Capital requirements and portfolio optimization under solvency constraints: a dynamical approach

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Outline

The setup

- A discrete-time framework with the set of trading dates $\mathcal{T} = \{t | t = 0, \ldots, T\}$.
- A portfolio of n assets with the gross return process over the period $[t, t+1]$ defined by $\mathbf{R}_{t+1} = (R_{1,t+1}, \ldots, R_{n,t+1})^T$.
- $\mathcal{F}_t = \sigma(\mathbf{R}_1, \ldots, \mathbf{R}_t)$ and we use the notation $E[\cdot|\mathcal{F}_t] = E_t[\cdot]$.
- Let $\mathbf{x}_t = (x_{1,t}, \ldots, x_{n,t})^T$ be the portfolio weights satisfying the budget constraint, $\sum_{i=1}^{n} x_{i,t} = 1$, and the no short sales constraint, $x_{i,t} \geq 0,\ i = 1, \ldots, n$.
- We introduce one-period optimization problems for a non-life insurance company over $[t, t+\tau]$, where τ is the solvency horizon.
- No rebalancing is allowed during the solvency period.

- p_t - the aggregate premium available for investment at time t.
- c_t - the regulatory initial capital provided by the shareholders.
- No other premiums are collected and no capital is issued or retired between t and $t+\tau$.
- The insurer's liability is modelled by a univariate random variable $Y_{t+\tau}$. It represents the aggregate claim amount over the solvency horizon which is assumed to be paid at time $t+\tau$.
- We define the insurer's net loss:

$$L_{t,t+\tau} := L(c_t, \mathbf{x}_t) = Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T \mathbf{x}_t.$$

- No other sources of risk other than the ones modelled through Y and $\mathbf{R}$.

- Each optimization problems characterized by minimizing the capital requirement c_t, subject to two key constraints:
  (i) *Solvency constraint:* capital requirement imposed by the insurer's regulator and based on one of the following criteria: Ruin Probability (*RP*), Conditional Value-at-Risk (*CVaR*) and Expected Policyholder Deficit (*EPD*).
  (ii) *Portfolio performance constraint:* target return on capital (*ROC*) provided:

$$ROC_{t,t+\tau} = -\frac{L_{t,t+\tau}}{c_t} \ .$$

- Santos, Nogales, Ruiz and Van Dijk (2012).
- Mankai and Bruneau (2012).
- Cummins and Phillips (2009).

Optimization with *RP* constraint

- Motivated by Solvency II Regime which applies to EU based insurance companies.
- Identify the capital required to maintain a target level for the ruin probability over a specified period of time.

$$\begin{aligned} & \min_{c_t, \mathbf{x}_t} c_t \\ \text{s.t.} \quad & E_t\left[\mathbb{1}_{\{L_{t,t+\tau}>0\}}\right] \leq 1-\alpha, \\ & E_t\left[ROC_{t,t+\tau}\right] \geq ROC^{\alpha}, \\ & \mathbf{1}^T\mathbf{x}_t = 1, \quad \mathbf{x}_t \geq \mathbf{0}, \quad c_t \geq 0. \end{aligned}$$

- α represents the specified solvency level (e.g. $\alpha = 99.5\%$).
- ROC^{α} is the lower bound for the expected return on capital.

- The solvency chance constraint can be reformulated as a Value-at-Risk constraint, where the *VaR* of a loss r.v. Z at α is defined:

$$VaR^{\alpha}(Z) := \inf\{z \in \Re : \Pr(Z \leq z) \geq \alpha\}.$$

- The constraint becomes:

$$VaR_t^{\alpha}(L_{t,t+\tau}) \leq 0.$$

- There are two streams of literature dealing with solving chance constrained optimization.
  1. Monte-Carlo estimators for the conditional expectation and perform further appropriate approximations.
  2. Solve the chance constraint using the *VaR* representation.

1. - Boyd and Vandenbergue (2004) - convex approximations by eliminating the indicator function.
   - Caliafore and Campi (2005), Nemirovski and Shapiro (2005) - scenario based approximations.
   - Nemirovski and Shapiro (2006) - convex approximation based on Bernstein scheme.
   - Luedtke and Ahmed (2008) - sample average approximation method and non-convex mixed-integer programming (MIP) reformulation, among others.
2. - Larsen, Mausser and Uryasev (2002) - algorithms based on iterative CVaR optimizations.
   - Gaivoronski and Pflug (2004) - scenario-based methods.
   - Wozabal, Hochreiter and Pflug (2008) - difference of convex functions reformulation.

