ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

# Best Estimate Reserves and the Claims Development Results in Consecutive Calendar Years

Annina Saluz and Alois Gisler

ETH Zurich

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# Overview


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2. Martingale Argument
3. Examples
   - One Jump Model
   - Normal-Normal Model

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Importance of claims reserves

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- Wrong reserves $\Rightarrow$ wrong premiums
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- Want to be sure that the actuarial reserves are 'best estimates'
- First check: look at the CDR
  - if it is negative over several years: actuaries are blamed to be responsible for the losses in the annual profit and loss statement
  - if it is positive over several years: actuaries are blamed that prices in the past were not competitive and that the company has missed profitable opportunities

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**Question:**
**Are claims development results with the same sign over several consecutive calendar years a contradiction to best estimate reserves?**

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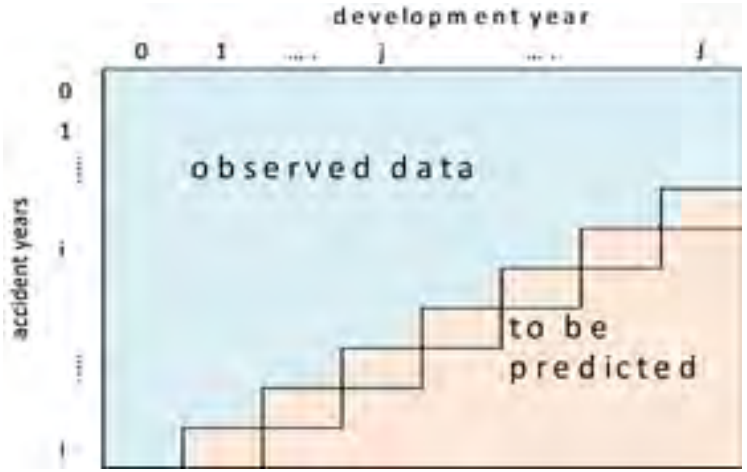

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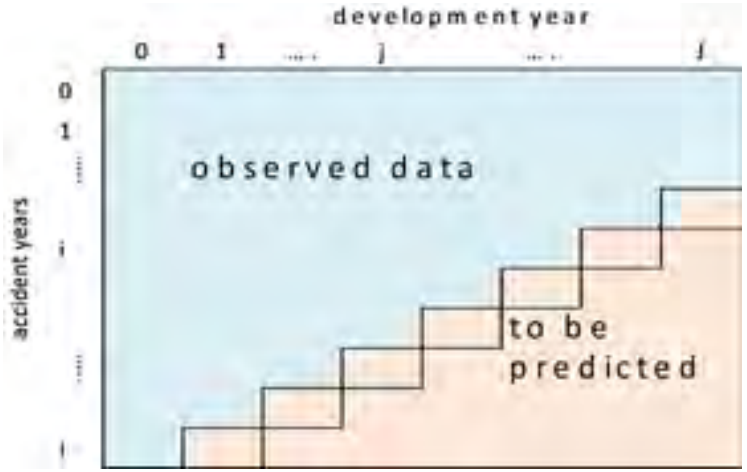

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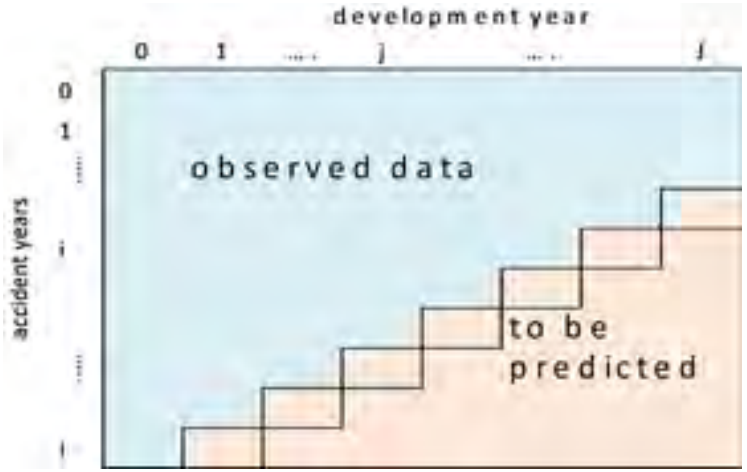
- Outstanding liabilities at time $I$:

$$R_i^{(I)} = \sum_{j=I-i+1}^{J} X_{i,j} = U_i - C_{i,I-i+1}, \qquad R^{(I)} = \sum_{i=I-J+1}^{I} R_i^{(I)}.$$



