On a Sparre-Andersen risk model with PH(n) interclaim times

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Model

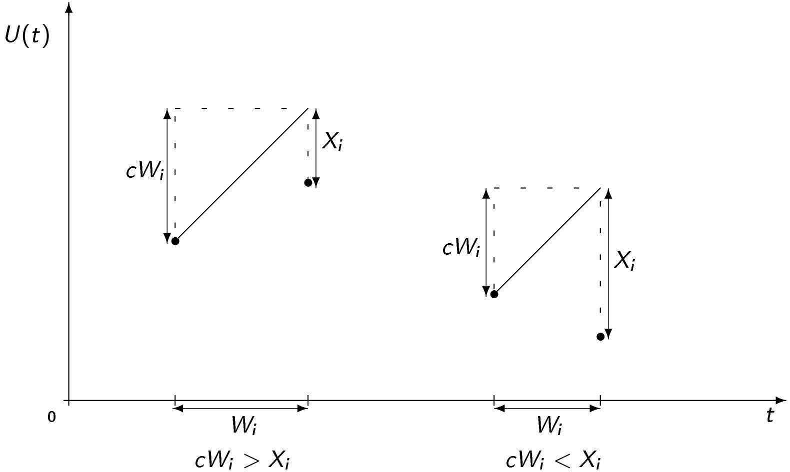
Sparre–Andersen model

$$U(t) = u + ct - \sum_{i=0}^{N(t)} X_i, \quad X_0 \equiv 0, \quad t \geq 0, \quad u \geq 0,$$

- u initial capital,
- t time variable,
- $c > 0$, loading factor,
- $\{X_i\}_{i=1}^{\infty}$, single claim amount sequence,
- $\{W_i\}_{i=1}^{\infty}$, interclaim time sequence,
- $N(t) = \max\{k : \; W_1 + W_2 + \cdots + W_k \leq t\}$, number of claims.

Assumptions on the model:

- $X_i's$ **i.i.d.**, $X_i \sim P(x)$ with density $p(x)$,
- $\hat{p}(s) = \int_0^\infty e^{-sx} p(x)dx$, Laplace transform of $p(x)$,
- $\mu_k = E[X_1^k]$ the k-th moment of X_1 ,
- $W_i's$ **i.i.d.** and independent from the $X_i's$,
- $W_i \sim K(t)$, $W_i \sim$ PH(n), with representation $(\alpha, \mathbf{B})$, density $k(t)$ and Laplace transform $\hat{k}(s)$,
- $cE(W_1) > E(X_1)$ **positive loading factor**.



$$\mathbf{E(cW_i) > E(X_i)}$$

Definitions

Time of ruin $T = \inf\{t > 0 : \; U(t) < 0\}, \; u \geq 0,$

$$T = \infty \;\text{ iff }\; U(t) \geq 0 \;\; \forall t > 0,$$

Ultimate ruin probability $\Psi(u) = P(T < \infty),$

Non-ruin probability $\Phi(u) = 1 - \Psi(u),$

Time to upcross barrier b $\tau_b = \inf\{t > 0 : U(0) = u, U(t) \geq b\},$

$R(u, b) = E[e^{-\delta\tau_b} | U(0) = u]$ the Laplace transform of τ_b, for $\delta \geq 0$.

