Moments of Dividends and Optimal Expected Dividends in the Erlang(n) dual risk model

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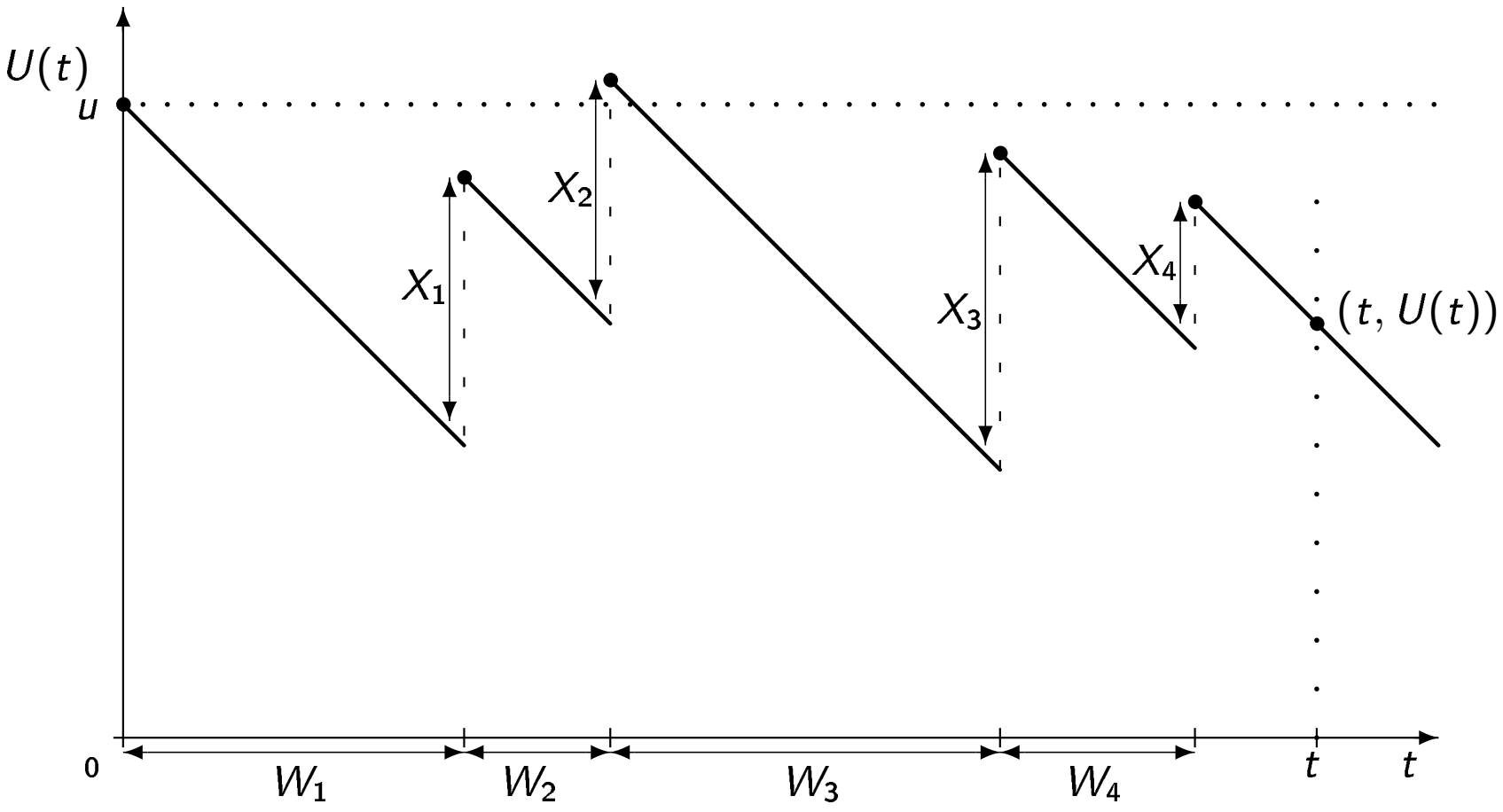
Model and Definitions

Renewal dual risk model

$$U(t) = u - ct + \sum_{i=0}^{N(t)} X_i, \quad X_0 \equiv 0, \quad t \geq 0, \quad u \geq 0,$$

- u, initial capital,
- t, time parameter,
- $c > 0$, rate of expenses,
- $\{X_i\}_{i=1}^{\infty}$, single gain amounts sequence,
- $\{W_i\}_{i=1}^{\infty}$, gain inter–arrival times sequence,
- $N(t) = \max\{k : \ W_1 + W_2 + \cdots + W_k \leq t\}$, number of gains up to time t,
- $\{N(t), t \geq 0\}$ is a renewal process.

