Best Estimate Reserves and the Claims Development Results in Consecutive Calendar Years

Annina Saluz and Alois Gisler

ETH Zurich

May 22, 2013
ASTIN Colloquium
The Hague, Netherlands
Overview

1 Introduction

2 Martingale Argument

3 Examples
   - One Jump Model
   - Normal-Normal Model
Introduction

Importance of claims reserves

- Most important item on the liability side of the balance sheet
- Wrong reserves $\Rightarrow$ wrong premiums
- Reserve risk = main insurance risk in solvency
Introduction

Importance of claims reserves

- Most important item on the liability side of the balance sheet
- Wrong reserves $\Rightarrow$ wrong premiums
- Reserve risk = main insurance risk in solvency

Perception of management

- Want to be sure that the actuarial reserves are ‘best estimates’
- First check: look at the CDR
  - if it is negative over several years: actuaries are blamed to be responsible for the losses in the annual profit and loss statement
  - if it is positive over several years: actuaries are blamed that prices in the past were not competitive and that the company has missed profitable opportunities

Question:
Are claims development results with the same sign over several consecutive calendar years a contradiction to best estimate reserves?

Annina Saluz and Alois Gisler

Best Estimates and the CDR

May, 2013
Introduction

Importance of claims reserves
- Most important item on the liability side of the balance sheet
- Wrong reserves $\Rightarrow$ wrong premiums
- Reserve risk = main insurance risk in solvency

Perception of management
- Want to be sure that the actuarial reserves are ‘best estimates’
- First check: look at the CDR
  - if it is negative over several years: actuaries are blamed to be responsible for the losses in the annual profit and loss statement
  - if it is positive over several years: actuaries are blamed that prices in the past were not competitive and that the company has missed profitable opportunities

Question:
Are claims development results with the same sign over several consecutive calendar years a contradiction to best estimate reserves?
Notation

- Incremental claims $X_{i,j}$
Notation

- Incremental claims $X_{i,j}$
- Cumulative claims $C_{i,j}$, $U_i = C_{i,J}$ ultimate claim
Notation

- Incremental claims \( X_{i,j} \)
- Cumulative claims \( C_{i,j}, U_i = C_{i,I} \) ultimate claim
- Information at time \( I \): \( D_I = \{ X_{i,j}; \ i + j \leq I \} \) and external information \( I_I \).

![Diagram showing the development of claims over accident years and development years with observed data and to be predicted data.](image-url)
Notation

- Incremental claims $X_{i,j}$
- Cumulative claims $C_{i,j}$, $U_i = C_{i,J}$ ultimate claim
- Information at time $I$: $D_I = \{X_{i,j}; i + j \leq I\}$ and external information $I_I$.
- Outstanding liabilities at time $I$:

$$R_{i}^{(I)} = \sum_{j=I-i+1}^{J} X_{i,j} = U_i - C_{i,I-i+1}, \quad R^{(I)} = \sum_{i=I-J+1}^{I} R_{i}^{(I)}.$$
Notation

- Incremental claims $X_{i,j}$
- Cumulative claims $C_{i,j}$, $U_i = C_{i,J}$ ultimate claim
- Information at time $I$: $\mathcal{D}_I = \{X_{i,j}; i + j \leq I\}$ and external information $\mathcal{I}_I$.
- Outstanding liabilities at time $I$:

$$R^{(I)}_i = \sum_{j=I-i+1}^{J} X_{i,j} = U_i - C_{i,I-i+1}, \quad R^{(I)} = \sum_{i=I-J+1}^{I} R^{(I)}_i.$$

- Claims Reserves:

$$\hat{R}^{(I)}_i = \hat{U}_i - C_{i,I-i+1} = \text{prediction of } R^{(I)}_i, \quad \hat{R}^{(I)} = \sum_{i=I-J+1}^{I} \hat{R}^{(I)}_i.$$
Claims Development Result (CDR)

After one year: New predictions $\hat{R}_i^{(I+1)}$, $\hat{U}_i^{(I+1)}$ and $\hat{R}^{(I+1)}$.

$$CDR^{(I+1)} = \sum_{i=I-J+1}^{I} \hat{U}_i^{(I)} - \hat{U}_i^{(I+1)}$$

$$= \sum_{i=I-J+1}^{I} \left( \hat{R}_i^{(I)} - X_{i,I-i+1} - \hat{R}_i^{(I+1)} \right).$$
Best Estimate Reserves

- Neither optimistic nor pessimistic, neither on the prudent nor on the aggressive side.

\[ \hat{R}(I_i) = E[R(I_i) | I_i, D_i] \]
Best Estimate Reserves

- Neither optimistic nor pessimistic, neither on the prudent nor on the aggressive side.
- For best estimate reserves: Expected value of the CDR is zero.
Best Estimate Reserves

