Closed-form solutions for options in incomplete markets

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What we do

- Derive closed-form solutions for European and perpetual American call options in incomplete markets

- **Main finding**: contrary to Black-Scholes (1973) option pricing model, an increase in volatility of underlying asset does not always lead to an increase in option price

- In incomplete markets:
  - increased volatility $\Rightarrow$ higher hedgeable risk, but also higher unhedgeable risk, which can decrease option value
  - incentive to exercise a perpetual American call option early as soon as increased unhedgeable risk erodes option value

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Black-Scholes (1973) option pricing model

• Assumes:
  o underlying asset is continuously traded
  o all sources of risk can be perfectly hedged against (i.e. market is complete)

• Predicts:
  o an increase in volatility of underlying asset always leads to an increase in option price
Problem

• Assume option is written on non-traded or infrequently traded underlying asset

• Then:
  - difficult to observe price of underlying
  - difficult to observe market price of risk of underlying $(\mu - r)/\sigma$
  - underlying no longer continuously traded $\Rightarrow$ violation of main assumption of Black-Scholes (1973) option pricing model
Consequences

- **Market becomes incomplete**: sources of risk from underlying asset are unhedgeable or partly hedgeable

- If we cannot trade units of the underlying asset at any given point in time:
  - cannot construct replicating portfolio of the option
  - cannot compute probabilities or option price using Black-Scholes (1973) model
Think of discrete-time binomial tree...

- $S_0$ - price of a traded asset
- $f$ - price of an option written on asset $S$
- there also exists a riskless asset yielding return $r$

- Because $S$ is traded, we can construct a replicating portfolio of stocks and bonds

- Then, apply the investment strategy: long in the option and short in the replicating portfolio, with a zero payoff

- By absence of arbitrage, the initial price of the strategy must also be zero $\Rightarrow$ price of option equals cost of replicating portfolio:

$$f = e^{-r\Delta t}[qf_u + (1 - q)f_d]$$

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Think of discrete-time binomial tree…

- Assume $S$ is no longer a traded asset, but either non-traded or infrequently traded

- If we cannot buy \textbf{units} of $S$ at any given point in time, we cannot construct the replicating portfolio and compute the probabilities or the option price

- We need a different approach, one that is able to price the remaining unhedgeable risk as well
**The quadrinomial tree...**

- To reflect both hedgeable and unhedgeable sources of risk, the binomial tree will become a quadrinomial tree.

- $S_0$:
  - $g$: good state of the world
  - $b$: bad state of the world
  - $g$, $b$ cannot be hedged
  - $u$, $d$ can be hedged with a traded asset correlated with non-traded underlying
  - As on a complete market
    \[ u = e^{\sigma \sqrt{\Delta t}} ; \quad d = e^{-\sigma \sqrt{\Delta t}} \]
  - Similarly, $g$ and $b$ will be of the generic form
    \[ g = e^{\varepsilon \sqrt{\Delta t}} ; \quad b = e^{-\varepsilon \sqrt{\Delta t}} \]

- The larger $\sigma$ and $\varepsilon$, the larger the gap between potential outcomes of price of option, particularly between unhedgeable states $g$ and $b$.

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Our solution: good-deal bounds for options in incomplete markets

• Following Cochrane and Saa-Requejo (2000), we use the good-deal bounds (GDB) to incorporate unhedgeable risk in the pricing mechanism

• However, we give a different interpretation to the GDB by applying a change of measure to the process of the underlying asset in order to price the option
Market setting and assumptions

- The market consists of a riskless asset $B$, a risky non-traded asset $V$ and a risky traded asset $S$ correlated with $V$

\[
\begin{align*}
  dB &= rB dt \\
  dV &= \mu_V V dt + \sigma_V V \left( \rho dz + \sqrt{1 - \rho^2} dw \right) \\
  dS &= \mu_S S dt + \sigma_S S dz
\end{align*}
\]

- $dz$ and $dw$ are independent Brownian motions

- Assets $V$ and $S$ are correlated with correlation coefficient $\rho$

- European call option written on asset $V$
How do the good-deal bounds work?

- To compute the price of the call option, we need to calculate the discounted expected value of the option payoff:

\[ C_0 = E^P \left[ \frac{\Lambda_T}{\Lambda_0} (V_T - K)^+ \right] = e^{-rT} E^{Q_{GDB}} [(V_T - K)^+] \]

- \( Q_{GDB} \) new probability measure via Radon-Nikodym derivative

- \( V_T \) is the value of the underlying asset at time \( T \)

- \( \Lambda \) is a stochastic discount factor

- \( K \) is a constant strike price

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How do the good-deal bounds work?

- The stochastic discount factor in a complete market must price only hedgeable sources of risk

\[ d\Lambda = -r\Lambda dt - \frac{\mu - r}{\sigma}\Lambda dz \]

- The volatility term is actually a Sharpe ratio

- So, restricting the volatility of the stochastic discount factor is equivalent to restricting the Sharpe ratio of a traded risky asset (Hansen and Jagannathan, 1991)
How do the good-deal bounds work?