Model and Definitions



$$\mathbf{U(t) = u - ct + \sum_{i=1}^{4} X_i, \quad N(t) = 4}$$

Model and Definitions

Assumptions of the model:

- X_i **i.i.d.**, $X_i \sim P(x)$ with density $p(x)$,
- $\hat{p}(s) = \int_0^\infty e^{-sx} p(x)dx,\ s \in \mathbb{C}$ Laplace transform of $p(x)$,
- $\mu_k = E[X_1^k]$ the k-th moment of X_1 , at least μ_1 must exist.
- W_i **i.i.d.** and independent from the $X_i{}'s$,
- $W_i \sim K_n(t)$, $W_i \sim$ Erlang(n, λ) with density

$$k_n(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad t \geq 0, \lambda > 0, n \in \mathbb{N}^+, \text{ and } \quad K_n(t) = 1 - \sum_{i=0}^{n-1} \frac{(\lambda t)^i e^{-\lambda t}}{(i)!}$$

- $cE(W_1) < E(X_1)$ **negative loading factor condition**,

  $\Leftrightarrow cn < \lambda \mu_1$.

Model and Definitions

Time of ruin $T_u = \inf\{t > 0 : \ U(t) = 0\}, \ u \geq 0,$

$$T_u = \infty \ \text{ iff } \ U(t) > 0 \ \ \forall t > 0.$$

Ultimate ruin probability $\psi(u) = P(T_u < \infty)$.

Survival probability $\phi(u) = 1 - \psi(u)$.

For a constant $\delta \geq 0$, the Laplace transform of the time of ruin is

$$\psi(u, \delta) = E(e^{-\delta T_u} \mathbb{I}(T_u < \infty)).$$

Remark: δ can be interpreted as an interest force.

Model and Definitions

Among pioneer works on the subject we can cite Cramér (1955), Gerber (1979), Seal (1969), Tákacs (1967). Recent works include those by Avanzi *et al.* (2007), Cheung and Drekic (2008), Ng (2009), Afonso *et al.* (2011).

From those, the works particularly focusing the dual model and the problem of discounted dividends assume that interarrival times follow an exponential distribution.

We work on a dual risk model with Erlang(n) interclaim times and study the moments of the expected discounted dividends. Then we use our results for the calculation of the optimal dividend barrier. For this we will assume that the gain amounts X_i follow a Phase Type PH(m) distribution.

Model and Definitions

The following equation, called the generalized Lundberg's equation, plays an important role in the computation of the expected dividends

$$E[e^{-\delta W_i} e^{-s(X_i - cW_i)}] = 1$$

.

For the Erlang(n) dual risk model, this equation becomes

$$\left(1 + \frac{\delta}{\lambda} - \left(\frac{c}{\lambda}\right) s\right)^n = \hat{p}(s), \quad s \in \mathbb{C}.$$

It can be proven that this equation has n roots with positive real parts, which we denote by $\rho_1, \ldots, \rho_n$, and that they are all distinct.

The following closed formulas for the ultimate ruin probability $\psi(u)$

$$\psi(u) = \sum_{k=1}^{n} \left[\prod_{i=1, i \neq k}^{n} \frac{\rho_i}{(\rho_i - \rho_k)} \right] e^{-\rho_k u},$$

and the Laplace transform of the time of ruin $\psi(u, \delta)$

$$\psi(u, \delta) = \sum_{k=1}^{n} \left[\prod_{i=1, i \neq k}^{n} \frac{\left(\rho_i - \frac{\delta}{c}\right)}{(\rho_i - \rho_k)} \right] e^{-\rho_k u},$$

can be found in Rodríguez *et al* (2012).