- Neither optimistic nor pessimistic, neither on the prudent nor on the aggressive side.
- For best estimate reserves: Expected value of the CDR is zero.
- In mathematical terms: $\hat{R}_i^{(I)} = E[R_i^{(I)} | \mathcal{I}_I, \mathcal{D}_I]$. 
Martingale Argument

- Best estimate predictions of the ultimate claims:

\[ \hat{U}_i^{(I)} = E[U_i | I, D_I]. \]
Martingale Argument

- Best estimate predictions of the ultimate claims:

\[ \hat{U}^{(I)}_{i} = E[U_i | \mathcal{I}_I, \mathcal{D}_I]. \]

- The sequence \( \{ \hat{U}^{(I)}_{i} : I = i, i+1, \ldots \} \) is a martingale.

\[ \Rightarrow \{ \text{CDR}^{(I)}_{i} = \hat{U}^{(I-1)}_{i} - \hat{U}^{(I)}_{i} : I = i, i+1, \ldots \} \]

are uncorrelated with \( E \left[ \text{CDR}^{(I)}_{i} \right] = 0. \)
Martingale Argument

- Best estimate predictions of the ultimate claims:

\[ \hat{U}^{(I)}_i = E[U_i|\mathcal{I}_I, \mathcal{D}_I] \]

- The sequence \( \{\hat{U}^{(I)}_i : I = i, i+1, \ldots\} \) is a martingale.

\[ \Rightarrow \{\text{CDR}^{(I)}_i = \hat{U}^{(I-1)}_i - \hat{U}^{(I)}_i : I = i, i+1, \ldots\} \]

are uncorrelated with \( E\left[\text{CDR}^{(I)}_i\right] = 0 \).

- Often drawn conclusion: CDR’s with the same sign (losses or gains) over several consecutive calendar years are a contradiction to best estimate reserves.
Example from Practice

Motor Liability in Switzerland after the mid nineties.

- Most companies: negative CDR over several consecutive years

Situation:
- Emergence of a new phenomenon: ‘whiplash cases’
- Creeping change of legislation leading to higher indemnities for such claims
- Reserving actuary at the beginning of this period: increase of incurred claims in newest diagonal (but not in paid triangle); not clear whether this was systematic or random;
- Not full weight given to observations in newest diagonal; only with time ‘new situation’ is fully reflected by the reserves;
- Consequence of this behaviour: negative CDR’s over several years.

Questions:
- Did all these actuaries violate the principle of best estimates?
- Or did we oversee something in the martingale argument?

Answer:
- Yes we have overseen something in the martingale argument
- Nothing wrong in the mathematics
- But we have to be careful in the interpretation what the martingale property exactly means.
Example from Practice

Motor Liability in Switzerland after the mid nineties.

- Most companies: negative CDR over several consecutive years
- Situation:
  - emergence of a new phenomenon: ‘whiplash cases’
  - creeping change of legislation leading to higher indemnities for such claims
  - reserving actuary at the beginning of this period:
    - increase of incurred claims in newest diagonal (but not in paid triangle);
    - not clear whether this was systematic or random;
  - not full weight given to observations in newest diagonal;
    - only with time ‘new situation’ is fully reflected by the reserves;
  - consequence of this behaviour: negative CDR’s over several years.

Questions:

- Did all these actuaries violate the principle of best estimates?
- Or did we oversee something in the martingale argument?

Answer:

- Yes we have overseen something in the martingale argument
- Nothing wrong in the mathematics
- But we have to be careful in the interpretation what the martingale property exactly means.
Example from Practice

Motor Liability in Switzerland after the mid nineties.
- Most companies: negative CDR over several consecutive years

Situation:
- emergence of a new phenomenon: ‘whiplash cases’
- creeping change of legislation leading to higher indemnities for such claims
- reserving actuary at the beginning of this period:
  - increase of incurred claims in newest diagonal (but not in paid triangle);
  - not clear whether this was systematic or random;
- not full weight given to observations in newest diagonal;
  - only with time ‘new situation’ is fully reflected by the reserves;
- consequence of this behaviour: negative CDR’s over several years.

Questions:
- Did all these actuaries violate the principle of best estimates?
- Or did we oversee something in the martingale argument?

Answer:
- Yes we have overseen something in the martingale argument
- Nothing wrong in the mathematics
- But we have to be careful in the interpretation what the martingale property exactly means.
Example from Practice

Motor Liability in Switzerland after the mid nineties.