• In an **incomplete market**, the stochastic discount factor must price both hedgeable and unhedgeable sources of risk

  \[ d\Lambda = -r\Lambda dt - \kappa_1\Lambda dz - \kappa_2\Lambda dw \]

• \( \kappa_1 \) is market price of hedgeable risk
  - can be observed (Sharpe ratio of the traded asset \( S \))

• \( \kappa_2 \) is market price of unhedgeable risk
  - cannot be observed
How do the good-deal bounds work?

- We are looking for a high enough value of the total market price of risk to induce trade, without including extremely high Sharpe ratios and arbitrage opportunities.

- Place a “reasonable” upper bound $k$ on the total volatility of the stochastic discount factor:

  $$\kappa_1^2 + \kappa_2^2 \leq k^2$$

- Under the new probability measure $Q_{GDB}$, the non-traded asset $V$ has a new process obtained via a Girsanov transformation:

  $$dV = (\mu_V - \sigma_V \rho \frac{\mu_S - r}{\sigma_S} - \sigma_V \sqrt{1 - \rho^2} \kappa_2) V dt + \sigma_V \sqrt{1 - \rho^2} dw^*$$

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How do the good-deal bounds work?

• Placing an upper bound on the volatility of the stochastic discount factor translates into a minimization over the set of risk measures

\[ C_0 = \min_{Q_{GDB} \leq k^2} e^{-rT} E^{Q_{GDB}} [(V_T - K)^+] \]

• We choose all measures $Q_{GDB}$ lower than $k^2$

• So, mathematically, the good-deal bounds are equivalent to the coherent risk measure of Artzner et al. (1999), with:
  
  o minimum leading to a lower bound
  
  o maximum (or minimum of the negative payoff) leading to an upper bound
**Results** (fix market price of risk and all other parameters except volatility of underlying asset)

- **Volatility very low**: BS and GDB prices converge (the only source of uncertainty comes from traded asset)

- **As volatility increases**: range of price possibilities becomes bigger, but option value doesn’t always increase; consistent with Henderson (2007) and Miao and Wang (2007) via utility indifference pricing

![Graph showing the relationship between option price and volatility of underlying asset. The graph includes three lines: Upper bound, BS, and Lower bound. The x-axis represents volatility of underlying asset (σv), and the y-axis represents option price.](image)
Interpretation

- **Reservation prices**: upper bound seller’s minimum valuation; lower bound buyer’s maximum valuation of the option.
- The interesting case is the one of the lower bound, which decreases as volatility increases.
- When economic agent does not own the option, he is willing to pay less and less for it as uncertainty increases.
- Feature not explained by complete market models, which take into account only hedgeable sources of risk.
Sensitivity analysis – correlation coefficient $\rho$

- Perfect correlation: GDB prices = BS price

- Lower bound price $\rho=0$: here buyer is willing to pay the least for the option (cannot hedge any part of the risk)

- Large gap between prices on almost complete market and that on complete market: GDB prices approach BS price at speed $\sqrt{1 - \rho^2}$ consistent with results of Davis (2006) via utility indifference pricing approach

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**Sensitivity analysis - volatility restriction \( k \)**

- Variance of the stochastic discount factor \( \Leftrightarrow \) “distance” between physical measure \( P \) and new probability measure \( Q_{GDB} \)
- The lower the restriction \( k \), the closer measure \( P \) is to measure \( Q_{GDB} \) and the tighter the bounds
- \( k = \) Sharpe ratio of traded asset: GDB prices converge to BS price (market price of unhedgeable risk is 0)
Perpetual American call option

- Assume the call option on the non-traded underlying asset $V$ is now a perpetual American option (i.e. maturity is infinite).

- Advantage is that option can be exercised at any point in time and exercise can be timed.

- Early exercise can happen once the value of the underlying asset is at least as large as a general threshold $V^*$, where $V^* > K$.

- In a complete market, an American call option written on an underlying asset which does not pay any dividends is never exercised early.
Early exercise of perpetual American call option

- In an incomplete market, increasing unhedgeable risk erodes the option value => American call option should be exercised early to lock in value while it still exists

- At the lower bound, the increase in volatility of underlying asset $\sigma_V$ leads to a decrease in option value and investment threshold
Calibrating the total volatility restriction \( k \)

- Use Sharpe ratio on equity market as a guideline:
  - set \( k \) equal to highest Sharpe ratio on this market
  - highest Sharpe ratio ever documented on this market was 2 (Murgoci, 2012)

- Imply the value of \( k \) from market data:
  - for example, using weather derivatives
Applications

• The good-deal bounds pricing mechanism can be applied to various market situations, like:
  
  o pricing of real options (i.e. options on non-financial assets)
  
  o valuation of investments using option pricing techniques (here, perpetual American options can be useful)
  
  o the valuation of long-dated contracts offered by life insurance companies or pension funds (for maturities beyond 30 years we no longer have bonds traded in the market)
Conclusion

- We price European and American call options using the good-deal bounds incomplete market technique.

- We find that an increase in the volatility of the underlying asset triggers a decrease in the lower bound option prices.

- Early exercise of a perpetual American call option can happen for the lower bound option prices.