- The Monte Carlo approximation of the solvency constraint:

$$\frac{1}{m}\sum_{j=1}^{m}\mathbb{1}_{\{Y_{t+\tau}(j)-(p_t+c_t)\mathbf{R}_{t+\tau}^T(j)\mathbf{x}_t>0\}}\leq 1-\alpha.$$

- This can be reformulated as a MIP problem. However, implementation becomes less efficient when m is large.
- Our approach: *Semiparametric method;*

$$\frac{1}{m}\sum_{j=1}^{m}E_t^{(j)}\left[\mathbb{1}_{\{Y_{t+\tau}-(p_t+c_t)\mathbf{R}_{t+\tau}^T(j)\mathbf{x}_t>0\}}\right]\leq 1-\alpha.$$

We used: $E\left[\,\cdot\,|\mathcal{F}_t\bigcup\{\mathbf{R}_{t+\tau}=\mathbf{R}_{t+\tau}(j)\}\right]=E_t^{(j)}\left[\,\cdot\,\right]$.

- A sufficient condition for convexity of our problem: $Y_{t+\tau}$ has a conditional convex survival function.

Optimization with *CVaR* constraint

- Rockafellar and Uryasev (2000) - alternative coherent risk measure to *VaR*; quantifies the loss severity in case of default.
- Defined as a weighted average of the corresponding *VaR* and conditional expected losses which strictly exceed *VaR*.
- *CVaR* coincides with the Expected Shortfall (*ES*) for continuous distributions.
- *ES* constitutes the basis for the target capital according to the Swiss Solvency Test (SST) (EIOPA, 2011). We take $\beta = 99\%$.

$$\begin{aligned} & \min_{c_t, \mathbf{x}_t} c_t \\ \text{s.t.} \quad & CVaR_t^{\beta}\left(L_{t,t+\tau}\right) \leq 0, \\ & E_t\left[ROC_{t,t+\tau}\right] \geq ROC^{\beta}, \\ & \mathbf{1}^T \mathbf{x}_t = 1, \quad \mathbf{x}_t \geq \mathbf{0}, \quad c_t \geq 0. \end{aligned}$$

- Rockafellar and Uryasev (2000) define *CVaR*:

$$CVaR^{\beta}(Z) = \inf_{s \in \Re} \left\{ s + \frac{1}{1-\beta} E\left[(Z-s)_+ \right] \right\}.$$

- The optimization problem becomes (only solvency constraint):

$$\begin{aligned} &\min_{s, c_t, \mathbf{x}_t} \; c_t \\ \text{s.t.} \quad & s + \tfrac{1}{1-\beta} E_t\left[(L_{t,t+\tau} - s)_+ \right] \leq 0. \end{aligned}$$

- The traditional approach: use MC estimator and reformulate as a Linear Programming (LP) problem. Less efficient when m is large.
- Our approach: *Semiparametric method:*

$$s + \frac{1}{m(1-\beta)} \sum_{j=1}^{m} E_t^{(j)}\left[(Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T(j)\mathbf{x}_t - s)_+ \right] \leq 0.$$

Optimization with *EPD* constraint

- Introduced by Butsic (1994) as an alternative method to the ruin probability for measuring insolvency risk.
- Constitutes a useful tool in establishing the US Risk Based Capital (RBC) regulatory system (e.g. see NAIC, 2009).
- Defined as the expected loss in the event of insolvency:

$$EPD(L_{t,t+\tau}) = E_t\Big[(Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T \mathbf{x}_t)_+\Big].$$

- Solvency criteria based on a target level of a deficit ratio:

$$\frac{EPD\Big(L_{t,t+\tau}\Big)}{E_t\Big[Y_{t+\tau}\Big]} \leq f.$$

- f is the maximum level for the *EPD* ratio with $0 \leq f < 1$. We arbitrarily take $f = 0.25\%$.

- A similar LP reformulations is available.
- For consistency, we consider our semiparametric representation:

$$\begin{aligned} &\min_{c_t, \mathbf{x}_t} c_t \\ \text{s.t.} \quad &\frac{1}{m}\sum_{j=1}^{m} E_t^{(j)}\Big[(Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T(j)\mathbf{x}_t)_+\Big] \leq f E_t\Big[Y_{t+\tau}\Big], \\ &E_t\Big[ROC_{t,t+\tau}\Big] \geq ROC^f, \\ &\mathbf{1}^T\mathbf{x}_t = 1, \quad \mathbf{x}_t \geq \mathbf{0}, \quad c_t \geq 0. \end{aligned}$$

- A sufficient condition for convexity: $E_t^{(j)}\Big[(Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T(j)\mathbf{x}_t)_+\Big]$ is a convex function in c_t and $\mathbf{x}_t$.
- This depends on the conditional distribution of $Y_{t+\tau}$.