- Most companies: negative CDR over several consecutive years

**Situation:**
- emergence of a new phenomenon: ‘whiplash cases’
- creeping change of legislation leading to higher indemnities for such claims
- reserving actuary at the beginning of this period:
  increase of incurred claims in newest diagonal (but not in paid triangle);
  not clear whether this was systematic or random;
- not full weight given to observations in newest diagonal;
  only with time ‘new situation’ is fully reflected by the reserves;
- consequence of this behaviour: negative CDR’s over several years.

**Questions:**
- Did all these actuaries violate the principle of best estimates?
- Or did we oversee something in the martingale argument?

**Answer:**
- Yes we have overseen something in the martingale argument
- Nothing wrong in the mathematics
- But we have to be careful in the interpretation what the martingale property exactly means.
Interpretation of the Martingale Property

- Claims development depends on ‘hidden’ characteristics (e.g. economic environment, legislation)

These ‘hidden’ characteristics are best modelled in a Bayesian way as random variables. Martingale argument holds in the average over these state space characteristics.

What we observe in a claims development triangle are conditional observations (given one specific trajectory of the state space process). These conditional observations are no longer increments of a martingale.

Consequence: CDR’s with same sign over several consecutive calendar years are not (necessarily) a contradiction to best estimate reserves.
Interpretation of the Martingale Property

- Claims development depends on ‘hidden’ characteristics (e.g. economic environment, legislation)
- These ‘hidden’ characteristics are best modelled in a Bayesian way as random variables.
Interpretation of the Martingale Property

- Claims development depends on ‘hidden’ characteristics (e.g. economic environment, legislation)
- These ‘hidden’ characteristics are best modelled in a Bayesian way as random variables.
- Martingale argument holds in the average over these state space characteristics.

What we observe in a claims development triangle are conditional observations (given one specific trajectory of the state space process). These conditional observations are no longer increments of a martingale.

Consequence: CDR’s with same sign over several consecutive calendar years are not (necessarily) a contradiction to best estimate reserves.
Interpretation of the Martingale Property

- Claims development depends on ‘hidden’ characteristics (e.g. economic environment, legislation)
- These ‘hidden’ characteristics are best modelled in a Bayesian way as random variables.
- Martingale argument holds in the average over these state space characteristics.
- What we observe in a claims development triangle are conditional observations (given one specific trajectory of the state space process).
Interpretation of the Martingale Property

- Claims development depends on ‘hidden’ characteristics (e.g. economic environment, legislation)
- These ‘hidden’ characteristics are best modelled in a Bayesian way as random variables.
- Martingale argument holds in the average over these state space characteristics.
- What we observe in a claims development triangle are conditional observations (given one specific trajectory of the state space process).
- These conditional observations are no longer increments of a martingale.
Interpretation of the Martingale Property

- Claims development depends on ‘hidden’ characteristics (e.g. economic environment, legislation).
- These ‘hidden’ characteristics are best modelled in a Bayesian way as random variables.
- Martingale argument holds in the average over these state space characteristics.
- What we observe in a claims development triangle are conditional observations (given one specific trajectory of the state space process).
- These conditional observations are no longer increments of a martingale.
- Consequence: CDR’s with same sign over several consecutive calendar years are not (necessarily) a contradiction to best estimate reserves.
Basic Assumptions

Bayesian framework:

- Each calendar year $t$ characterised by a calendar year effect $\Psi_t$
- $\Psi_t$ modelled as hidden random variable
- $\Psi = (\Psi_0, \Psi_1, \ldots)$
- $E[X_{i,j} | \Psi] = \mu_i \gamma_j \Psi_{i+j}$
One Jump Model

Situation considered:

- Calendar years $t \leq s_0 - 1$:
  - ‘stable’ situation
  - no diagonal effects visible in the triangle
  - no indication of a change in the future

- Calendar years $t \geq s_0 + 1$:
  - stable situation, no further changes in claims environment
One Jump Model

Situation considered:

- Calendar years $t \leq s_0 - 1$:
  - ‘stable’ situation
  - no diagonal effects visible in the triangle
  - no indication of a change in the future
- Calendar year $s_0$
  - external information: change of legislation
  - impact:
    - increase of expected value of future claim payments by some not exactly known factor $\Psi_{s_0}$
  - no indication of other changes in the future
One Jump Model

Situation considered:

- **Calendar years** $t \leq s_0 - 1$:
  - ‘stable’ situation
  - no diagonal effects visible in the triangle
  - no indication of a change in the future

- **Calendar year** $s_0$
  - external information: change of legislation
  - impact: increase of expected value of future claim payments by some not exactly known factor $\Psi_{s_0}$
  - no indication of other changes in the future