Time of the first upcrossing of the level 0 after ruin occurs

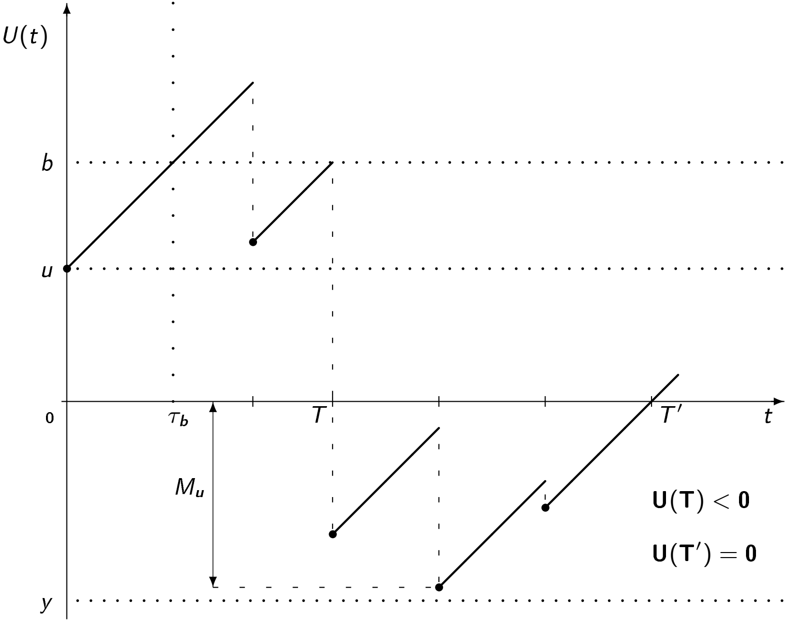
$$T' = \inf\{t: \ t > T, \ U(t) \geq 0\}, \ T < \infty,$$

Maximum severity of ruin

$$M_u = \sup\{|U(t)|: \ U(0) = u, \ T \leq t \leq T'\}, \ u \geq 0,$$

Distribution of the maximum severity

$$J(z; u) = P(M_u \leq z \mid T < \infty), \ u, z \geq 0,$$



$\mathbf{U(T) < 0}$

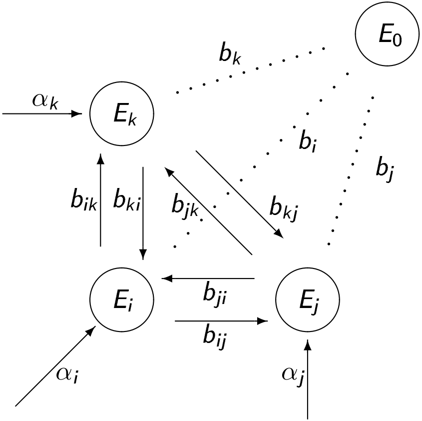
$\mathbf{U(T') = 0}$

As mentioned before, the interclaim times W_i follow a Phase–Type(n) distribution with representation $(\alpha, \mathbf{B})$. This means that W_i corresponds to the time of absorption in a terminating continuous time Markov chain $\{J(t)\}_{t\geq 0}$ with n transient states $\{E_1, E_2, \ldots, E_n\}$ and one absorbing state $\{E_0\}$.

The $n \times n$ intensity matrix $\mathbf{B} = (b_{i,j})_{i,j=1}^{n}$ denotes the transition rates between the n transient states, with $b_{i,i} < 0$, $b_{i,j} \geq 0$ for $i \neq j$, and $\sum_{j=1}^{n} b_{i,j} \leq 0$ for $i = 1, \ldots, n$.

The vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ denotes the initial distribution with $\alpha_i \geq 0$ for $i = 1, \ldots, n$, and $\sum_{i=1}^{n} \alpha_i = 1$.

The vector $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ denotes the exit rates to the absorbing state $\{E_0\}$.


E_0
b_k
α_k
E_k
b_i
b_j
b_{ik}
b_{ki}
b_{jk}
b_{kj}
E_i
b_{ji}
E_j
b_{ij}
α_i
α_j

Then the density, the distribution, the Laplace transform and the expected value of W_i are given by

$$
\begin{aligned}
k(t) &= \alpha e^{\mathbf{B}t}\mathbf{b}^{\mathsf{T}}, \quad K(t) = 1 - \alpha e^{\mathbf{B}t}\mathbf{e}^{\mathsf{T}}, \quad t \geq 0, \\
\hat{k}(s) &= \alpha(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^{\mathsf{T}}, \quad E[W_1] = -\alpha\mathbf{B}\mathbf{e}^{\mathsf{T}},
\end{aligned}
$$

where $\mathbf{b}^{\mathsf{T}} = -\mathbf{B}\mathbf{e}^{\mathsf{T}}$, $\mathbf{e} = (1, 1, \ldots, 1)$ is a $1 \times n$ vector and $\mathbf{I}$ is the $n \times n$ identity matrix.