Moments of the Discounted Dividends

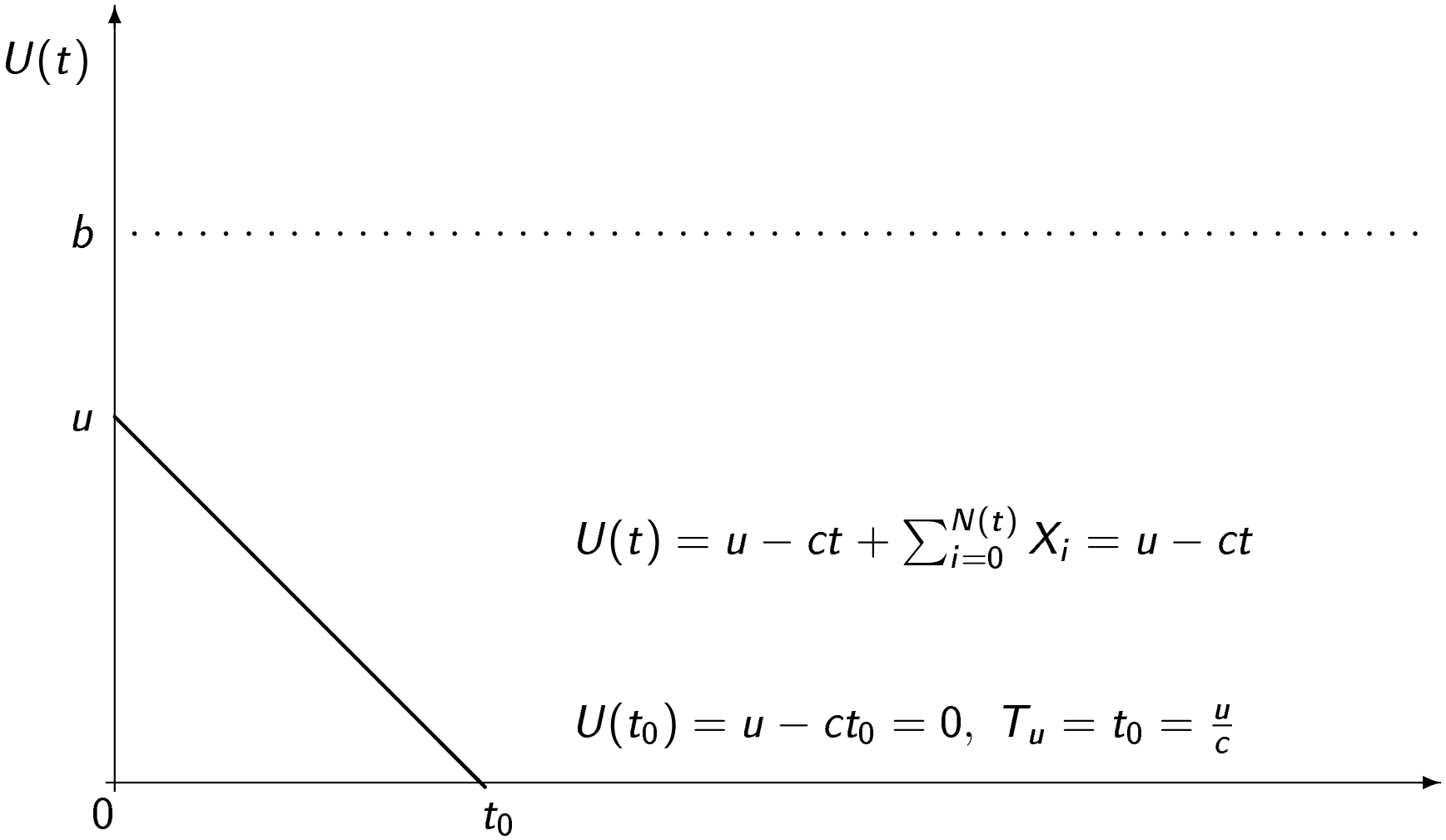
We introduce an upper barrier into the model and let b denote its level.

Each time the surplus process upcrosses level b the excess gain is paid out immediately to the capital holder as a dividend, prior to ruin.

Let $\{D_i\}_{i=1}^{\infty}$ be the sequence of the dividend payments and let $D(u,b)$ be the aggregate discounted dividends, at force of interest δ and from initial surplus u.

We denote by $V_k(u,b) = E[D(u,b)^k]$, $k \geq 1$, the k-th order moment of $D(u,b)$, for simplicity denote $V(u,b) = V_1(u,b)$.

