Moments of Dividends and Optimal Expected Dividends in the Erlang\((n)\) dual risk model

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<table>
<thead>
<tr>
<th>Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Model and Definitions</td>
</tr>
</tbody>
</table>
Table of contents

1. Model and Definitions
2. Moments of the Discounted Dividends
Table of contents

1 Model and Definitions
2 Moments of the Discounted Dividends
3 The Optimal Dividend Barrier
4 Conclusions
Model and Definitions

Renewal dual risk model

\[ U(t) = u - ct + \sum_{i=0}^{N(t)} X_i, \quad X_0 = 0, \quad t \geq 0, \quad u \geq 0, \]

- \(u\), initial capital,
- \(t\), time parameter,
- \(c > 0\), rate of expenses,
- \(\{X_i\}_{i=1}^{\infty}\), single gain amounts sequence,
- \(\{W_i\}_{i=1}^{\infty}\), gain inter–arrival times sequence,
- \(N(t) = \max\{k : W_1 + W_2 + \cdots + W_k \leq t\}\), number of gains up to time \(t\),
- \(\{N(t), t \geq 0\}\) is a renewal process.
$U(t) = u - ct + \sum_{i=1}^{4} X_i, \quad N(t) = 4$
Assumptions of the model:

• $X_i \text{ i.i.d.}, X_i \sim P(x)$ with density $p(x)$,

• $\hat{p}(s) = \int_0^\infty e^{-sx} p(x)dx$, $s \in \mathbb{C}$ Laplace transform of $p(x)$,

• $\mu_k = E[X_1^k]$ the $k$-th moment of $X_1$, at least $\mu_1$ must exist.

• $W_i \text{ i.i.d.}$ and independent from the $X_i$’s ,

• $W_i \sim K_n(t)$, $W_i \sim$ Erlang($n, \lambda$) with density

$$k_n(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad t \geq 0, \lambda > 0, n \in \mathbb{N}^+, \text{ and } \quad K_n(t) = 1 - \sum_{i=0}^{n-1} \frac{(\lambda t)^i e^{-\lambda t}}{(i)!}$$

• $cE(W_1) < E(X_1)$ negative loading factor condition,

\[ \iff cn < \lambda \mu_1. \]
Model and Definitions

**Time of ruin** \( T_u = \inf \{ t > 0 : U(t) = 0 \}, \ u \geq 0, \)
\[ T_u = \infty \text{ iff } U(t) > 0 \ \forall t > 0. \]

**Ultimate ruin probability** \( \psi(u) = P(T_u < \infty). \)

**Survival probability** \( \phi(u) = 1 - \psi(u). \)

For a constant \( \delta \geq 0, \) the Laplace transform of the time of ruin is
\[ \psi(u, \delta) = E(e^{-\delta T_u \mathbb{1}(T_u < \infty)}). \]

Remark: \( \delta \) can be interpreted as an interest force.
Among pioneer works on the subject we can cite Cramér (1955), Gerber (1979), Seal (1969), Tákacs (1967). Recent works include those by Avanzi et al. (2007), Cheung and Drekic (2008), Ng (2009), Afonso et al. (2011).

From those, the works particularly focusing the dual model and the problem of discounted dividends assume that interarrival times follow an exponential distribution.

We work on a dual risk model with Erlang(n) interclaim times and study the moments of the expected discounted dividends. Then we use our results for the calculation of the optimal dividend barrier. For this we will assume that the gain amounts $X_i$ follow a Phase Type PH(m) distribution.
Model and Definitions

The following equation, called the generalized Lundberg’s equation, plays an important role in the computation of the expected dividends

\[ E[e^{-\delta W_i} e^{-s(X_i-cW_i)}] = 1 \]

For the Erlang($n$) dual risk model, this equation becomes

\[ \left( 1 + \frac{\delta}{\lambda} - \left( \frac{c}{\lambda} \right) s \right)^n = \hat{p}(s), \quad s \in \mathbb{C}. \]