Motivation

The Sparre–Andersen model with PH(n) interclaim times has been subject of many studies.

In the works presented by Albrecher and Boxma (2005), Gerber and Shiu (2005), Schmidli (2005), Ren (2007) and Li (2008) they perform the calculation of the expected discounted penalty at ruin, the ultimate and finite time ruin probabilities, the probability of arrival to a barrier prior to ruin, severity of ruin and its maximum, the expected discounted future dividends, among others important quantities.

For this purpose they studied an special equation, called the fundamental Lundberg's equation

$$\hat{k}(\delta - cs)\hat{p}(s) = 1, \; s \in \mathbb{C}, \; \delta \; \text{non negative constant.} \tag{1}$$

Gerber and Shiu (2005) shows that this equation has exactly n roots with positive real parts, which we denote by $\rho_1, \rho_2, \ldots, \rho_n$.

It turns out that in all of the existing papers on this subject it is assumed that these roots are distinct, and that the calculation of the quantities mentioned before is based on this assumption.

However, we can find a great variety of examples where multiple roots can arise, specially double roots.

Our main objective in this presentation is to find cases where the fundamental Lundberg's equation has double roots, and show alternative ways to compute the quantities of interest in those cases.

As an illustration we compute the Laplace transform of the time to upcross barrier b, τ_b, in the presence of a double root.

We recall that the Laplace transform of the $k(t)$ is

$$\hat{k}(s) = \alpha(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^{\mathsf{T}}$$

In order to look for multiple roots of the fundamental Lundberg's equation $\hat{k}(\delta - cs)\hat{p}(s) = 1$ we need to find a more tractable expression for $\hat{k}(s)$.

Definition

Let $\mathbf{A} = (a_{i,j})_{i,j=1}^{n}$ be a $n \times n$ matrix.
Define, for $1 \leq i_1 < i_2 < \ldots < i_k \leq n$

$$\mathbf{M}_{i_1,i_2\ldots i_k} = \begin{pmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \ldots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \ldots & a_{i_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k,i_1} & a_{i_k,i_2} & \ldots & a_{i_k,i_k} \end{pmatrix}, \ 1 \leq k \leq n,$$

then

$$tr_k(\mathbf{A}) = \sum_{1\leq i_1<i_2<\ldots<i_k\leq n} det(\mathbf{M}_{i_1,i_2\ldots i_k}).$$

Example

For $k = 1$

$$tr_1(\mathbf{A}) = \sum_{i=1}^{n} det(\mathbf{M}_i) = \sum_{i=1}^{n} a_{ii} = tr(\mathbf{A}).$$

For $k = 2$

$$tr_2(\mathbf{A}) = \sum_{1 \le i < j \le n} det(\mathbf{M}_{ij}) = \sum_{1 \le i < j \le n} (a_{ii}a_{jj} - a_{ij}a_{ji}).$$

For $k = n - 1$

$$tr_{n-1}(\mathbf{A}) = \sum_{i=1}^{n} det(\mathbf{M}_{1,\dots,i-1,i+1,\dots,n}) = det(\mathbf{A})tr(\mathbf{A}^{-1}).$$

For $k = n$

$$tr_n(\mathbf{A}) = det(\mathbf{M}_{1\dots n}) = det(\mathbf{A}).$$

By convention we set $tr_0(A) = 1$.