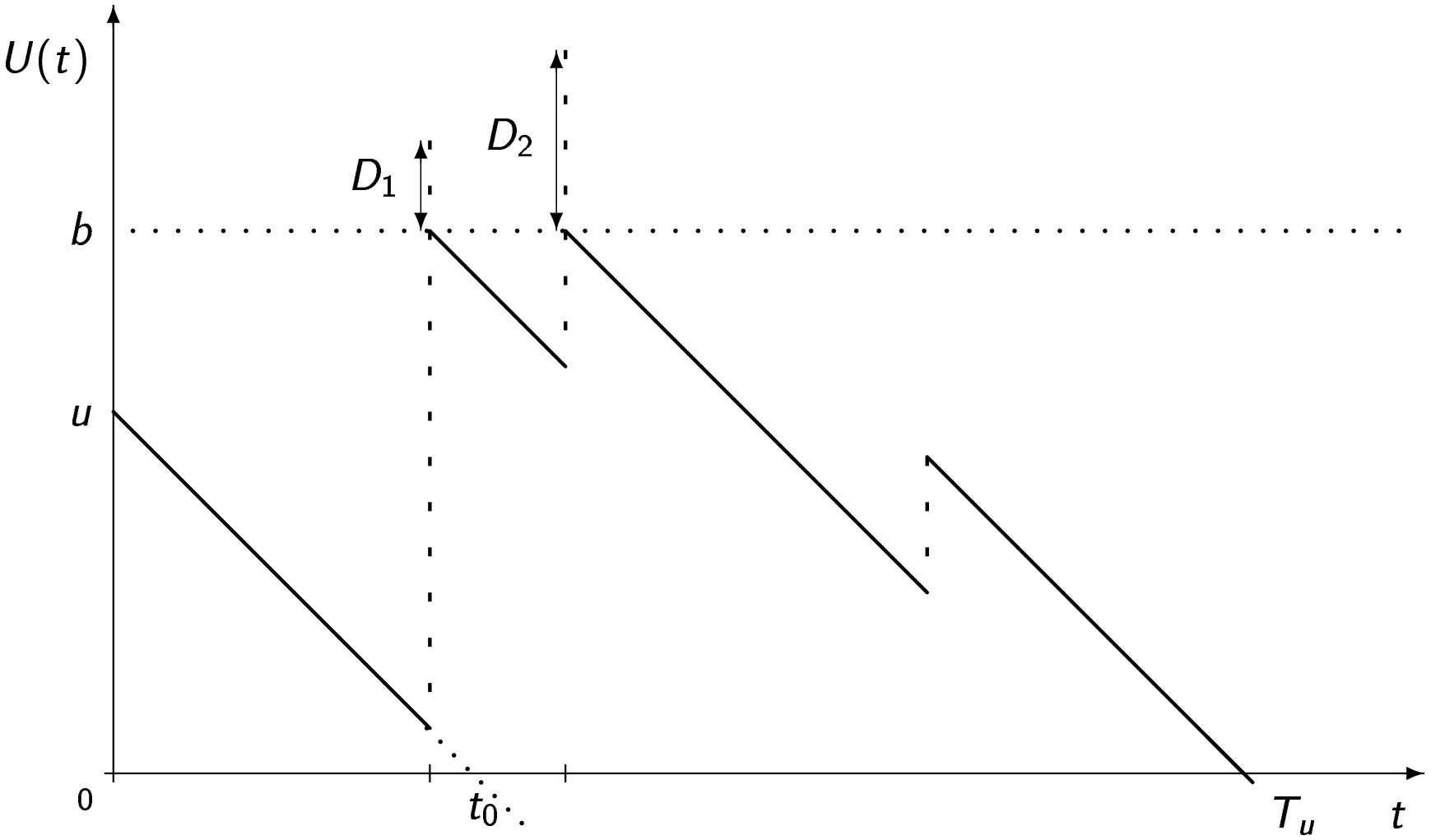
Moments of the Discounted Dividends



$$U(t) = u - ct + \sum_{i=0}^{N(t)} X_i = u - ct$$

$$U(t_0) = u - ct_0 = 0, \;\; T_u = t_0 = \frac{u}{c}$$

Moments of the Discounted Dividends



Moments of the Discounted Dividends

Note that

$$V(u,b) = E[u - b + D(b,b)] = u - b + V(b,b), \quad u \geq b.$$

and in general

$$V_k(u,b) = E[(u - b + D(b,b))^k] = \sum_{j=0}^{k} \binom{k}{j} (u-b)^j V_{k-j}(b,b), \quad u \geq b.$$

Moments of the Discounted Dividends

The expected discounted dividends $V(u,b)$ satisfy the renewal equation

$$\begin{aligned} V(u,b) &= \int_0^{\frac{u}{c}} k_n(t)e^{-\delta t}\left[\int_0^{b-u+ct} V(u-ct+y,b)p(y)dy+\right. \\ &\quad \left.\int_{b-u+ct}^{\infty} \widetilde{V}(u-ct+y,b)p(y)dy\right]dt, \quad u<b, \end{aligned}$$

with

$$\widetilde{V}(x,b) = E[D(x,b)] = E[x-b+D(b,b)] = x-b+V(b,b), \quad x \geq b.$$

Moments of the Discounted Dividends

$V_k(u,b)$ satisfies the renewal equation

$$V_k(u,b) = \int_0^{\frac{u}{c}} k_n(t)e^{-k\delta t}\left[\int_0^{b-u+ct} V_k(u-ct+y,b)p(y)dy + \int_{b-u+ct}^{\infty} \widetilde{V}_k(u-ct+y,b)p(y)dy\right]dt, \quad u<b,$$

with

$$\widetilde{V}_k(x,b) = E[(x-b+D(b,b))^k] = \sum_{j=0}^{k}\binom{k}{j}(x-b)^j V_{k-j}(b,b), \quad x \geq b.$$

In the above expression we assume the convention $V_0(u,b) \equiv 1$.