Modelling assets and liabilities

- Assets follow the DCC - GARCH model of Engle (2002)

$$\begin{aligned}
\log \mathbf{R}_{t+1} &= \mathbf{m}_{t+1} + \varepsilon_{t+1}, \quad \varepsilon_{t+1}|\mathcal{F}_t \sim \mathbf{MVN}(\mathbf{0}, H_{t+1}), \\
H_{t+1} &= D_{t+1}^{1/2} \Sigma_{t+1} D_{t+1}^{1/2}, \\
D_{t+1} &= diag(h_{1,t+1}, \ldots, h_{n,t+1}), \\
\Sigma_{t+1} &= diag(q_{11,t+1}^{-1/2}, \ldots, q_{nn,t+1}^{-1/2}) Q_{t+1} diag(q_{11,t+1}^{-1/2}, \ldots, q_{nn,t+1}^{-1/2}), \\
Q_{t+1} &= (1 - \theta_1 - \theta_2)\bar{Q} + \theta_1 \mathbf{u}_t \mathbf{u}_t^T + \theta_2 Q_t.
\end{aligned}$$

- D_{t+1} is the $n \times n$ diagonal matrix formed with the univariate conditional variances GARCH(1,1).
- Σ_{t+1} is the time-varying conditional correlation matrix of $\mathbf{R}_{t+1}$.
- Liabilities are LogNormal distributed:

$$Y_{t+\tau} \sim \mathbf{LGN}(\mu_{t+\tau}, \sigma_{t+\tau}).$$

- $Y_{t+\tau}$ is independent of the enlarged filtration $\mathcal{F}_t \bigcup \sigma(\mathbf{R}_{t+\tau})$.

Data description and estimation

- Assets: 3-asset portfolios formed with T-Bills, NASDAQ and NYSE.

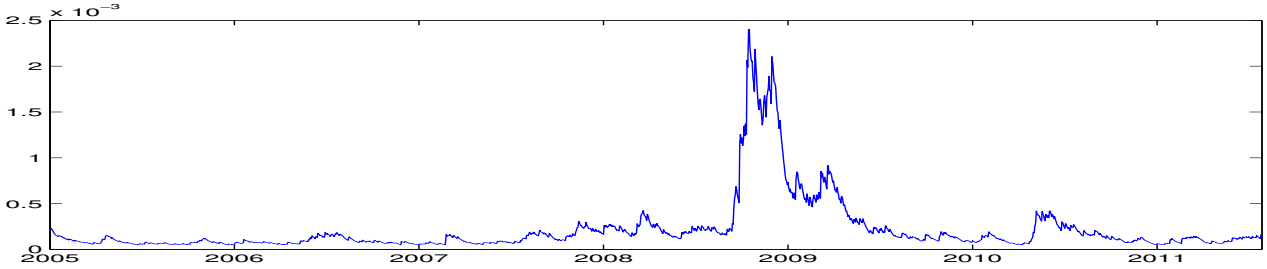
Table: Descriptive statistics for NASDAQ and NYSE log-returns from January 3, 2005 - July 29, 2011 for a total of 1,656 observations.

Index	Min	Max	Mean	Std	Skewness	Kurtosis
NASDAQ	-0.0959	0.1116	0.0001	0.0149	-0.1670	10.2725
NYSE	-0.1023	0.1153	0.0001	0.0150	-0.3480	12.7329

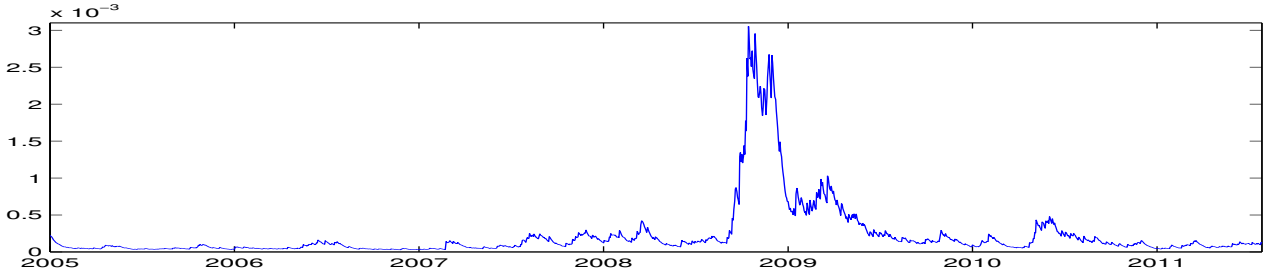
- Liabilities: aggregate monthly claim amounts on property insurance for the same period used in the assets case.

Table: Descriptive statistics for monthly claim amounts from January 3, 2005 - July 29, 2011 for a total of 79 observations (figures are in thousands Euros).