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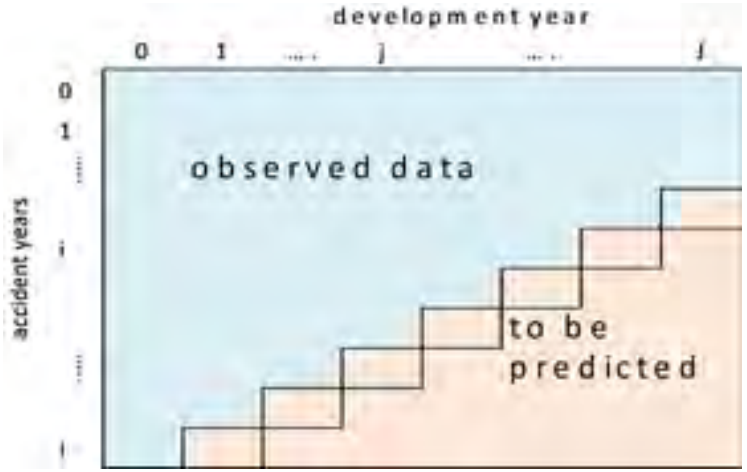
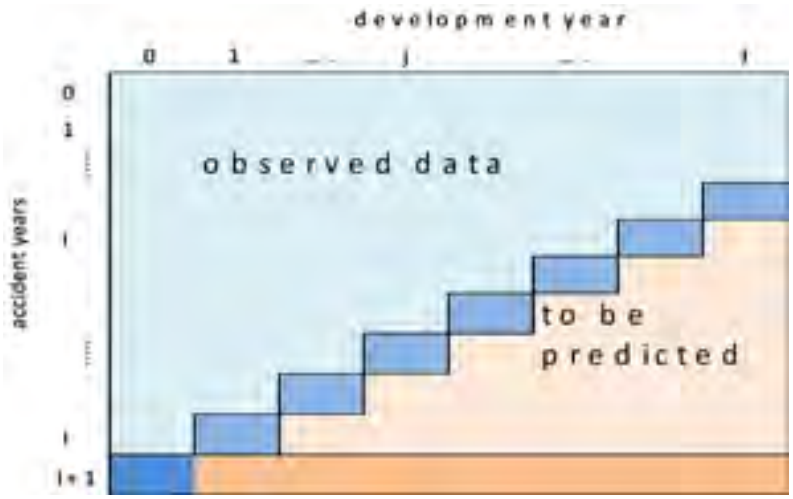
- Claims Reserves:

$$\hat{R}_i^{(I)} = \hat{U}_i - C_{i,I-i+1} = \text{prediction of } R_i^{(I)}, \qquad \hat{R}^{(I)} = \sum_{i=I-J+1}^{I} \hat{R}_i^{(I)}.$$



# Claims Development Result (CDR)

After one year: New predictions $\hat{R}_i^{(I+1)}$, $\hat{U}_i^{(I+1)}$ and $\hat{R}^{(I+1)}$.




$$\mathrm{CDR}^{(I+1)} = \sum_{i=I-J+1}^{I} \hat{U}_i^{(I)} - \hat{U}_i^{(I+1)}$$

$$= \sum_{i=I-J+1}^{I} \left( \hat{R}_i^{(I)} - X_{i,I-i+1} - \hat{R}_i^{(I+1)} \right).$$

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- For best estimate reserves: Expected value of the CDR is zero.
- In mathematical terms: $\hat{R}_i^{(I)} = E[R_i^{(I)} | \mathcal{I}_I, \mathcal{D}_I]$.

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$$\hat{U}_i^{(I)} = E[U_i | \mathcal{I}_I, \mathcal{D}_I].$$

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- The sequence $\left\{\hat{U}_i^{(I)} : I = i, i+1, \ldots\right\}$ is a martingale.
  $\Rightarrow \left\{\mathrm{CDR}_i^{(I)} = \hat{U}_i^{(I-1)} - \hat{U}_i^{(I)} : I = i, i+1, \ldots\right\}$
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  are uncorrelated with $E\left[\mathrm{CDR}_i^{(I)}\right] = 0$.
- Often drawn conclusion: CDR's with the same sign (losses or gains) over several consecutive calendar years are a contradiction to best estimate reserves.

# Example from Practice

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  - emergence of a new phenomenon: 'whiplash cases'
  - creeping change of legislation leading to higher indemnities for such claims
  - reserving actuary at the beginning of this period:
    increase of incurred claims in newest diagonal (but not in paid triangle);
    not clear whether this was systematic or random;
  - not full weight given to observations in newest diagonal;
    only with time 'new situation' is fully reflected by the reserves;
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- Questions:
  - Did all these actuaries violate the principle of best estimates?
  - Or did we oversee something in the martingale argument?
- Answer:
  - Yes we have overseen something in the martingale argument
  - Nothing wrong in the mathematics
  - But we have to be careful in the interpretation what the martingale property exactly means.

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- Martingale argument holds in the average over these state space characteristics.
- What we observe in a claims development triangle are conditional observations (given one specific trajectory of the state space process).
- These conditional observations are no longer increments of a martingale.
- Consequence: CDR's with same sign over several consecutive calendar years are not (necessarily) a contradiction to best estimate reserves.