It can be proven that this equation has $n$ roots with positive real parts, which we denote by $\rho_1, \ldots, \rho_n$, and that they are all distinct.
The following closed formulas for the ultimate ruin probability $\psi(u)$

$$
\psi(u) = \sum_{k=1}^{n} \left[ \prod_{i=1, i \neq k}^{n} \frac{\rho_i}{(\rho_i - \rho_k)} \right] e^{-\rho_k u},
$$

and the Laplace transform of the time of ruin $\psi(u, \delta)$

$$
\psi(u, \delta) = \sum_{k=1}^{n} \left[ \prod_{i=1, i \neq k}^{n} \frac{(\rho_i - \delta_c)}{(\rho_i - \rho_k)} \right] e^{-\rho_k u},
$$

can be found in Rodríguez et al (2012).
We introduce an upper barrier into the model and let $b$ denote its level.

Each time the surplus process upcrosses level $b$ the excess gain is paid out immediately to the capital holder as a dividend, prior to ruin.

Let $\{D_i\}_{i=1}^{\infty}$ be the sequence of the dividend payments and let $D(u, b)$ be the aggregate discounted dividends, at force of interest $\delta$ and from initial surplus $u$.

We denote by $V_k(u, b) = E[D(u, b)^k]$, $k \geq 1$, the $k$-th order moment of $D(u, b)$, for simplicity denote $V(u, b) = V_1(u, b)$.
Moments of the Discounted Dividends

\[ U(t) = u - ct + \sum_{i=0}^{N(t)} X_i = u - ct \]

\[ U(t_0) = u - ct_0 = 0, \quad T_u = t_0 = \frac{u}{c} \]
Moments of the Discounted Dividends

\[ U(t) \]

\[ \begin{align*}
U(0) &= b \\
U(T_u) &= u
\end{align*} \]

\[ U(t) \quad t_0 \quad T_u \]

\[ D_1 \quad D_2 \]

\[ \text{Eugenio Rodríguez (ISEG, UTL)} \]
Moments of the Discounted Dividends

Note that

\[ V(u, b) = E[u - b + D(b, b)] = u - b + V(b, b), \quad u \geq b. \]

and in general

\[
V_k(u, b) = E[(u - b + D(b, b))^k] = \sum_{j=0}^{k} \binom{k}{j} (u - b)^j V_{k-j}(b, b), \quad u \geq b.
\]
Moments of the Discounted Dividends

The expected discounted dividends $V(u, b)$ satisfy the renewal equation

$$V(u, b) = \int_{0}^{\frac{u}{c}} k_n(t)e^{-\delta t} \left[ \int_{0}^{b-u+ct} V(u-ct+y, b)p(y)dy + \int_{b-u+ct}^{\infty} \tilde{V}(u-ct+y, b)p(y)dy \right] dt, \quad u < b,$$

with

$$\tilde{V}(x, b) = E[D(x, b)] = E[x - b + D(b, b)] = x - b + V(b, b), \quad x \geq b.$$
Moments of the Discounted Dividends

\( V_k(u, b) \) satisfies the renewal equation

\[
V_k(u, b) = \int_0^u k_n(t)e^{-k\delta t} \left[ \int_0^{b-u+ct} V_k(u - ct + y, b)p(y)dy + \int_{b-u+ct}^{\infty} \tilde{V}_k(u - ct + y, b)p(y)dy \right] dt, \quad u < b,
\]

with

\[
\tilde{V}_k(x, b) = E[(x - b + D(b, b))^k] = \sum_{j=0}^{k} \binom{k}{j} (x - b)^j V_{k-j}(b, b), \quad x \geq b.
\]

In the above expression we assume the convention \( V_0(u, b) \equiv 1. \)
Moments of the Discounted Dividends