Moments of the Discounted Dividends

Differentiating the renewal equation with respect to u we can obtain an integro–differential equation for $V_k(u,b)$

$$\left(\left(1+\frac{k\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V_k(u,b)=W_{k\delta}(u,b), \quad u<b, \tag{1}$$

with boundary conditions

$$\left.\frac{d^i}{du^i}V_k(u,b)\right|_{u=0}=0, \quad i=0,\ldots,n-1, \tag{2}$$

where

$$W_{k\delta}(u,b)=\int_u^b V_k(x,b)p(x-u)dx+\int_b^\infty \widetilde{V_k}(x,b)p(x-u)dx,$$

is the integral term.

One way to solve the integro–differential equation is the annihilator method. Let $A(\mathcal{D})$ be a differential operator such that

$$\mathbf{A}(\mathcal{D})\left[\left(\left(1+\frac{k\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n} V_k(u,b)\right]=\mathbf{A}(\mathcal{D})\left[W_{k\delta}(u,b)\right]=0.$$

Then $A(\mathcal{D})$ is called an annihilator, since it annihilates the integral term $W_{k\delta}(u,b)$.

In consequence our original integro–differential equation

$$\left(\left(1+\frac{k\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n} V_k(u,b) = W_{k\delta}(u,b), \quad u < b,$$

becomes into an homogeneous differential equation of a higher order of the form

$$\widetilde{A}(\mathcal{D})V_k(u,b) = 0. \tag{3}$$

Solutions of the first equation are solutions of the second equation. The converse is true only with some additional constrains.

Now recall that the integral term

$$W_{k\delta}(u,b) = \int_u^b V_k(x,b)p(x-u)dx + \int_b^\infty \widetilde{V_k}(x,b)p(x-u)dx,$$

depends on the density $p(x)$ of the gain amounts.

Therefore in order to follow this approach and find the annihilator we must at least specify which is the density $p(x)$ of the gains.

Moments of the Discounted Dividends with Phase Type gains

Consider the case when the gains X_i follow a Phase-Type PH(m) distribution.

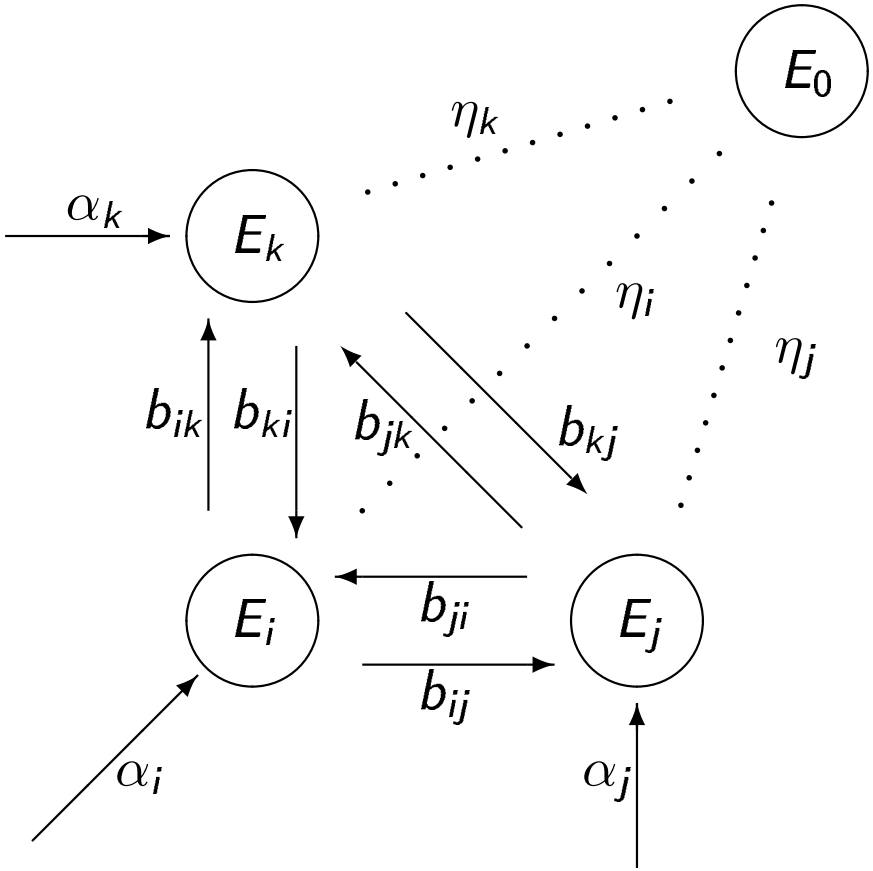
Let $\mathbf{B} = (b_{ij})_{1 \leq i,j \leq m}$ be the matrix of transition rates between m transient states $\{E_1, E_2, \ldots, E_m\}$, let $\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be the vector of the initial probabilities, $\eta' = (\eta_1, \eta_2, \ldots, \eta_m)$ the vector of the exit rates to the absorbing state E_0, and $\mathbf{1}' = (1, 1, \ldots, 1)$ the $1 \times m$ vector. Let $\mathbf{I_m}$ denote the identity matrix $m \times m$.