# Basic Assumptions

Bayesian framework:

- Each calendar year $t$ characterised by a calendar year effect $\Psi_t$
- $\Psi_t$ modelled as hidden random variable
- $\mathbf{\Psi} = (\Psi_0, \Psi_1, \ldots)$
- $E[X_{i,j}|\mathbf{\Psi}] = \mu_i \gamma_j \Psi_{i+j}$

# One Jump Model

Situation considered:

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- Calendar year $s_0$
  - external information: change of legislation
  - impact:
    increase of expected value of future claim payments by some not exactly known factor $\Psi_{s_0}$
  - no indication of other changes in the future
- Calendar years $t \geq s_0 + 1$
  - stable situation, no further changes in claims environment

# One Jump Model

**Model Assumptions:**

i) Given $\mathbf{\Psi}$, $X_{i,j}$ are independent and normally distributed with

$$E[X_{i,j}|\mathbf{\Psi}] = \mu_i \gamma_j \Psi_{i+j}, \qquad \text{Var}(X_{i,j}|\mathbf{\Psi}) = \mu_i \gamma_j \sigma^2,$$

where $\sum_{j=0}^{J} \gamma_j = 1$.

ii) For $t \leq s_0 - 1$:
Conditional on $\{\mathcal{I}_t, \mathcal{D}_t\}$: $\Psi_l = 1$ for $l \leq t$ and
$E[\Psi_{t+l}|\mathcal{I}_t, \mathcal{D}_t] = E[\Psi_t|\mathcal{I}_t, \mathcal{D}_t] = 1$ for $l \geq 1$.

iii) For $t \geq s_0$:
Conditional on $\{\mathcal{I}_t, \mathcal{D}_t\}$

$$\begin{aligned}
\Psi_l &= 1 \text{ for } l \leq s_0 - 1; \\
\Psi_l &= \Psi_{s_0} \text{ for } l = s_0, \ldots, t; \\
E\left[\Psi_{t+l}|\mathcal{I}_t, \mathcal{D}_t\right] &= E\left[\Psi_t|\mathcal{I}_t, \mathcal{D}_t\right] = E\left[\Psi_{s_0}|\mathcal{I}_t, \mathcal{D}_t\right] \text{ for } l \geq 1,
\end{aligned}$$

where $\Psi_{s_0}|\{\mathcal{I}_t, \mathcal{D}_{s_0-1}\} \sim N(m, \tau^2)$.

# Best Estimates

- Best estimate reserves for $t \geq s_0$:

$$\hat{R}^{(t)} = \sum_{k=1}^{J} \left( E[\Psi_{s_0} | \mathcal{I}_t, \mathcal{D}_t] \sum_{j=k}^{J} \gamma_j \mu_{t+k-j} \right)$$

Let $\Psi_{s_0}^{(t)} = E[\Psi_{s_0} | \mathcal{I}_t, \mathcal{D}_t]$.

- Exact Credibility case with

$$\begin{aligned} \Psi_{s_0}^{(s_0+l)} &= E[\Psi_{s_0} | \mathcal{I}_{s_0+l}, \mathcal{D}_{s_0+l}] = \alpha_{s_0+l} \bar{D}_{s_0+l} + (1 - \alpha_{s_0+l}) m, \\ q_{s_0}^{(s_0+l)} &= \mathsf{Var}(\Psi_{s_0} | \mathcal{I}_{s_0+l}, \mathcal{D}_{s_0+l}) = (1 - \alpha_{s_0+l}) \tau^2, \end{aligned}$$

where

$$\bar{D}_{s_0+l} = \frac{\sum_{t=s_0}^{s_0+l} \sum_{j=0}^{J} X_{t-j,j}}{\sum_{t=s_0}^{s_0+l} \sum_{j=0}^{J} \mu_{t-j} \gamma_j} \quad \text{and} \quad \alpha_{s_0+l} = \frac{\sum_{t=s_0}^{s_0+l} \sum_{j=0}^{J} \mu_{t-j} \gamma_j}{\sum_{t=s_0}^{s_0+l} \sum_{j=0}^{J} \mu_{t-j} \gamma_j + \frac{\sigma^2}{\tau^2}}.$$

# Recursive estimation of $\Psi_{s_0}$

i) For $t = s_0$

$$\Psi_{s_0}^{(s_0)} = \tilde{\alpha}_{s_0} \frac{\sum_{j=0}^{J} X_{s_0-j,j}}{w_{s_0}} + (1-\tilde{\alpha}_{s_0})m, \quad q_{s_0}^{(s_0)} = (1-\tilde{\alpha}_{s_0})\tau^2,$$

where

$$\tilde{\alpha}_{s_0} = \frac{w_{s_0}}{w_{s_0} + \frac{\sigma^2}{\tau^2}} \quad \text{and} \quad w_{s_0} = \sum_{j=0}^{J} \mu_{s_0-j}\gamma_j.$$

ii) For $t > s_0$

$$\Psi_{s_0}^{(t)} = \tilde{\alpha}_t \frac{\sum_{j=0}^{J} X_{t-j,j}}{w_t} + (1-\tilde{\alpha}_t)\Psi_{s_0}^{(t-1)}, \quad q_{s_0}^{(t)} = (1-\tilde{\alpha}_t)q_{s_0}^{(t-1)},$$

