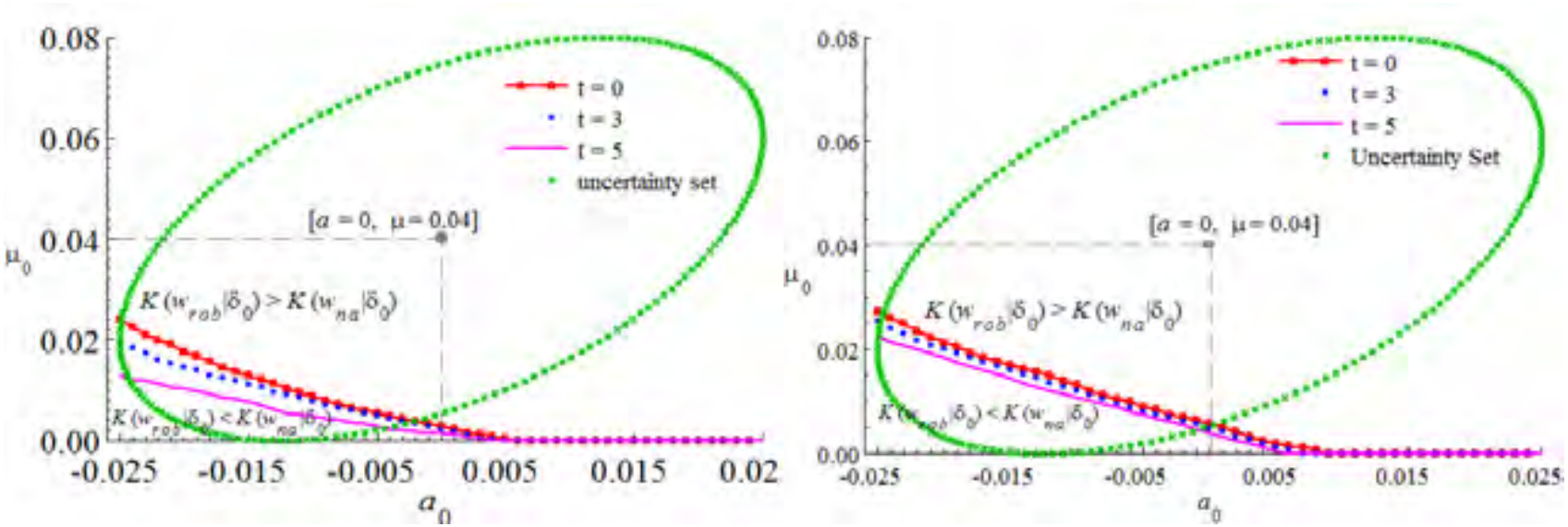
Figure: Region beneath the curve, the robust policy outperforms the naive one and in the region above is another way around.

Dynamic Robust Optimization



- If $C_t < 1$, increase the risk exposure on stock market over time.
- If $C_t > 1$, decrease his risk exposure over time.

Policy Evaluation



(a) $C_t = 80\%$ (b) $C_t = 90\%$

Figure: Dynamic loss function equivalent curve.

- The beneficial region of robust policy decreases over time.

Hansen, L. and Sargent, T. (2007). *Robustness*. Princeton University Press Princeton, NJ.