Then the density of the gain amounts is given by

$$p(x) = \alpha' \mathbf{e}^{\mathbf{B}x} \eta,$$

and the distribution is

$$P(x) = 1 - \alpha' \mathbf{e}^{\mathbf{B}x} \mathbf{1}.$$


E_0
η_k
α_k
E_k
η_i
η_j
b_{ik}
b_{ki}
b_{jk}
b_{kj}
E_i
b_{ji}
E_j
b_{ij}
α_i
α_j

Moments of the Discounted Dividends with Phase Type gains

Let $q_{\mathbf{B}}(y) = Det(\mathbf{B} - y\mathbf{I_m})$ be the characteristic polynomial of the intensity matrix $\mathbf{B}$ and $\mathcal{D} = \frac{d}{du}$ the differentiation with respect to u.

Define the differential operator $q_{\mathbf{B}}(-\mathcal{D})$

$$q_{\mathbf{B}}(-\mathcal{D}) = \sum_{i=0}^{m} q_i \frac{d^{\,i}}{du^{\,i}},$$

for some coefficients q_i that depend on the entries of $\mathbf{B}$.

Theorem

$q_{\mathbf{B}}(-\mathcal{D})$ is an annihilator for the integro–differential equation

$$\left(\left(1+\frac{k\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n} V_k(u,b)=W_{k\delta}(u,b), \quad u<b,$$

Moments of the Discounted Dividends with Phase Type gains

After applying $q_{\mathbf{B}}(-\mathcal{D})$ to the integro–differential equation (1) we get a linear homogeneous differential equation of degree $m+n$ of the following form

$$
\begin{aligned}
0 \quad = \quad & \sum_{l=0}^{n+m} \left[\sum_{i+j=l} q_i \binom{n}{n-j} \left(1+\frac{k\delta}{\lambda}\right)^{n-j} \left(\frac{c}{\lambda}\right)^{j} \right] \frac{d^l}{du^l} V_k(u,b) + \\
& \sum_{j=0}^{m-1} \left[\sum_{i=j+1}^{m} q_i \alpha'(-\mathbf{B})^{i-j}\mathbf{1} \right] \frac{d^j}{du^j} V_k(u,b).. \qquad (4)
\end{aligned}
$$

Moments of the Discounted Dividends with Phase Type gains

Solutions of (4) are of the form

$$V_k(u,b) = \sum_{l=1}^{n+m} a_l e^{-r_l u}. \tag{5}$$

We replace this expression in our original integro–differential equation

$$\left(\left(1+\frac{k\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V_k(u,b) = W_{k\delta}(u,b), \quad u < b,$$

Moments of the Discounted Dividends with Phase Type gains

The result is the following

$$
\begin{aligned}
0 \quad = \quad & \sum_{l=1}^{n+m} a_l \left[\left(1 + \frac{k\delta}{\lambda} - \left(\frac{c}{\lambda} \right) r_l \right)^n - \hat{p}(r_l) \right] e^{-r_l u} \\
& -\alpha' \left[\sum_{l=1}^{n+m} a_l e^{-r_l b} \left((r_l \mathbf{I_m} - \mathbf{B})^{-1} \mathbf{B} + \mathbf{I_m} \right) \right. \\
& \left. + \sum_{j=1}^{k} j \binom{k}{j} V_{k-j}(b,b) (-\mathbf{B})^{-j} \right] e^{\mathbf{B}(b-u)} \mathbf{1}.
\end{aligned}
$$