- **Calendar years** $t \geq s_0 + 1$
  - stable situation, no further changes in claims environment
One Jump Model

Model Assumptions:

i) Given $\Psi$, $X_{i,j}$ are independent and normally distributed with

$$E[X_{i,j} | \Psi] = \mu_i \gamma_j \Psi_{i+j}, \quad \text{Var}(X_{i,j} | \Psi) = \mu_i \gamma_j \sigma^2,$$

where $\sum_{j=0}^{J} \gamma_j = 1$.

ii) For $t \leq s_0 - 1$:
Conditional on $\{I_t, D_t\}$: $\Psi_l = 1$ for $l \leq t$ and
$E[\Psi_{t+l} | I_t, D_t] = E[\Psi_t | I_t, D_t] = 1$ for $l \geq 1$.

iii) For $t \geq s_0$:
Conditional on $\{I_t, D_t\}$

$$\Psi_l = 1 \text{ for } l \leq s_0 - 1;$$
$$\Psi_l = \Psi_{s_0} \text{ for } l = s_0, \ldots, t;$$

$$E[\Psi_{t+l} | I_t, D_t] = E[\Psi_t | I_t, D_t] = E[\Psi_{s_0} | I_t, D_t] \text{ for } l \geq 1,$$

where $\Psi_{s_0} | \{I_t, D_{s_0-1}\} \sim N(m, \tau^2)$. 
Best Estimates

- Best estimate reserves for $t \geq s_0$:

$$
\hat{R}(t) = \sum_{k=1}^{J} \left( E[\Psi_{s_0}|\mathcal{I}_t, \mathcal{D}_t] \sum_{j=k}^{J} \gamma_j \mu_{t+k-j} \right)
$$

Let $\Psi_{s_0}^{(t)} = E[\Psi_{s_0}|\mathcal{I}_t, \mathcal{D}_t]$.

- Exact Credibility case with

$$
\Psi_{s_0}^{(s_0+l)} = E[\Psi_{s_0}|\mathcal{I}_{s_0+l}, \mathcal{D}_{s_0+l}] = \alpha_{s_0+l} \bar{D}_{s_0+l} + (1 - \alpha_{s_0+l})m,
$$

$$
q_{s_0}^{(s_0+l)} = \text{Var}(\Psi_{s_0}|\mathcal{I}_{s_0+l}, \mathcal{D}_{s_0+l}) = (1 - \alpha_{s_0+l})\tau^2,
$$

where

$$
\bar{D}_{s_0+l} = \frac{\sum_{t=s_0}^{s_0+l} \sum_{j=0}^{J} X_{t-j,j}}{\sum_{t=s_0}^{s_0+l} \sum_{j=0}^{J} \mu_{t-j}\gamma_j} \quad \text{and} \quad \alpha_{s_0+l} = \frac{\sum_{t=s_0}^{s_0+l} \sum_{j=0}^{J} \mu_{t-j}\gamma_j}{\sum_{t=s_0}^{s_0+l} \sum_{j=0}^{J} \mu_{t-j}\gamma_j + \frac{\sigma^2}{\tau^2}}.
$$
Recursive estimation of $\Psi_{s_0}$

i) For $t = s_0$

$$\Psi_{s_0}^{(s_0)} = \tilde{\alpha}_{s_0} \sum_{j=0}^{J} \frac{X_{s_0-j,j}}{w_{s_0}} + (1 - \tilde{\alpha}_{s_0}) m, \quad q_{s_0}^{(s_0)} = (1 - \tilde{\alpha}_{s_0}) \tau^2,$$

where

$$\tilde{\alpha}_{s_0} = \frac{w_{s_0}}{w_{s_0} + \frac{\sigma^2}{\tau^2}} \quad \text{and} \quad w_{s_0} = \sum_{j=0}^{J} \mu_{s_0-j} \gamma_j.$$

ii) For $t > s_0$

$$\Psi_{s_0}^{(t)} = \tilde{\alpha}_{t} \sum_{j=0}^{J} \frac{X_{t-j,j}}{w_{t}} + (1 - \tilde{\alpha}_{t}) \Psi_{s_0}^{(t-1)}, \quad q_{s_0}^{(t)} = (1 - \tilde{\alpha}_{t}) q_{s_0}^{(t-1)},$$

where

$$\tilde{\alpha}_{t} = \frac{w_{t}}{w_{t} + \frac{\sigma^2}{q_{s_0}^{(t-1)}}} \quad \text{and} \quad w_{t} = \sum_{j=0}^{J} \mu_{t-j} \gamma_j.$$
Claims Development Result