where

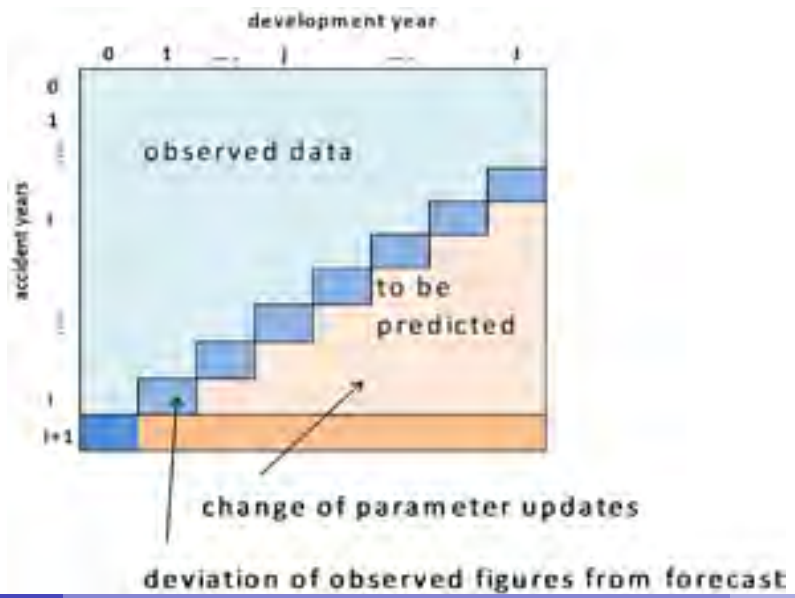
$$\tilde{\alpha}_t = \frac{w_t}{w_t + \frac{\sigma^2}{q_{s_0}^{(t-1)}}} \quad \text{and} \quad w_t = \sum_{j=0}^{J} \mu_{t-j}\gamma_j.$$

# Claims Development Result

Assume $\mu_i \equiv \mu$:

$$\mathrm{CDR}^{(t+1)} = \underbrace{\Psi_{s_0}^{(t+1)}\mu(1-\beta_0) - \sum_{j=1}^{J} X_{t+1-j,j}}_{\text{difference updated forecast and observations}} + \underbrace{(\Psi_{s_0}^{(t)} - \Psi_{s_0}^{(t+1)})\mu \sum_{k=1}^{J}(1-\beta_{k-1})}_{\text{change in update}}$$

$$= \underbrace{\Psi_{s_0}^{(t)}\mu(1-\beta_0) - \sum_{j=1}^{J} X_{t+1-j,j}}_{\text{difference forecast and observations}} + \underbrace{(\Psi_{s_0}^{(t)} - \Psi_{s_0}^{(t+1)})\mu \sum_{k=2}^{J}(1-\beta_{k-1})}_{\text{change future}},$$



# Sign of the CDR

- Observations: Conditional data, given a specific realisation of $\Psi_{s_0}$.
- Expected Value of the CDR for $t+1 \geq s_0+1$

$$E[\text{CDR}^{(t+1)}|\mathcal{I}_{t+1}, \Psi_{s_0}]$$
$$= (m - \Psi_{s_0})\mu \left( (1-\beta_0)(1-\alpha_t) + (\alpha_{t+1} - \alpha_t) \sum_{k=2}^{J} (1-\beta_{k-1}) \right)$$

- If $\beta_j \leq 1$ for all $j$:
  - Sign of $E[\text{CDR}^{(t+1)}|\mathcal{I}_{t+1}, \Psi_{s_0}]$ is the same for all $t+1 \geq s_0+1$.
  - Absolute value is the bigger, the bigger $|m - \Psi_{s_0}|$ is.
  - Absolute value is the bigger, the longer the development is.
  - Absolute value is decreasing in $t$ since $\alpha_t$ is increasing in $t$.
  - For $t \to \infty$ the CDR is conditionally unbiased since $\alpha_t \to 1$ $(t \to \infty)$.

# Simulation Example

- Parameter choices: $\mu = 10'000$, $m = 1.02$, $\sigma = 0.04$, $\tau = 0.02$.

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\gamma_j$ | 0.2 | 0.23 | 0.12 | 0.11 | 0.09 | 0.07 | 0.058 | 0.05 | 0.043 | 0.029 |

Table: Development pattern $\gamma_j$.

- Estimation of $\Psi_{s_0}$:

$$\Psi_{s_0}^{(t)} = \alpha_t \frac{1}{\mu(t - s_0 + 1)} \sum_{l=s_0}^{t} \sum_{j=0}^{J} X_{l-j,j} + (1 - \alpha_t)m,$$

where

$$\alpha_t = \frac{t - s_0 + 1}{(t - s_0 + 1) + \frac{\sigma^2}{\tau^2}}.$$

- Assume jump occurs at time $s_0 = 10$, $\Psi_{10} = 1.035$.