Then we obtain the following expression for $\hat{k}(s)$

Theorem

$$\hat{k}(s) = \alpha(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^T = \frac{N(s,n)}{det(s\mathbf{I} - \mathbf{B})},$$

where,

$$det(s\mathbf{I} - \mathbf{B}) = \sum_{i=0}^{n}(-1)^{n-i} tr_{n-i}(\mathbf{B}) s^i,$$

and $N(s,n)$ is a polynomial of degree at most $n-1$

$$N(s,n) \quad = \quad \alpha \left[\sum_{i=0}^{n-1} a_i s^i\right] \mathbf{1}^T,$$

where, for n even

$$
\begin{aligned}
a_i &= -\sum_{k=0}^{i} \mathbf{B}^{-k} tr_{n+k-j}(\mathbf{B}), \quad 0 \le i \le \frac{n-2}{2} \\
a_{n-j} &= \sum_{k=1}^{j} \mathbf{B}^{k} tr_{j-k}(\mathbf{B}), \quad 1 \le j \le \frac{n}{2}
\end{aligned}
$$

and for n odd

$$
\begin{aligned}
a_i &= -\sum_{k=0}^{i} \mathbf{B}^{-k} tr_{n+k-j}(\mathbf{B}), \quad 0 \le i \le \frac{n-3}{2} \\
a_{n-j} &= -\sum_{k=1}^{j} \mathbf{B}^{k} tr_{j-k}(\mathbf{B}), \quad 1 \le j \le \frac{n+1}{2}
\end{aligned}
$$

Example

For $n = 1$, $\alpha = (1)$, $\mathbf{B} = (-b)$, $\mathbf{1} = (1)$, $\mathbf{I} = (1)$, then

$$\hat{k}(s) = \frac{\alpha[-\mathbf{I}det(\mathbf{B})]\mathbf{1}^{\mathsf{T}}}{s - det(\mathbf{B})} = \frac{b}{s+b}.$$

For $n = 2$

$$\hat{k}(s) = \frac{\alpha[\mathbf{B}s - \mathbf{I}det(\mathbf{B})]\mathbf{1}^{\mathsf{T}}}{s^2 - tr(\mathbf{B})s + det(\mathbf{B})}.$$

For $n = 3$

$$\hat{k}(s) = \frac{\alpha[-\mathbf{B}s^2 - (\mathbf{B}^2 - \mathbf{B}\,tr(\mathbf{B}))s - \mathbf{I}det(\mathbf{B})]\mathbf{1}^{\mathsf{T}}}{s^3 - tr(\mathbf{B})s^2 + tr_2(\mathbf{B})s - det(\mathbf{B})}.$$

Now we recall the fundamental Lundberg's equation $\hat{k}(\delta - cs)\hat{p}(s) = 1$.

We restrict our attention to the right half of the complex plane, more specifically on the positive real axis, and we look for the possibility of having a double real root.

Lemma

For $s \in \mathbb{R}^+$, the Laplace transform $\hat{p}(s)$ is a positive and decreasing function of s, with $p(0) = 1$ and $\lim_{s \to \infty} \hat{p}(s) = 0$. Therefore $\hat{p}(s)$ has no zeros or poles in $s \in \mathbb{R}^+$.

The function $\hat{k}(\delta - cs)$ is the quotient of the polynomial $N(\delta - cs, n)$, which has degree at most $n - 1$, and the polynomial $det((\delta - cs)\mathbf{I} - \mathbf{B})$, which has degree n.

The poles of $\hat{k}(\delta - cs)$ are the numbers $s = \frac{\delta - \zeta}{c}$, where ζ ranges over all the eigenvalues of $\mathbf{B}$.

Theorem

Let s_1 and s_2, with $s_1 < s_2$, be two real poles of $\hat{k}(\delta - cs)$, and suppose that there is no other real pole or zero of $\hat{k}(\delta - cs)$ in the interval (s_1, s_2). If $\hat{k}(\delta - cs)$ is positive in the interval (s_1, s_2) then the fundamental Lundbeg's equation has one of the following:
- Two real roots in the interval (s_1, s_2).
- A double root in this interval.
- Two complex conjugate roots, where the real part of them is in the interval.