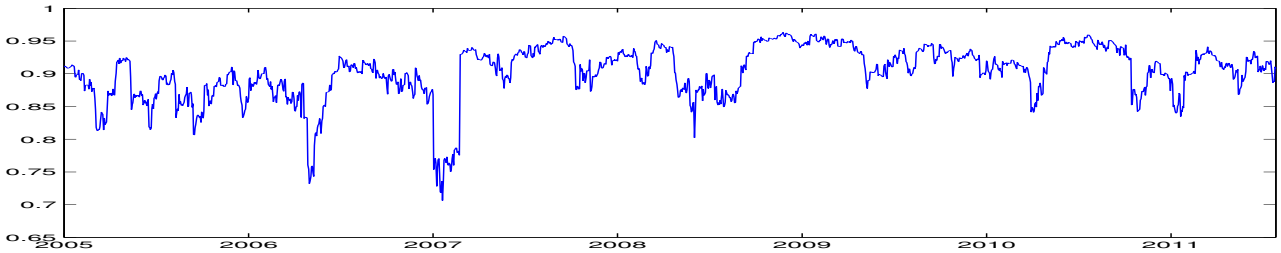
Min	Max	Mean	StDev	Skewness	Kurtosis
8.2465	2049.2119	603.2802	375.1311	1.2434	5.5068