# Simulated Data

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2308 | 2151 | 1083 | 1067 | 701 | 641 | 502 | 657 | 520 | 324 |
| 1 | 1731 | 2612 | 1189 | 1050 | 915 | 807 | 854 | 470 | 573 | 321 |
| 2 | 1757 | 2036 | 1014 | 1106 | 926 | 634 | 651 | 373 | 446 | 239 |
| 3 | 1868 | 2190 | 1098 | 746 | 680 | 654 | 773 | 501 | 354 | 227 |
| 4 | 1970 | 2342 | 1091 | 1098 | 1021 | 626 | 533 | 468 | 329 | 370 |
| 5 | 1915 | 1959 | 1066 | 1241 | 1011 | 763 | 764 | 661 | 415 | 284 |
| 6 | 1766 | 2293 | 1028 | 1070 | 841 | 844 | 711 | 516 | 431 | 230 |
| 7 | 2027 | 2039 | 1476 | 1344 | 904 | 618 | 490 | 524 | 465 | 145 |
| 8 | 2041 | 2278 | 986 | 1209 | 1014 | 685 | 594 | 591 | 441 | 286 |
| 9 | 2236 | 2325 | 1008 | 1369 | 1002 | 687 | 563 | 540 | 419 | 395 |
| 10 | 2008 | 2507 | 1289 | 1081 | 1046 | 718 | 551 | 680 | 381 | 334 |
| 11 | 2197 | 2399 | 1449 | 1139 | 1008 | 764 | 667 | 614 | 430 | 302 |
| 12 | 2098 | 2362 | 1366 | 883 | 855 | 851 | 529 | 723 | 256 | 431 |
| 13 | 1918 | 2448 | 1301 | 1133 | 935 | 699 | 593 | 692 | 457 | |
| 14 | 1741 | 2693 | 1417 | 1106 | 935 | 649 | 470 | 631 | | |
| 15 | 2203 | 2294 | 1280 | 1283 | 1225 | 687 | 637 | | | |
| 16 | 2122 | 2523 | 1581 | 1201 | 845 | 765 | | | | |
| 17 | 2238 | 2176 | 1232 | 1165 | 813 | | | | | |
| 18 | 2128 | 1960 | 1297 | 1235 | | | | | | |
| 19 | 2091 | 2121 | 1264 | | | | | | | |
| 20 | 2143 | 2833 | | | | | | | | |
| 21 | 2004 | | | | | | | | | |