Moments of the Discounted Dividends with Phase Type gains

Since the last equation is valid $\forall u$ then we must have

$$\left(1+\frac{k\delta}{\lambda}-\left(\frac{c}{\lambda}\right)r_l\right)^n-\hat{p}(r_l)=0, \quad l=1,\ldots,n+m,$$

so the exponents r_l, $l=1,...,n+m$, are all the $m+n$ roots of the generalized Lundberg's equation, from those n roots have positive real parts, namely $\rho_1,\rho_2,\ldots,\rho_n$, and m roots have negative real parts, $\rho_{n+1},\rho_{n+2},\ldots,\rho_{n+m}$. Also, we must have

$$\alpha'\left[\sum_{l=1}^{n+m}a_l e^{-r_l b}\left((r_l\mathbf{I_m}-\mathbf{B})^{-1}\mathbf{B}+\mathbf{I_m}\right)+\sum_{j=1}^{k}j\binom{k}{j}V_{k-j}(b,b)(-\mathbf{B})^{-j}\right]=\mathbf{0}. \quad (6)$$

We obtain the coefficients $a_l, l = 1, \ldots, m+n$ from the boundary conditions

$$\frac{d^i}{du^i} V_k(u,b)\Big|_{u=0} = 0, \quad i = 0, \ldots, n-1,$$

together with (6), giving a system of $m+n$ equations on $m+n$ unknowns that we can solve.

Moments of the Discounted Dividends with Phase Type gains

Example

We want to compute $V(u,b)$. Assume that the times between jumps are Erlang$(2,\lambda)$ distributed and the jump amounts are Erlang$(2,\beta)$ distributed. Then the negative loading condition is $c < \frac{\lambda}{\beta}$ and the generalized Lundberg's equation becomes

$$(\lambda+\beta-cs)^2(\beta+s)^2 = \lambda^2\beta^2 \tag{7}$$

Let

$$V(u,b) = \sum_{l=1}^{4} a_l e^{-r_l u}$$

Then the exponents r_l's are the four roots of (7). Say that they are ρ_1, ρ_2 with positive real parts and ρ_3, ρ_4 with negative real parts.

Moments of the Discounted Dividends with Phase Type gains

Example

The coefficients a_l's are obtained using the corresponding boundary conditions $V(0,b) = V'(0,b) = 0$

$$\sum_{l=1}^{4} a_l = 0, \text{ and } \sum_{l=1}^{4} a_l \rho_l = 0,$$

and from (6)

$$\sum_{l=1}^{4} a_l e^{-\rho_l b} \frac{\rho_l}{\rho_l + \beta} = -\frac{1}{\beta}, \quad \sum_{l=1}^{4} a_l e^{-\rho_l b} \frac{\rho_l \beta}{(\rho_l + \beta)^2} = -\frac{1}{\beta},$$

Moments of the Discounted Dividends with Phase Type gains

Example

Set the values for the parameters $\lambda = \beta = 1$, $c = 0.75$, $\delta = 0.02$. Then $\rho_1 = 0.423$, $\rho_2 = 1.831$, $\rho_3 = -0.063$ and $\rho_4 = -1.471$.

$u \backslash b$	3	5	6	7	8	10	15	20
2	3.079	4.107	4.390	4.507	4.489	4.212	3.187	2.333
3	4.533	6.033	6.450	6.621	6.595	6.188	4.682	3.428
5	6.533	8.773	9.374	9.622	9.584	8.993	6.805	4.981
10	11.533	13.773	14.501	14.825	14.770	13.829	10.468	7.663
15	16.533	18.773	19.501	19.825	19.770	18.829	14.478	10.603
20	21.533	23.773	24.501	24.825	24.770	23.829	19.478	14.537