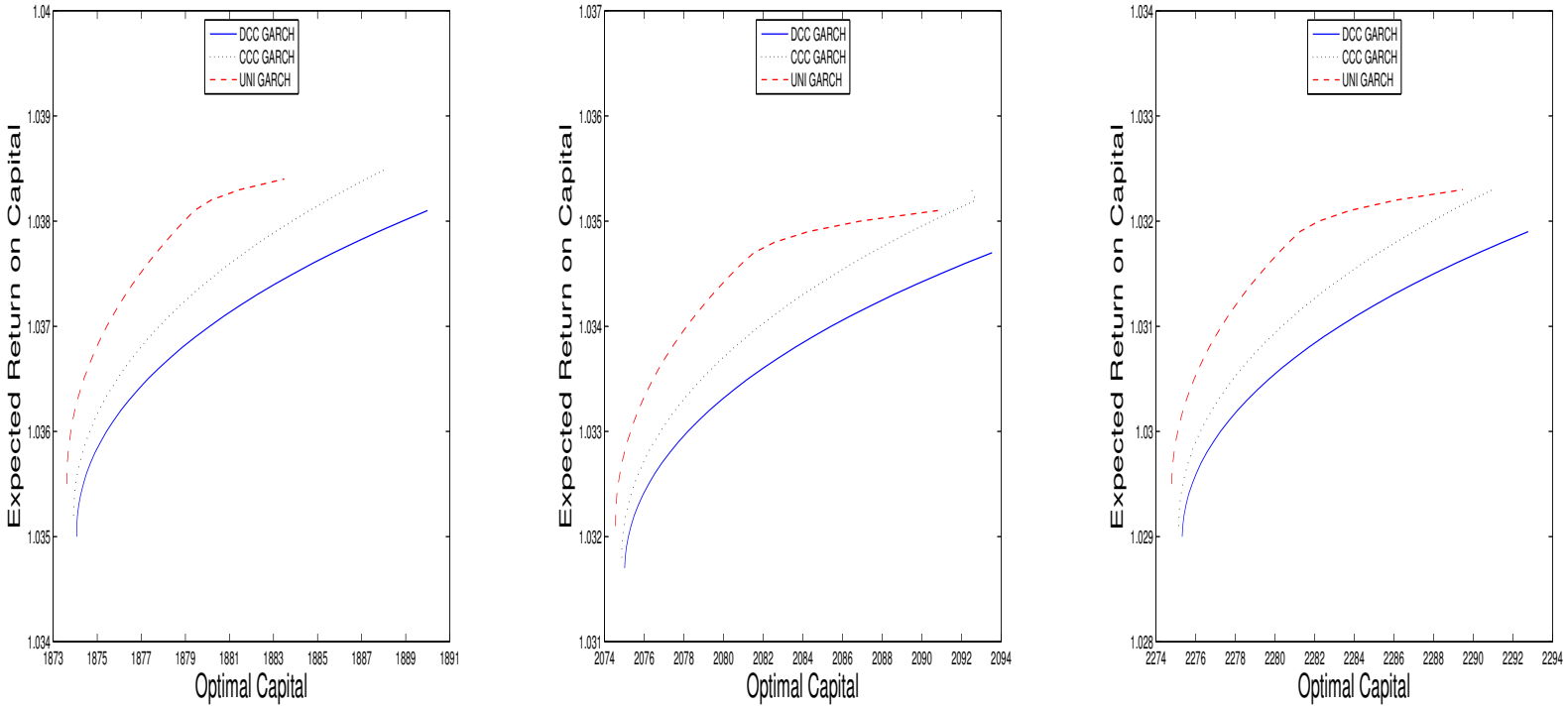
(a) Conditional variance for NASDAQ



(b) Conditional variance for NYSE

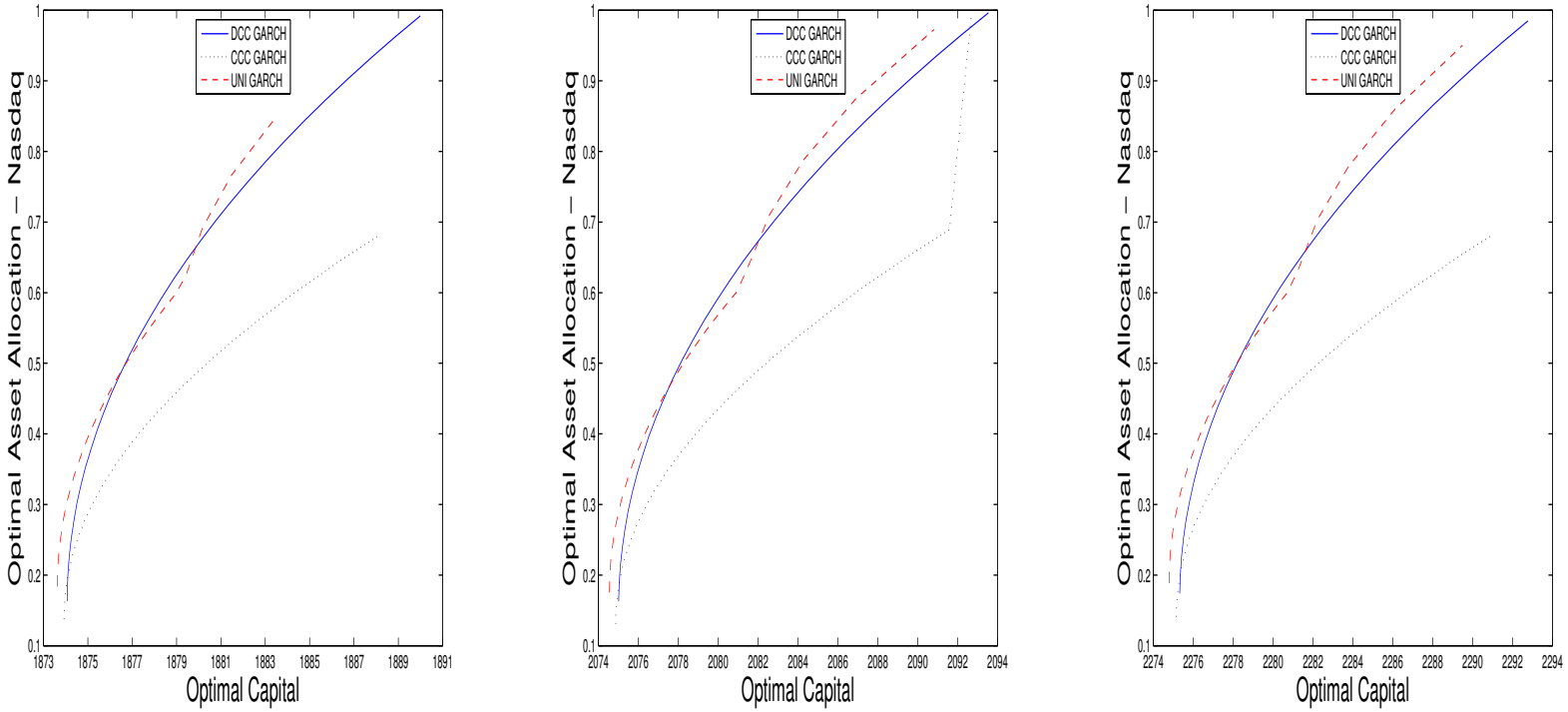


(c) Conditional correlation