Assume \( \mu_i \equiv \mu \):

\[
CDR^{(t+1)} = \Psi_{s_0}^{(t+1)} \mu (1 - \beta_0) - \sum_{j=1}^{J} X_{t+1-j,j} + (\Psi_{s_0}^{(t)} - \Psi_{s_0}^{(t+1)}) \mu \sum_{k=1}^{J} (1 - \beta_{k-1})
\]

\[
\text{difference updated forecast and observations}
\]

\[
\text{change in update}
\]

\[
= \Psi_{s_0}^{(t)} \mu (1 - \beta_0) - \sum_{j=1}^{J} X_{t+1-j,j} + (\Psi_{s_0}^{(t)} - \Psi_{s_0}^{(t+1)}) \mu \sum_{k=2}^{J} (1 - \beta_{k-1}),
\]

\[
\text{difference forecast and observations}
\]

\[
\text{change future}
\]
Sign of the CDR

- Observations: Conditional data, given a specific realisation of $\Psi_{s_0}$.
- Expected Value of the CDR for $t + 1 \geq s_0 + 1$

$$E[CDR^{(t+1)}|I_{t+1}, \Psi_{s_0}]$$

$$= (m - \Psi_{s_0})\mu \left((1 - \beta_0)(1 - \alpha_t) + (\alpha_{t+1} - \alpha_t) \sum_{k=2}^{J}(1 - \beta_{k-1})\right)$$

- If $\beta_j \leq 1$ for all $j$:
  - Sign of $E[CDR^{(t+1)}|I_{t+1}, \Psi_{s_0}]$ is the same for all $t + 1 \geq s_0 + 1$.
  - Absolute value is the bigger, the bigger $|m - \Psi_{s_0}|$ is.
  - Absolute value is the bigger, the longer the development is.
  - Absolute value is decreasing in $t$ since $\alpha_t$ is increasing in $t$.
  - For $t \to \infty$ the CDR is conditionally unbiased since $\alpha_t \to 1$ ($t \to \infty$).
Simulation Example

- Parameter choices: $\mu = 10'000$, $m = 1.02$, $\sigma = 0.04$, $\tau = 0.02$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_j$</td>
<td>0.2</td>
<td>0.23</td>
<td>0.12</td>
<td>0.11</td>
<td>0.09</td>
<td>0.07</td>
<td>0.058</td>
<td>0.05</td>
<td>0.043</td>
<td>0.029</td>
</tr>
</tbody>
</table>

| $T$ | 0.2 | 0.23 | 0.12 | 0.11 | 0.09 | 0.07 | 0.058 | 0.05 | 0.043 | 0.029 |

**Table**: Development pattern $\gamma_j$.

- Estimation of $\Psi_{s_0}$:

$$\Psi_{s_0}(t) = \alpha_t \frac{1}{\mu(t - s_0 + 1)} \sum_{l=s_0}^{t} \sum_{j=0}^{J} X_{l-j,j} + (1 - \alpha_t)m,$$

where

$$\alpha_t = \frac{t - s_0 + 1}{(t - s_0 + 1) + \sigma^2 \tau^2}.$$

- Assume jump occurs at time $s_0 = 10$, $\Psi_{10} = 1.035$. 

Annina Saluz and Alois Gisler
Best Estimates and the CDR
May, 2013
Simulated Data