Table: Incremental claims

# Predictions of $\Psi_{s_0}$ over time

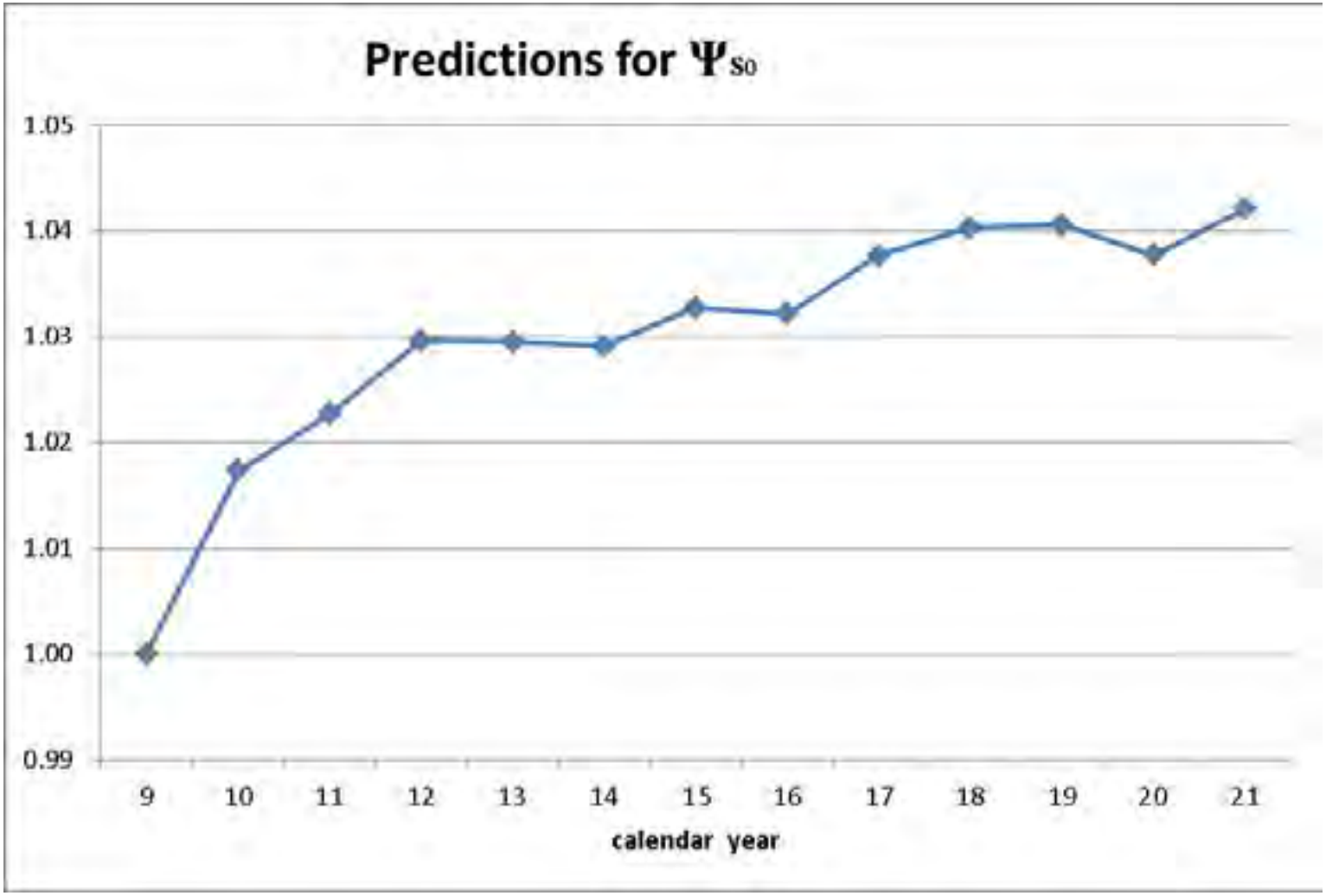



Figure: Predictions $\Psi_{s_0}^{(t)}$ of $\Psi_{s_0} = 1.035$.

# CDR's

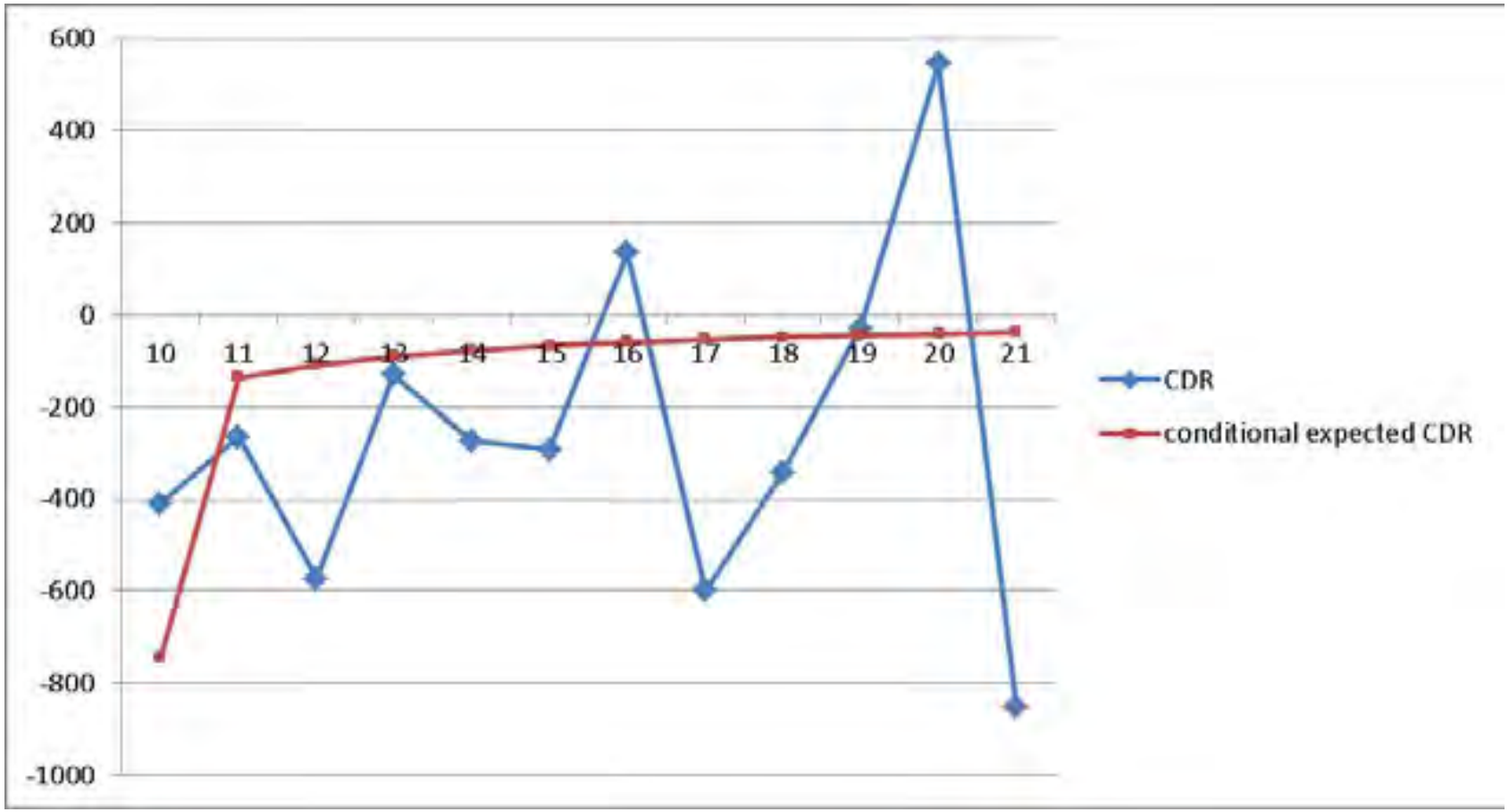



Figure: CDR and $E[\text{CDR}_t|\Psi_{s_0}, \mathcal{I}_t]$.