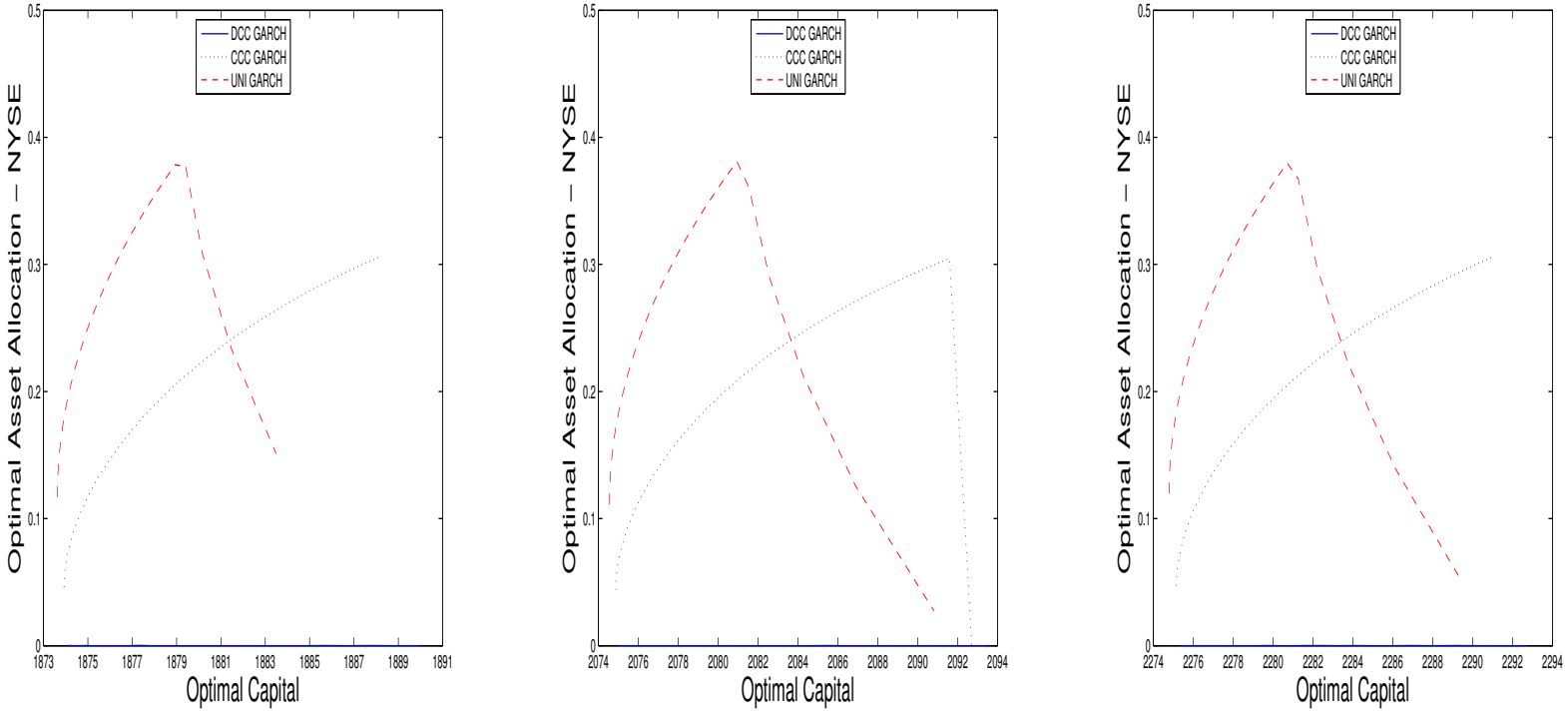
(a) *RP* (b) *CVaR* (c) *EPD*

Figure: Efficient frontiers for DCC, CCC and UNI-GARCH models under the *RP*, *CVaR* and *EPF*-constrained problems.