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2308</td>
<td>2151</td>
<td>1083</td>
<td>1067</td>
<td>701</td>
<td>641</td>
<td>502</td>
<td>657</td>
<td>520</td>
<td>324</td>
</tr>
<tr>
<td>1</td>
<td>1731</td>
<td>2612</td>
<td>1189</td>
<td>1050</td>
<td>915</td>
<td>807</td>
<td>854</td>
<td>470</td>
<td>573</td>
<td>321</td>
</tr>
<tr>
<td>2</td>
<td>1757</td>
<td>2036</td>
<td>1014</td>
<td>1106</td>
<td>926</td>
<td>634</td>
<td>651</td>
<td>373</td>
<td>446</td>
<td>239</td>
</tr>
<tr>
<td>3</td>
<td>1868</td>
<td>2190</td>
<td>1098</td>
<td>746</td>
<td>680</td>
<td>654</td>
<td>773</td>
<td>501</td>
<td>354</td>
<td>227</td>
</tr>
<tr>
<td>4</td>
<td>1970</td>
<td>2342</td>
<td>1091</td>
<td>1098</td>
<td>1021</td>
<td>626</td>
<td>533</td>
<td>468</td>
<td>329</td>
<td>370</td>
</tr>
<tr>
<td>5</td>
<td>1915</td>
<td>1959</td>
<td>1066</td>
<td>1241</td>
<td>1011</td>
<td>763</td>
<td>764</td>
<td>661</td>
<td>415</td>
<td>284</td>
</tr>
<tr>
<td>6</td>
<td>1766</td>
<td>2293</td>
<td>1028</td>
<td>1070</td>
<td>841</td>
<td>844</td>
<td>711</td>
<td>516</td>
<td>431</td>
<td>230</td>
</tr>
<tr>
<td>7</td>
<td>2027</td>
<td>2039</td>
<td>1476</td>
<td>1344</td>
<td>904</td>
<td>618</td>
<td>490</td>
<td>524</td>
<td>465</td>
<td>145</td>
</tr>
<tr>
<td>8</td>
<td>2041</td>
<td>2278</td>
<td>986</td>
<td>1209</td>
<td>1014</td>
<td>685</td>
<td>594</td>
<td>591</td>
<td>441</td>
<td>286</td>
</tr>
<tr>
<td>9</td>
<td>2236</td>
<td>2325</td>
<td>1008</td>
<td>1369</td>
<td>1002</td>
<td>687</td>
<td>563</td>
<td>540</td>
<td>419</td>
<td>395</td>
</tr>
<tr>
<td>10</td>
<td>2008</td>
<td>2507</td>
<td>1289</td>
<td>1081</td>
<td>1046</td>
<td>718</td>
<td>551</td>
<td>680</td>
<td>381</td>
<td>334</td>
</tr>
<tr>
<td>11</td>
<td>2197</td>
<td>2399</td>
<td>1449</td>
<td>1139</td>
<td>1008</td>
<td>764</td>
<td>667</td>
<td>614</td>
<td>430</td>
<td>302</td>
</tr>
<tr>
<td>12</td>
<td>2098</td>
<td>2362</td>
<td>1366</td>
<td>883</td>
<td>855</td>
<td>851</td>
<td>529</td>
<td>723</td>
<td>256</td>
<td>431</td>
</tr>
<tr>
<td>13</td>
<td>1918</td>
<td>2448</td>
<td>1301</td>
<td>1133</td>
<td>935</td>
<td>699</td>
<td>593</td>
<td>692</td>
<td>457</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>1741</td>
<td>2693</td>
<td>1417</td>
<td>1106</td>
<td>935</td>
<td>649</td>
<td>470</td>
<td>631</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>2203</td>
<td>2294</td>
<td>1280</td>
<td>1283</td>
<td>1225</td>
<td>687</td>
<td>637</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>2122</td>
<td>2523</td>
<td>1581</td>
<td>1201</td>
<td>845</td>
<td>765</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>2238</td>
<td>2176</td>
<td>1232</td>
<td>1165</td>
<td>813</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>2128</td>
<td>1960</td>
<td>1297</td>
<td>1235</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>2091</td>
<td>2121</td>
<td>1264</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>2143</td>
<td>2833</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>2004</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table: Incremental claims
Predictions of $\Psi_{s0}$ over time

Figure: Predictions $\Psi_{s0}^{(t)}$ of $\Psi_{s0} = 1.035$. 
CDR’s

Figure: CDR and $E[\text{CDR}_t | \Psi_{s_0}, I_t]$. 
Figure: CDR and split in ‘difference forecast and observations’ plus ‘change in future’.
Results

We observe

- negative CDR’s in 6 consecutive years
- expected value of the CDR is negative
- 9 times a loss in the 10 years from calendar year 10 to 19.
AR(1) Process

Idea: Based on external information $\mathcal{I}_t$ assume

- AR structure for future diagonal effects

$$\{\Psi_{t+l} : l \geq 0\} \overset{d}{=} \{\tilde{\Psi}_{t+l} = \rho(t)\tilde{\Psi}_{t+l-1} + (1 - \rho(t))m(t) + \Delta_{t+l}^{(t)}, \ l \geq 0\}$$

- $\Delta_{t+l}^{(t)}|\mathcal{I}_t \overset{iid}{\sim} N\left(0, \delta^2(t)\right)$
- Given $\mathcal{I}_t$, $\Delta_{t+l}^{(t)}$, $l \geq 1$, independent from $\tilde{\Psi}_t$
- Given $\mathcal{I}_t$, $\Delta_t^{(t)}$ independent from $\tilde{\Psi}_{t-1}$
- $\Psi_t|\mathcal{I}_t$ normally distributed.