Moments of the Discounted Dividends with Phase Type gains

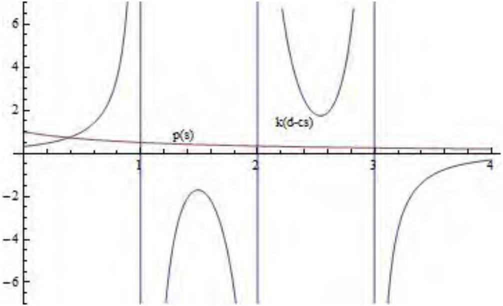


Figure: Looking for solutions of $\hat{k}(\delta - cs)\hat{p}(s) = 1$.

Moments of the Discounted Dividends with Phase Type gains

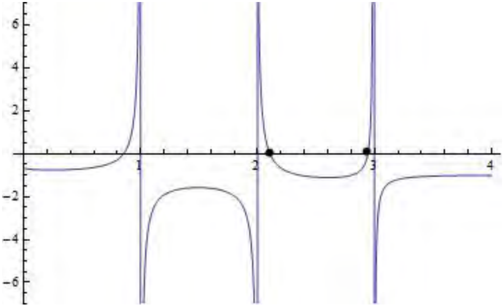


Figure: Two real roots in the interval.

Moments of the Discounted Dividends with Phase Type gains

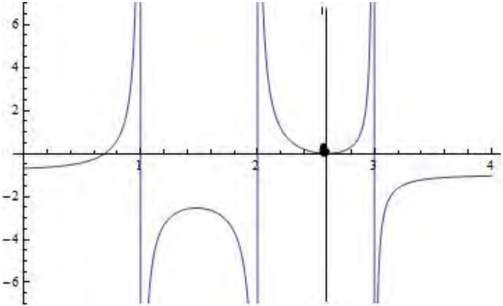


Figure: A double root in this interval.

Moments of the Discounted Dividends with Phase Type gains

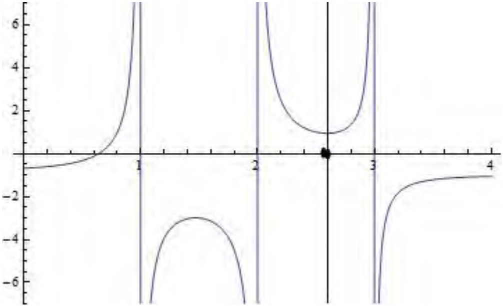


Figure: Two complex conjugate roots, where the real part of them is in the interval.

Example

Suppose that the interclaim times W_i follow a generalized Erlang(3) distribution, with intensity matrix

$$\mathbf{B} = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0 & -1.5 & 1.5 \\ 0 & 0 & -2.5 \end{pmatrix}$$

and $\alpha = (1, 0, 0)$, $\mathbf{b} = (0, 0, 2.5)$. Then $E[W_i] = 3.067$. Suppose that the claim amounts X_i are Exponentially distributed with parameter $\beta \geq 0.5$. Then we choose $c = 1$ to satisfy the positive loading condition and let $\delta = 0.5$.

Example

The fundamental Lundberg's equation becomes

$$\left(\frac{1.875}{(1-s)(2-s)(3-s)}\right)\left(\frac{\beta}{\beta+s}\right)=1$$

The function $\hat{k}(\delta - cs) = \hat{k}(0.5 - s) = \frac{1.875}{(1-s)(2-s)(3-s)}$ has no zeros and 3 poles at $s = 1, 2, 3$, furthermore it is positive in the interval $(2, 3)$. Then it is easy to verify that the fundamental Lundberg's equation has
- Two real roots in $(2, 3)$ for $0.5 \leq \beta < 0.67$.
- A double root 2.61 in $(2, 3)$ for $\beta = 0.67$.
- Two complex conjugate roots, where the real part of them is in $(2, 3)$ for $\beta > 0.67$.

The following matrix

$$\mathbf{L}_\delta(s) = \left(s - \frac{\delta}{c}\right)\mathbf{I} + \frac{1}{c}\mathbf{B} + \frac{1}{c}\mathbf{b}^\mathsf{T}\alpha\hat{p}(s),$$

which we call the Lundberg's matrix, have been subject of study in several works, like Albrecher and Boxma (2005), Schmidli (2005), Ren (2007), Li (2008), Ji and Zhang (2011), among others. In the expression δ stands for a non negative constant.