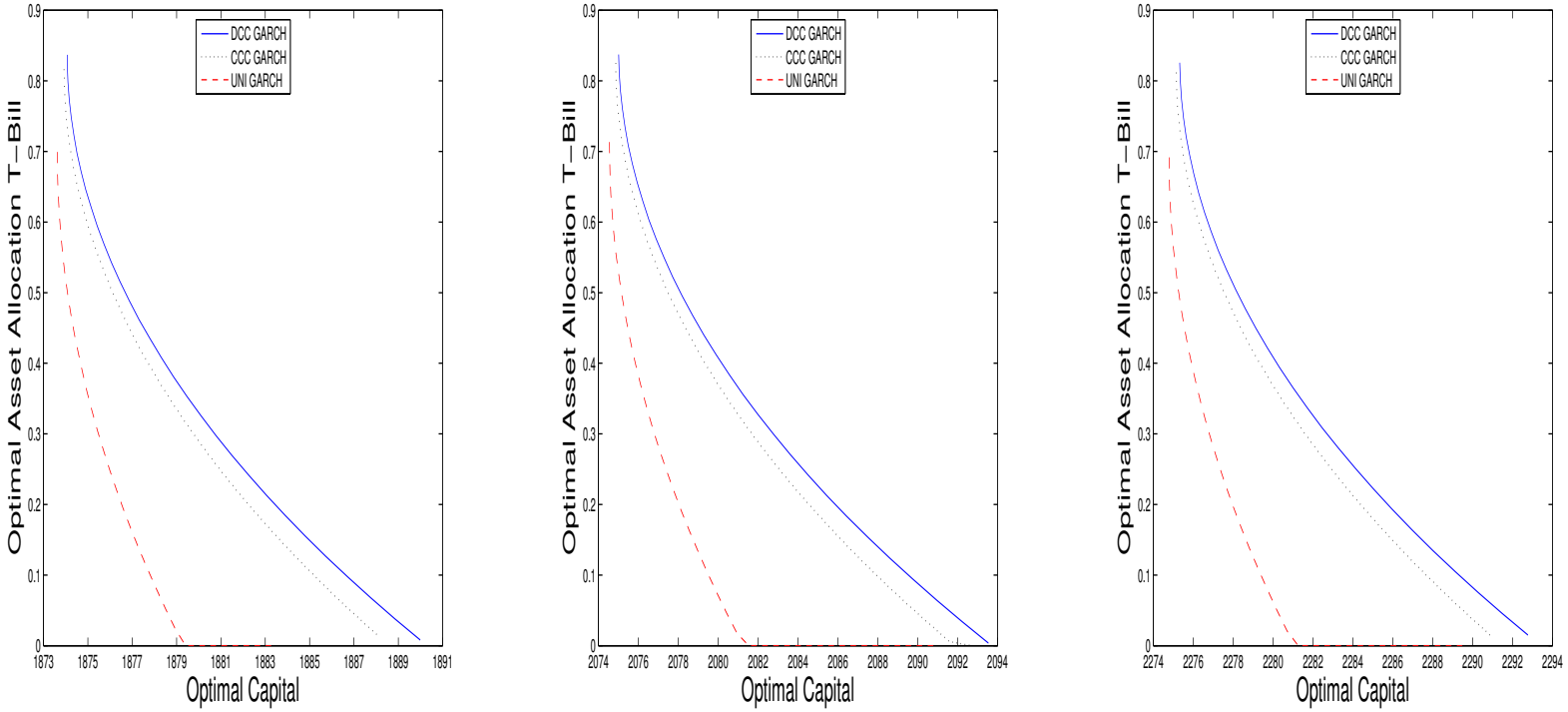
(a) *RP* (b) *CVaR* (c) *EPD*

Figure: Optimal allocation in Nasdaq.



(a) *RP* (b) *CVaR* (c) *EPD*

Figure: Optimal allocation in NYSE.

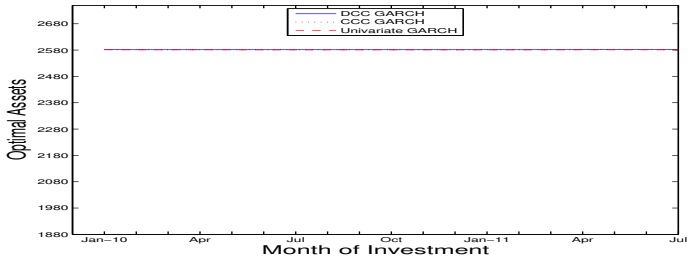


(a) *RP* (b) *CVaR* (c) *EPD*

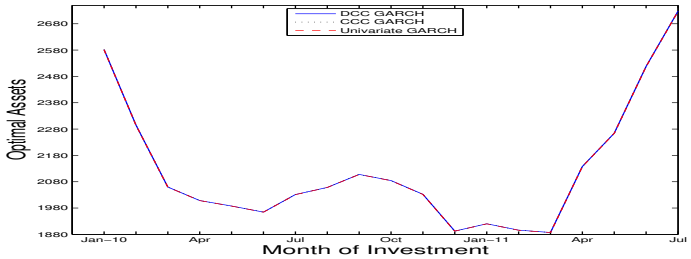
Figure: Optimal allocation in T-Bills.

Rolling window implementation

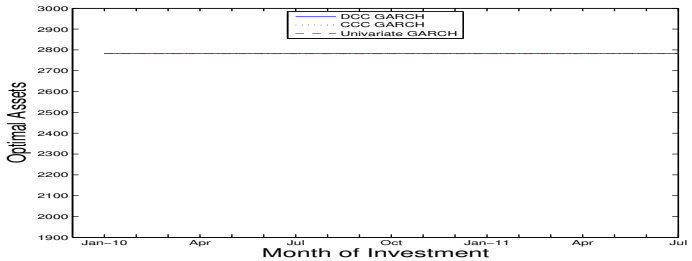
- Assets (sampled daily): Sample A: January 3, 2005 - December 32, 2009 ($l_A = 1,259$) and Sample B: January 1, 2010 - July 29, 2011 ($l_B = 397$).
- Liabilities (sampled monthly): Sample A': January 3, 2005 - December 32, 2009 ($l_{A'} = 60$) and Sample B': January 1, 2010 - July 29, 2011 ($l_{B'} = 19$).
- Given the estimates based on Samples A and A', find the optimal solution $(c_t^*, \mathbf{x}_t^*)$.
- *Single rolling window:* drop the first τ obs. from Sample A and replace them with the first τ observations from Sample B.
- *Double rolling window:* additionally, drop the first obs. from Sample A' and replace it with the first obs. from B'.
- Repeat the estimation and optimization steps until the length of the out-of-sample data set is reached (i.e. $l_{B'} = 19$ *times*).