Differentiating the renewal equation with respect to \( u \) we can obtain an integro–differential equation for \( V_k(u, b) \)

\[
\left( \left( 1 + \frac{k \delta}{\lambda} \right) I + \left( \frac{c}{\lambda} \right) D \right)^n V_k(u, b) = W_{k\delta}(u, b), \quad u < b, \tag{1}
\]

with boundary conditions

\[
\left. \frac{d^i}{du^i} V_k(u, b) \right|_{u=0} = 0, \quad i = 0, \ldots, n - 1, \tag{2}
\]

where

\[
W_{k\delta}(u, b) = \int_u^b V_k(x, b)p(x - u)dx + \int_b^\infty \widetilde{V}_k(x, b)p(x - u)dx,
\]

is the integral term.
One way to solve the integro–differential equation is the annihilator method. Let $A(\mathcal{D})$ be a differential operator such that

\[ A(\mathcal{D}) \left[ \left( \left( 1 + \frac{k\delta}{\lambda} \right) \mathcal{I} + \left( \frac{c}{\lambda} \right) \mathcal{D} \right)^n V_k(u, b) \right] = A(\mathcal{D}) [W_k\delta(u, b)] = 0. \]

Then $A(\mathcal{D})$ is called an annihilator, since it annihilates the integral term $W_k\delta(u, b)$. 
In consequence our original integro–differential equation

\[
\left( \left( 1 + \frac{k\delta}{\lambda} \right) I + \left( \frac{c}{\lambda} \right) D \right)^n V_k(u, b) = W_{k\delta}(u, b), \quad u < b,
\]

becomes into an homogeneous differential equation of a higher order of the form

\[
\tilde{A}(D)V_k(u, b) = 0.
\] (3)

Solutions of the first equation are solutions of the second equation. The converse is true only with some additional constrains.
Now recall that the integral term

\[ W_{k\delta}(u, b) = \int_{u}^{b} V_k(x, b)p(x - u)\,dx + \int_{b}^{\infty} \tilde{V}_k(x, b)p(x - u)\,dx, \]

depends on the density \( p(x) \) of the gain amounts.

Therefore in order to follow this approach and find the annihilator we must at least specify which is the density \( p(x) \) of the gains.
Consider the case when the gains $X_i$ follow a Phase-Type PH($m$) distribution.

Let $B = (b_{ij})_{1 \leq i, j \leq m}$ be the matrix of transition rates between $m$ transient states $\{E_1, E_2, \ldots, E_m\}$, let $\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be the vector of the initial probabilities, $\eta' = (\eta_1, \eta_2, \ldots, \eta_m)$ the vector of the exit rates to the absorbing state $E_0$, and $1' = (1, 1, \ldots, 1)$ the $1 \times m$ vector. Let $I_m$ denote the identity matrix $m \times m$.

Then the density of the gain amounts is given by

$$p(x) = \alpha' e^{Bx} \eta,$$

and the distribution is

$$P(x) = 1 - \alpha' e^{Bx} 1.$$
Moments of the Discounted Dividends with Phase Type gains

Let $q_B(y) = Det(B - yI_m)$ be the characteristic polynomial of the intensity matrix $B$ and $\mathcal{D} = \frac{d}{du}$ the differentiation with respect to $u$.

Define the differential operator $q_B(-\mathcal{D})$

$$q_B(-\mathcal{D}) = \sum_{i=0}^{m} q_i \frac{d^i}{du^i},$$

for some coefficients $q_i$ that depend on the entries of $B$. 
Theorem

$q_B(-\mathcal{D})$ is an annihilator for the integro–differential equation

\[
\left( \left( 1 + \frac{k\delta}{\lambda} \right) \mathcal{I} + \left( \frac{c}{\lambda} \right) \mathcal{D} \right)^n V_k(u, b) = W_{k\delta}(u, b), \quad u < b,
\]
Moments of the Discounted Dividends with Phase Type gains

After applying $q_B (-D)$ to the integro–differential equation (1) we get a linear homogeneous differential equation of degree $m + n$ of the following form

\[
0 = \sum_{l=0}^{n+m} \left[ \sum_{i+j=l} q_i \binom{n}{n-j} \left( 1 + \frac{k\delta}{\lambda} \right)^{n-j} \left( \frac{c}{\lambda} \right)^j \right] \frac{d^l}{du^l} V_k(u, b) + \\
\sum_{j=0}^{m-1} \left[ \sum_{i=j+1}^{m} q_i \alpha' (-B)^{i-j} 1 \right] \frac{d^j}{du^j} V_k(u, b) \ldots
\]  