According to Ren (2007), the solutions of

$$det(\mathbf{L}_{\delta}(s)) = 0, \tag{2}$$

and the solutions of the fundamental Lundberg's equation

$$\hat{k}(\delta - cs)\hat{p}(s) = 1$$

are identical.

Previously we denoted by $\rho_1, \rho_2, \ldots, \rho_n$ the n solutions (of both equations) which have positive real parts.

Consider the Lundberg's matrices $\mathbf{L}_\delta(\rho_i)$, $i = 1, 2, \ldots, n$.

Since $det(\mathbf{L}_\delta(\rho_i)) = 0$, $i = 1, 2, \ldots, n$, all those matrices are singular, or equivalently all of them have 0 as an eigenvalue. Let $\mathbf{h}_i$ be an eigenvector of $\mathbf{L}_\delta(\rho_i)$ associated to the eigenvalue 0.

Theorem

Let $\rho_1, \rho_2, \ldots, \rho_m$ be distinct, $2 \leq m \leq n$. Then the eigenvectors $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_m$ are linearly independent.

Applications

In this section we want to apply our results to the calculation of the Laplace transform of the time to upcross a barrier. For a barrier level $b \geq u$ define

$$\tau_b = min\{t \geq 0 : U(t) = b\},$$

to be the first time the surplus reaches level b. For $\delta \geq 0$ define

$$R(u, b) = E[e^{-\delta \tau_b} | U(0) = u],$$

to be the Laplace transform of τ_b.

Assuming that the roots of the fundamental Lundberg's equation with positive real parts $\rho_1, \rho_2, \ldots, \rho_n$ are distinct, Li (2008) shows that

$$R(u,b) = \alpha \mathbf{H} e^{-\mathbf{D}(b-u)} \mathbf{H}^{-1} \mathbf{e}^{\mathsf{T}}, \quad u \leq b. \tag{3}$$

where $\mathbf{D} = diag(\rho_1, \rho_2, \ldots, \rho_n)$ and $\mathbf{H} = (\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n)$. The column vector $\mathbf{h}_i$ is an eigenvector of $L_\delta(\rho_i)$ corresponding to the eigenvalue 0.

If the roots $\rho_1, \rho_2, \ldots, \rho_n$ are not all distinct then the matrix $\mathbf{H}$ is not invertible and we can not apply formula (3) to find $R(u,b)$.

In the case of a double root we propose to replace one the appearances of such root by one of the negative roots of the fundamental Lundberg's equation.

Assuming the existence of the moment generating function of the claim amounts X_i, there is a negative root, we denote it by $\rho_0 = -r$ where $r > 0$ is the adjustment coefficient.

Example

We continue the last example. Choosing $\beta = 0,67$ the fundamental Lundberg's equation has the following roots

$$\rho_0 = -r = -0.58, \rho_1 = 0.69, \rho_2 = \rho_3 = 2.61,$$

the corresponding eigenvectors are

$$\mathbf{h}_0 = (0.15, 0.49, 0.85), \mathbf{h}_1 = (0.77, 0.47, 0.41), \mathbf{h}_2 = (0.27, -0.89, 0.36),$$

Example

Therefore

$$\mathbf{H} = \begin{pmatrix} 0.15 & 0.77 & 0.27 \\ 0.49 & 0.47 & -0.89 \\ 0.85 & 0.41 & 0.36 \end{pmatrix}$$

and we apply formula (3) to obtain

$$R(u,b) = 0.1e^{0.58(b-u)} + 0.93e^{-0.69(b-u)} - 0.034e^{-2.61(b-u)}$$

Remark

In the case of double roots we can apply the same method to compute other quantities like the ultimate and finite time ruin probabilities, severity of ruin and its maximum, the expected discounted future dividends, among others.

Conclusions

We studied the fundamental Lundberg's equation to find cases where double roots can arise and for such cases we provided a method to compute the Laplace Transform of the time to reach a certain level.

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Thank you for your attention!