Table : Values of $V(u, b)$

Moments of the Discounted Dividends with Phase Type gains

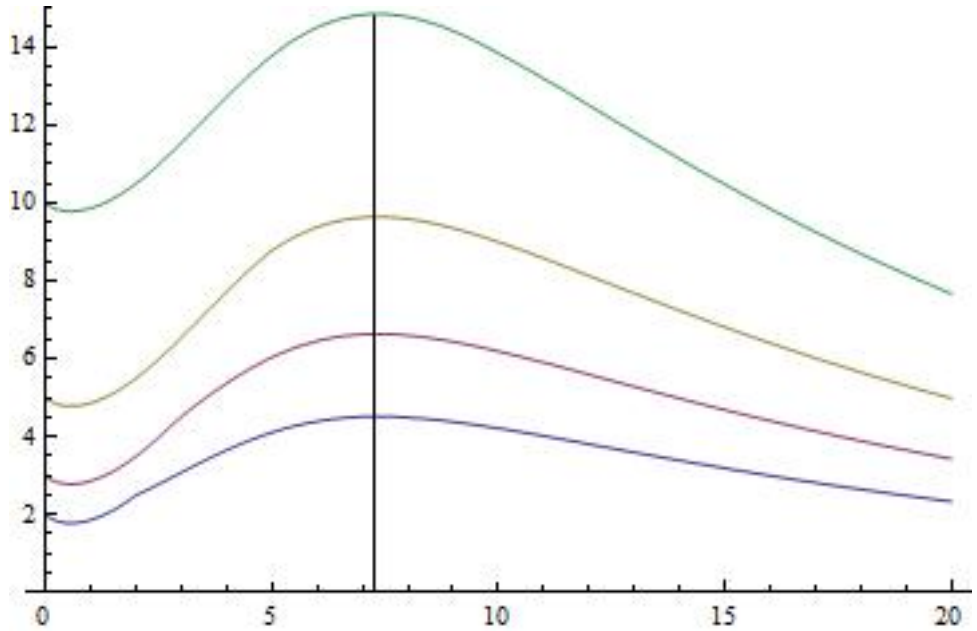


Figure : Values of $V(u,b)$ for $u = 2,3,5,10$ and $b \in (0,20)$

We noticed that for a fixed u the value of $V(u,b)$ increases until a certain value of b and then decreases. This behavior is similar to Afonso *et al* (2011) and Avanzi *et al* (2007).

The Optimal Dividend Barrier

For a given initial capital u, let b^* denote the optimal value of the barrier b that maximizes the expected discounted dividends $V(u, b)$.

Avanzi *et al* (2007)shows that for a dual model with exponentially distributed gain interarrival times the value of b^* is independent of u.

The Optimal Dividend Barrier

We have observed that the same situation occur for a dual model with Erlang(n) distributed gain interarrival times and Phase–Type(m) distributed gain amounts. Let b_k^* be the value that maximizes $V_k(u, b)$, for $k \geq 1$.

Theorem

b_k^* is independent of the initial surplus u.

Example

In the example given in the previous section, the value of b that maximizes $V(u, b)$ is $b^* = 7,33$.

The Optimal Dividend Barrier

Consider the case $k = 1$. Since $V(u, b)$ is maximal at $b = b^*$ for each u, we have

$$\left.\frac{\partial V(u,b)}{\partial b}\right|_{b=b^*} = 0.$$

For the optimal value b^* the smooth pasting condition must be satisfied. For a dual model with Erlang(n) distributed interclaim times this condition states that

$$\begin{aligned}
\left.\frac{\partial V(u,b^*)}{\partial u}\right|_{u=b^*-} &= \left.\frac{\partial V(u,b^*)}{\partial u}\right|_{u=b^*+} = 1 \\
\left.\frac{\partial^k V(u,b^*)}{\partial u^k}\right|_{u=b^*-} &= \left.\frac{\partial^k V(u,b^*)}{\partial u^k}\right|_{u=b^*+} = 0, \quad 2 \le k \le n,
\end{aligned}$$

The Optimal Dividend Barrier

this means that the left and right partial derivatives have to coincide in the optimal value. Using this we get the following result

$$V(b^*, b^*) = \frac{\lambda^n \mu_1 - n(\lambda+\delta)^{n-1} c}{(\lambda+\delta)^n - \lambda^n}.$$

For $n = 1$ this agrees with the results of Avanzi *et al* (2007) and others, where $V(b^*, b^*) = \frac{\lambda \mu_1 - c}{\delta}$.

From the equation

$$V(b^*, b^*) = \sum_{l=1}^{n+m} a_l e^{-\rho_l b^*} = \frac{\lambda^n \mu_1 - n(\lambda+\delta)^{n-1} c}{(\lambda+\delta)^n - \lambda^n},$$

it is possible to obtain the value of b^* numerically.

Clearly, this value is independent from u.

Conclusions

Using the annihilator approach we found expressions for the k–th moments of the aggregate discounted dividends $V_k(u, b)$, $k \geq 1$.

There is an optimal barrier level b^* that maximizes the expected discounted dividends $V(u, b)$ prior to ruin. The value of b^* is independent from the initial surplus u and can be found numerically.

The same can be told about the barrier level b_k^* that maximizes $V_k(u, b)$.

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Thank you for your attention!