(4)
Moments of the Discounted Dividends with Phase Type gains

Solutions of (4) are of the form

\[ V_k(u, b) = \sum_{l=1}^{n+m} a_l e^{-r_l u}. \]  

(5)

We replace this expression in our original integro–differential equation

\[ \left( \left( 1 + \frac{k\delta}{\lambda} \right) I + \left( \frac{c}{\lambda} \right) D \right)^n V_k(u, b) = W_{k\delta}(u, b), \quad u < b, \]
Moments of the Discounted Dividends with Phase Type gains

The result is the following

$$0 = \sum_{l=1}^{n+m} a_l \left[ \left( 1 + \frac{k \delta}{\lambda} - \left( \frac{c}{\lambda} \right) r_l \right)^n - \hat{p}(r_l) \right] e^{-r_l u}$$

$$-\alpha' \left[ \sum_{l=1}^{n+m} a_l e^{-r_l b} \left( (r_l 1_m - B)^{-1} B + 1_m \right) \right]$$

$$+ \sum_{j=1}^{k} \binom{k}{j} V_{k-j}(b, b)(-B)^{-j} e^{B(b-u)} 1.$$
Since the last equation is valid \( \forall u \) then we must have

\[
\left( 1 + \frac{k\delta}{\lambda} - \left( \frac{c}{\lambda} \right) r_l \right)^n - \hat{p}(r_l) = 0, \quad l = 1, \ldots, n + m,
\]

so the exponents \( r_l, \ l = 1, \ldots, n + m \), are all the \( m + n \) roots of the generalized Lundberg’s equation, from those \( n \) roots have positive real parts, namely \( \rho_1, \rho_2, \ldots, \rho_n \), and \( m \) roots have negative real parts, \( \rho_{n+1}, \rho_{n+2}, \ldots, \rho_{n+m} \). Also, we must have

\[
\alpha' \left[ \sum_{l=1}^{n+m} a_l e^{-r_l} \left( (r_l I_m - B)^{-1}B + I_m \right) + \sum_{j=1}^{k} j \binom{k}{j} V_{k-j}(b, b) (-B)^{-j} \right] = 0. \tag{6}
\]
We obtain the coefficients $a_l$, $l = 1, \ldots, m + n$ from the boundary conditions

$$\frac{d^i}{du^i} V_k(u, b) \bigg|_{u=0} = 0, \quad i = 0, \ldots, n - 1,$$

together with (6), giving a system of $m + n$ equations on $m + n$ unknowns that we can solve.
Moments of the Discounted Dividends with Phase Type gains

Example

We want to compute \( V(u, b) \). Assume that the times between jumps are Erlang(2, \( \lambda \)) distributed and the jump amounts are Erlang(2, \( \beta \)) distributed. Then the negative loading condition is \( c < \frac{\lambda}{\beta} \) and the generalized Lundberg’s equation becomes

\[
(\lambda + \beta - cs)^2(\beta + s)^2 = \lambda^2 \beta^2
\]  

(7)

Let

\[
V(u, b) = \sum_{l=1}^{4} a_l e^{-\rho_l u}
\]

Then the exponents \( \rho_l \)'s are the four roots of (7). Say that they are \( \rho_1, \rho_2 \) with positive real parts and \( \rho_3, \rho_4 \) with negative real parts.
Moments of the Discounted Dividends with Phase Type gains

Example

The coefficients $a_l$’s are obtained using the corresponding boundary conditions $V(0, b) = V'(0, b) = 0$

$$\sum_{l=1}^{4} a_l = 0, \quad \text{and} \quad \sum_{l=1}^{4} a_l \rho_l = 0,$$

and from (6)

$$\sum_{l=1}^{4} a_l e^{-\rho_l b} \frac{\rho_l}{\rho_l + \beta} = -\frac{1}{\beta}, \quad \sum_{l=1}^{4} a_l e^{-\rho_l b} \frac{\rho_l \beta}{(\rho_l + \beta)^2} = -\frac{1}{\beta},$$
Moments of the Discounted Dividends with Phase Type gains

Example

Set the values for the parameters $\lambda = \beta = 1$, $c = 0.75$, $\delta = 0.02$. Then $\rho_1 = 0.423$, $\rho_2 = 1.831$, $\rho_3 = -0.063$ and $\rho_4 = -1.471$.