# CDR's

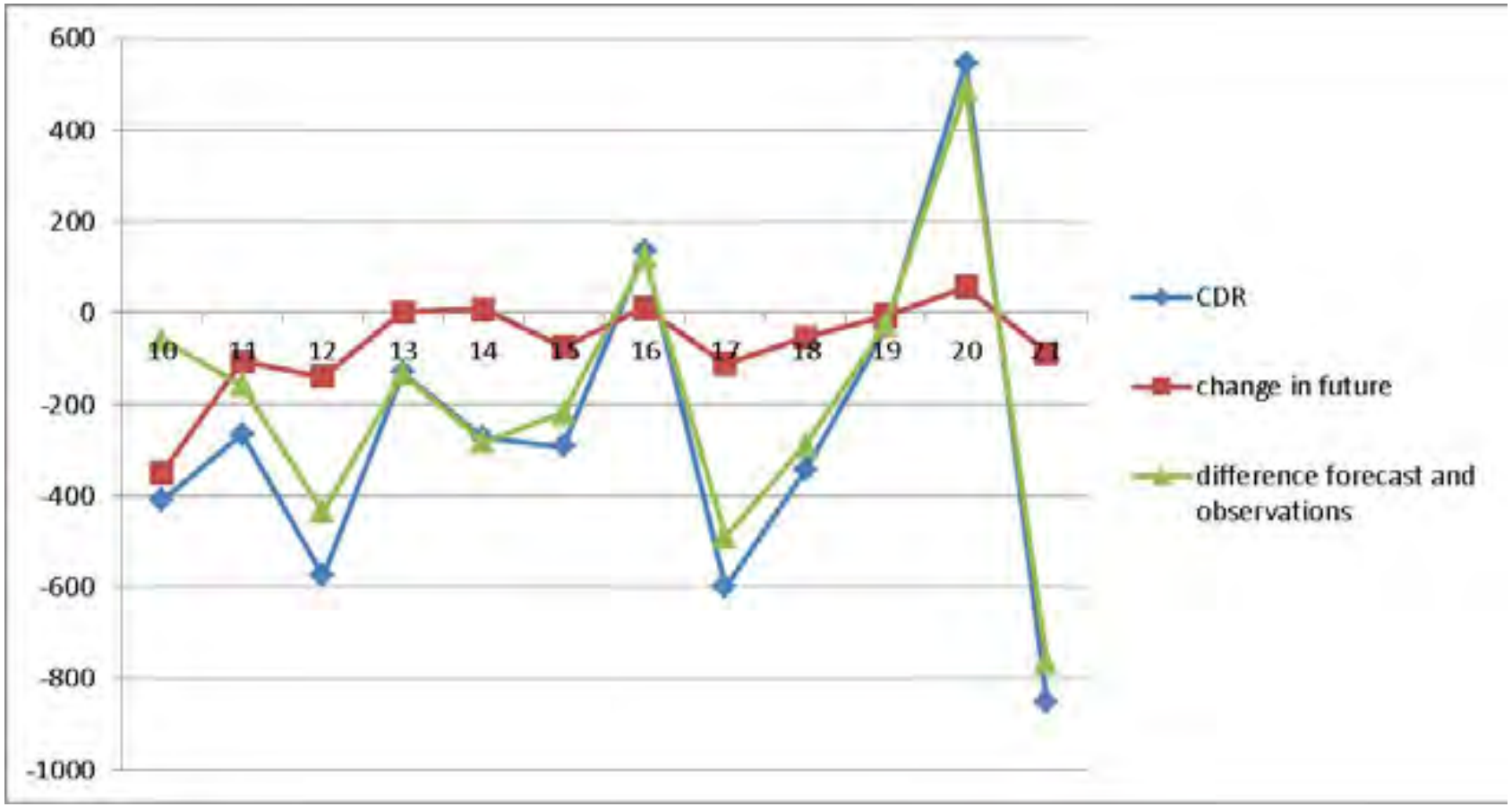



Figure: CDR and split in 'difference forecast and observations' plus 'change in future'.

# Results

We observe

- negative CDR's in $6$ consecutive years
- expected value of the CDR is negative
- $9$ times a loss in the $10$ years from calendar year $10$ to $19$.

# AR(1) Process

Idea: Based on external information $\mathcal{I}_t$ assume

- AR structure for future diagonal effects

$$\{\Psi_{t+l} : l \geq 0\} \stackrel{d}{=} \{\tilde{\Psi}_{t+l} = \rho^{(t)}\tilde{\Psi}_{t+l-1} + (1-\rho^{(t)})m^{(t)} + \Delta^{(t)}_{t+l},\ l \geq 0\}$$

  - $\Delta^{(t)}_{t+l}|\mathcal{I}_t \stackrel{iid}{\sim} N\left(0, \delta^{2^{(t)}}\right)$
  - Given $\mathcal{I}_t$, $\Delta^{(t)}_{t+l}$, $l \geq 1$, independent from $\tilde{\Psi}_t$
  - Given $\mathcal{I}_t$, $\Delta^{(t)}_t$ independent from $\tilde{\Psi}_{t-1}$
  - $\Psi_t|\mathcal{I}_t$ normally distributed.
- Updating of AR parameters for future diagonal effects is possible.
- Past diagonal effects not updated due to new external information: $\Psi_{t-l}|\mathcal{I}_t \stackrel{d}{=} \Psi_{t-l}|\mathcal{I}_{t-l}$