- Updating of AR parameters for future diagonal effects is possible.
- Past diagonal effects not updated due to new external information:

$$\Psi_{t-l}|\mathcal{I}_t \overset{d}{=} \Psi_{t-l}|\mathcal{I}_{t-l}$$
More General Model

- Idea: $\Psi_t = a_t(t)\Psi_{t-1} + b_t(t) + \Delta_t(t)$, where $\Delta_t(t)|\mathcal{I}_t \sim N(0, \delta_t^2(t))$ independent of $\mathcal{D}_{t-1}$ and $\Psi_{t-1}$.
  - Parameters $a_t(t)$, $b_t(t)$ based on updated a priori information $\mathcal{I}_t$ and on past data $\mathcal{D}_{t-1}$, and $\delta_t^2(t)$ based on $\mathcal{I}_t$.
  - Parameters $a_t(t)$, $b_t(t)$ and $\delta_t^2(t)$ will not change due to new external information $\mathcal{I}_{t+1}$.

- Future: $\Psi_{t+1} = a_{t+1}(t)\Psi_{t+1-1} + b_{t+1}(t) + \Delta_{t+1}(t)$, where the $\Delta_{t+1}(t)|\mathcal{I}_t \sim N(0, \delta_{t+1}^2(t))$ are mutually independent and independent of $\mathcal{D}_t$ and $\Psi_t$.
  - Parameters $a_{t+1}(t)$, $b_{t+1}(t)$ and $\delta_{t+1}^2(t)$ reflect a priori assumptions of process in later years based on $\mathcal{I}_t$ and $\mathcal{D}_t$.
  - Parameters $a_{t+1}(t)$, $b_{t+1}(t)$ and $\delta_{t+1}^2(t)$ will be replaced in later years due to new information.

- Updating the external ‘a priori’ information $\mathcal{I}_t$ is possible.
Normal-Normal Model

i) Conditionally, given $\Psi$ and $\mathcal{I}_t$, $X_{i,j}$ independent and normally distributed with

$$
E [ X_{i,j} | \Psi, \mathcal{I}_t ] = E [ X_{i,j} | \Psi_{i+j} ] = \mu_i \gamma_j \Psi_{i+j},
$$

$$
\text{Var} ( X_{i,j} | \Psi, \mathcal{I}_t ) = \mu_i \eta_j^2 \sigma^2,
$$

where $\sum_{j=0}^{J} \gamma_j = \sum_{j=0}^{J} \eta_j^2 = 1$. 
Normal-Normal Model

i) Conditionally, given $\Psi$ and $\mathcal{I}_t$, $X_{i,j}$ independent and normally distributed with

$$E \left[ X_{i,j} \mid \Psi, \mathcal{I}_t \right] = E \left[ X_{i,j} \mid \Psi_{i+j} \right] = \mu_i \gamma_j \Psi_{i+j},$$

$$\text{Var} \left( X_{i,j} \mid \Psi, \mathcal{I}_t \right) = \mu_i \eta_j^2 \sigma^2,$$

where $\sum_{j=0}^{J} \gamma_j = \sum_{j=0}^{J} \eta_j^2 = 1$.

ii) Conditionally, given $\mathcal{I}_t$ and $\mathcal{D}_{t-1}$, $\Psi_t$ normally distributed with

$$\Psi_{(t,t-1)} = E[\Psi_t \mid \mathcal{I}_t, \mathcal{D}_{t-1}] = a_t^{(t)} \Psi_{(t-1)} + b_t^{(t)}$$

$$q_{(t,t-1)} = \text{Var}(\Psi_t \mid \mathcal{I}_t, \mathcal{D}_{t-1}) = \left(a_t^{(t)} \right)^2 q_{(t-1)} + \delta_{(t)}^2,$$

for $t \geq 0$, and $\Psi_{(-1)} = 1$ and $q_{(-1)} = \tau^2$. 
Normal-Normal Model

i) Conditionally, given $\Psi$ and $I_t$, $X_{i,j}$ independent and normally distributed with

$$E \left[ X_{i,j} \mid \Psi, I_t \right] = E \left[ X_{i,j} \mid \Psi_{i+j} \right] = \mu_i \gamma_j \Psi_{i+j},$$

$$\text{Var} \left( X_{i,j} \mid \Psi, I_t \right) = \mu_i \eta_j^2 \sigma^2,$$

where $\sum_{j=0}^{J} \gamma_j = \sum_{j=0}^{J} \eta_j^2 = 1$.

ii) Conditionally, given $I_t$ and $D_{t-1}$, $\Psi_t$ normally distributed with

$$\Psi_{t(t,t-1)} = E[\Psi_t \mid I_t, D_{t-1}] = a_t^{(t)} \Psi_{t-1}^{(t-1)} + b_t^{(t)},$$

$$q_{t(t,t-1)} = \text{Var}(\Psi_t \mid I_t, D_{t-1}) = \left( a_t^{(t)} \right)^2 q_{t-1}^{(t-1)} + \delta_t^{2(t)},$$

for $t \geq 0$, and $\Psi_{-1} = 1$ and $q_{-1} = \tau^2$.