<table>
<thead>
<tr>
<th>$u$ \ $b$</th>
<th>3</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>10</th>
<th>15</th>
<th>20</th>
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</table>

Table: Values of $V(u, b)$
Moments of the Discounted Dividends with Phase Type gains

Figure: Values of $V(u, b)$ for $u = 2, 3, 5, 10$ and $b \in (0, 20)$

We noticed that for a fixed $u$ the value of $V(u, b)$ increases until a certain value of $b$ and then decreases. This behavior is similar to Afonso et al (2011) and Avanzi et al (2007).
For a given initial capital $u$, let $b^*$ denote the optimal value of the barrier $b$ that maximizes the expected discounted dividends $V(u, b)$.

Avanzi et al (2007) shows that for a dual model with exponentially distributed gain interarrival times the value of $b^*$ is independent of $u$. 
We have observed that the same situation occur for a dual model with Erlang\((n)\) distributed gain interarrival times and Phase–Type\((m)\) distributed gain amounts. Let \(b_k^*\) be the value that maximizes \(V_k(u, b)\), for \(k \geq 1\).

**Theorem**

\(b_k^*\) is independent of the initial surplus \(u\).

**Example**

In the example given in the previous section, the value of \(b\) that maximizes \(V(u, b)\) is \(b^* = 7.33\).
Consider the case $k = 1$. Since $V(u, b)$ is maximal at $b = b^*$ for each $u$, we have

$$\frac{\partial V(u, b)}{\partial b} \bigg|_{b=b^*} = 0.$$

For the optimal value $b^*$ the smooth pasting condition must be satisfied. For a dual model with Erlang($n$) distributed interclaim times this condition states that

$$\frac{\partial V(u, b^*)}{\partial u} \bigg|_{u=b^*-} = \frac{\partial V(u, b^*)}{\partial u} \bigg|_{u=b^*+} = 1$$

$$\frac{\partial^k V(u, b^*)}{\partial u^k} \bigg|_{u=b^*-} = \frac{\partial^k V(u, b^*)}{\partial u^k} \bigg|_{u=b^*+} = 0, \quad 2 \leq k \leq n,$$
this means that the left and right partial derivatives have to coincide in the optimal value. Using this we get the following result

\[ V(b^*, b^*) = \frac{\lambda^n \mu_1 - n(\lambda + \delta)^{n-1}c}{(\lambda + \delta)^n - \lambda^n}. \]

For \( n = 1 \) this agrees with the results of Avanzi et al (2007) and others, where \( V(b^*, b^*) = \frac{\lambda \mu_1 - c}{\delta} \).
From the equation

\[ V(b^*, b^*) = \sum_{l=1}^{n+m} a_l e^{-\rho_l b^*} = \frac{\lambda^n \mu_1 - n(\lambda + \delta)^{n-1} c}{(\lambda + \delta)^n - \lambda^n}, \]

it is possible to obtain the value of \( b^* \) numerically.

Clearly, this value is independent from \( u \).
Using the annihilator approach we found expressions for the $k$–th moments of the aggregate discounted dividends $V_k(u, b), k \geq 1$.

There is an optimal barrier level $b^*$ that maximizes the expected discounted dividends $V(u, b)$ prior to ruin. The value of $b^*$ is independent from the initial surplus $u$ and can be found numerically.

The same can be told about the barrier level $b_k^*$ that maximizes $V_k(u, b)$.


Thank you for your attention!