# More General Model

- Idea: $\Psi_t = a_t^{(t)}\Psi_{t-1} + b_t^{(t)} + \Delta_t^{(t)}$, where $\Delta_t^{(t)}|\mathcal{I}_t \sim N(0, \delta_t^{2^{(t)}})$ independent of $\mathcal{D}_{t-1}$ and $\Psi_{t-1}$.
  - Parameters $a_t^{(t)}$, $b_t^{(t)}$ based on updated a priori information $\mathcal{I}_t$ and on past data $\mathcal{D}_{t-1}$, and $\delta_t^{2^{(t)}}$ based on $\mathcal{I}_t$.
  - Parameters $a_t^{(t)}$, $b_t^{(t)}$ and $\delta_t^{2^{(t)}}$ will not change due to new external information $\mathcal{I}_{t+l}$.
- Future: $\Psi_{t+l} = a_{t+l}^{(t)}\Psi_{t+l-1} + b_{t+l}^{(t)} + \Delta_{t+l}^{(t)}$, where the $\Delta_{t+l}^{(t)}|\mathcal{I}_t \sim N(0, \delta_{t+l}^{2^{(t)}})$ are mutually independent and independent of $\mathcal{D}_t$ and $\Psi_t$.
  - Parameters $a_{t+l}^{(t)}$, $b_{t+l}^{(t)}$ and $\delta_{t+l}^{2^{(t)}}$ reflect a priori assumptions of process in later years based on $\mathcal{I}_t$ and $\mathcal{D}_t$.
  - Parameters $a_{t+l}^{(t)}$, $b_{t+l}^{(t)}$ and $\delta_{t+l}^{2^{(t)}}$ will be replaced in later years due to new information.
- Updating the external 'a priori' information $\mathcal{I}_t$ is possible.

# Normal-Normal Model

i) Conditionally, given $\mathbf{\Psi}$ and $\mathcal{I}_t$, $X_{i,j}$ independent and normally distributed with

$$E\left[X_{i,j} \middle| \mathbf{\Psi}, \mathcal{I}_t\right] = E\left[X_{i,j} \middle| \Psi_{i+j}\right] = \mu_i \gamma_j \Psi_{i+j},$$
$$\mathrm{Var}\left(X_{i,j} \middle| \mathbf{\Psi}, \mathcal{I}_t\right) = \mu_i \eta_j^2 \sigma^2,$$

where $\sum_{j=0}^{J} \gamma_j = \sum_{j=0}^{J} \eta_j^2 = 1$.

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ii) Conditionally, given $\mathcal{I}_t$ and $\mathcal{D}_{t-1}$, $\Psi_t$ normally distributed with

$$\Psi_t^{(t,t-1)} = E[\Psi_t|\mathcal{I}_t,\mathcal{D}_{t-1}] = a_t^{(t)}\Psi_{t-1}^{(t-1)} + b_t^{(t)}$$
$$q_t^{(t,t-1)} = \mathsf{Var}(\Psi_t|\mathcal{I}_t,\mathcal{D}_{t-1}) = \left(a_t^{(t)}\right)^2 q_{t-1}^{(t-1)} + \delta_t^{2^{(t)}},$$

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iv) For $0 \leq l \leq t$, $\Psi_{t-l}|(\mathcal{I}_t,\mathcal{D}_{t-l}) \stackrel{d}{=} \Psi_{t-l}|(\mathcal{I}_{t-l},\mathcal{D}_{t-l})$.

# Recursive Calculation of the $\Psi_t$'s, Normal-Normal Model

For $t \geq 0$:

a) Update in the newest diagonal:

$$\Psi_t^{(t)} = \alpha_t \bar{X}_t + (1 - \alpha_t)\, \Psi_t^{(t,t-1)},$$

where

$$\alpha_t = \frac{w_t}{w_t + \kappa_t}, \quad \text{with} \quad w_t = \sum_{j=0}^{J \wedge t} \mu_{t-j} \frac{\gamma_j^2}{\eta_j^2}, \quad \kappa_t = \frac{\sigma^2}{q_t^{(t,t-1)}},$$

$$\bar{X}_t = \frac{1}{w_t} \sum_{j=0}^{J \wedge t} \frac{\gamma_j}{\eta_j^2} X_{t-j,j},$$

and

$$q_t^{(t)} = (1 - \alpha_t) q_t^{(t,t-1)}.$$

b) Forecast of future diagonal effects: For $k \geq 1$

$$\Psi_{t+k}^{(t)} \quad = \quad a_{t+k}^{(t)} \Psi_{t+k-1}^{(t)} + b_{t+k}^{(t)}.$$

# Summary

- Models that allow to
  - describe diagonal effects,
  - update external a priori information.
- Conclusion:
  - Same sign of CDR over several consecutive calendar years is **not** a contradiction to best estimates.
  - Potential applications for pricing.