iii) Conditionally, given $I_t$ and $D_t$, $\{\Psi_{t+l} : l \geq 1\}$ are normally distributed with

$$\Psi_{t+t} := E[\Psi_{t+l} \mid I_t, D_t] = a_{t+l}^{(t)} \Psi_t^{(t)} + b_{t+l}^{(t)}$$

$$q_{t+t} := \text{Var} \left( \Psi_{t+l} \mid I_t, D_t \right) = \left( a_{t+l}^{(t)} \right)^2 q_{t+l-1}^{(t)} + \delta_{t+l}^2.$$
Normal-Normal Model

i) Conditionally, given \( \Psi \) and \( \mathcal{I}_t \), \( X_{i,j} \) independent and normally distributed with

\[
\begin{align*}
E[ X_{i,j} | \Psi, \mathcal{I}_t ] &= E[ X_{i,j} | \Psi_{i+j} ] = \mu_i \gamma_j \Psi_{i+j}, \\
\text{Var} ( X_{i,j} | \Psi, \mathcal{I}_t ) &= \mu_i \eta_j^2 \sigma^2,
\end{align*}
\]

where \( \sum_{j=0}^{J} \gamma_j = \sum_{j=0}^{J} \eta_j^2 = 1 \).

ii) Conditionally, given \( \mathcal{I}_t \) and \( \mathcal{D}_{t-1} \), \( \Psi_t \) normally distributed with

\[
\begin{align*}
\Psi_{t}^{(t,t-1)} &= E[ \Psi_t | \mathcal{I}_t, \mathcal{D}_{t-1} ] = a_t^{(t)} \Psi_{t-1}^{(t-1)} + b_t^{(t)} \\
q_{t}^{(t,t-1)} &= \text{Var}( \Psi_t | \mathcal{I}_t, \mathcal{D}_{t-1} ) = \left( a_t^{(t)} \right)^2 q_{t-1}^{(t-1)} + \delta_t^{2(t)},
\end{align*}
\]

for \( t \geq 0 \), and \( \Psi_{-1}^{(-1)} = 1 \) and \( q_{-1}^{(-1)} = \tau^2 \).

iii) Conditionally, given \( \mathcal{I}_t \) and \( \mathcal{D}_t \), \{\( \Psi_{t+l} : l \geq 1 \)\} are normally distributed with

\[
\begin{align*}
\Psi_{t+l}^{(t)} := E[ \Psi_{t+l} | \mathcal{I}_t, \mathcal{D}_t ] &= a_{t+l}^{(t)} \Psi_{t+l-1}^{(t)} + b_{t+l}^{(t)} \\
q_{t+l}^{(t)} := \text{Var} ( \Psi_{t+l} | \mathcal{I}_t, \mathcal{D}_t ) &= \left( a_{t+l}^{(t)} \right)^2 q_{t+l-1}^{(t)} + \delta_{t+l}^{2(t)}.
\end{align*}
\]

iv) For \( 0 \leq l \leq t \), \( \Psi_{t-l} | (\mathcal{I}_t, \mathcal{D}_{t-l}) \overset{d}{=} \Psi_{t-l} | (\mathcal{I}_{t-l}, \mathcal{D}_{t-l}) \).
Recursive Calculation of the $\Psi_t$’s, Normal-Normal Model

For $t \geq 0$:

a) Update in the newest diagonal:

$$
\Psi_t^{(t)} = \alpha_t \bar{X}_t + (1 - \alpha_t) \Psi_t^{(t,t-1)},
$$

where

$$
\alpha_t = \frac{w_t}{w_t + \kappa_t}, \quad \text{with} \quad w_t = \sum_{j=0}^{J \wedge t} \mu_{t-j} \frac{\gamma_j^2}{\eta_j^2}, \quad \kappa_t = \frac{\sigma^2}{q_{t,t-1}},
$$

$$
\bar{X}_t = \frac{1}{w_t} \sum_{j=0}^{J \wedge t} \frac{\gamma_j}{\eta_j^2} X_{t-j,j},
$$

and

$$
q_t^{(t)} = (1 - \alpha_t) q_t^{(t,t-1)}.
$$

b) Forecast of future diagonal effects: For $k \geq 1$

$$
\Psi_{t+k}^{(t)} = a_{t+k}^{(t)} \Psi_{t+k-1}^{(t)} + b_{t+k}^{(t)}.
$$
Summary

- Models that allow to
  - describe diagonal effects,
  - update external a priori information.

Conclusion:
- Same sign of CDR over several consecutive calendar years is not a contradiction to best estimates.
- Potential applications for pricing.