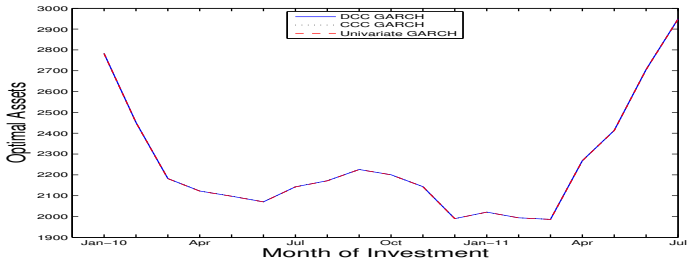
(a) *RP* and single rolling window



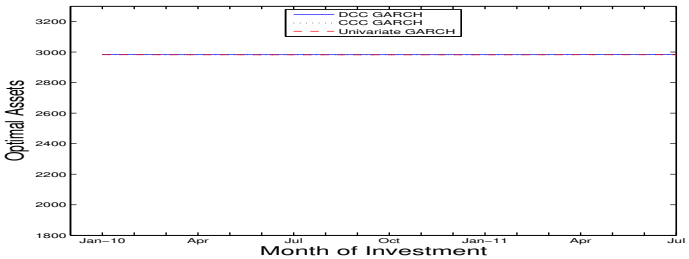
(b) *RP* and double rolling window



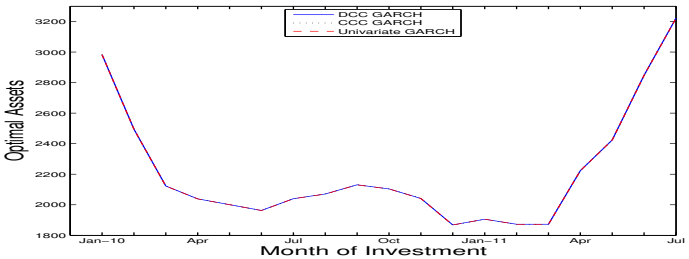
(c) *CVaR* and single rolling window



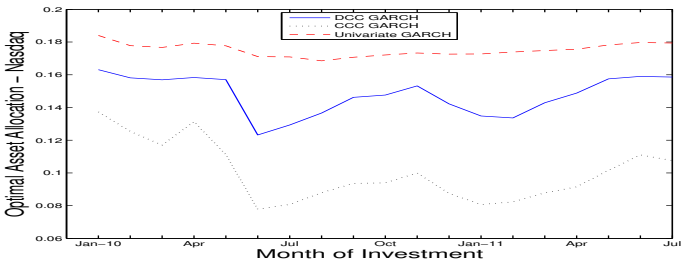
(d) *CVaR* and double rolling window



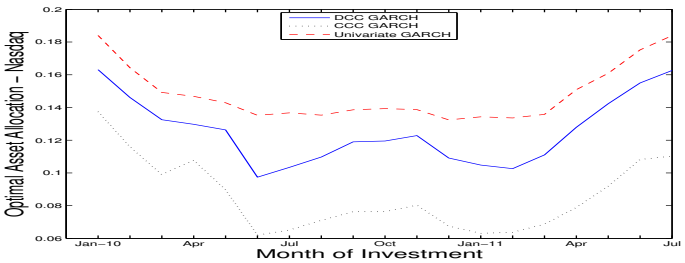
(e) *EPD* and single rolling window



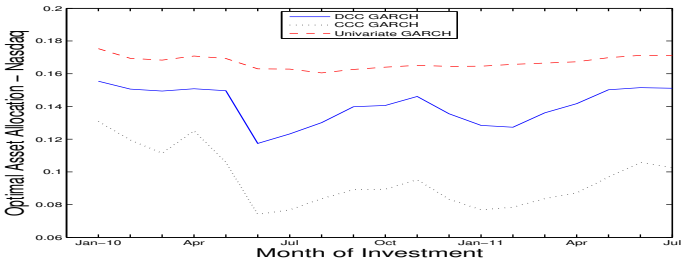
(f) *EPD* and double rolling window



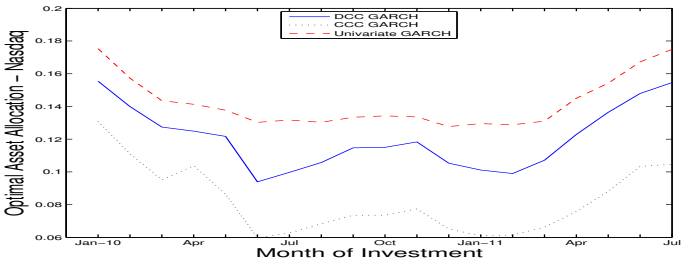
(a) *RP* and single rolling window



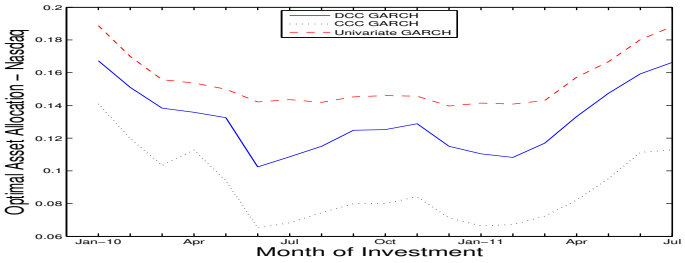
(b) *RP* and double rolling window



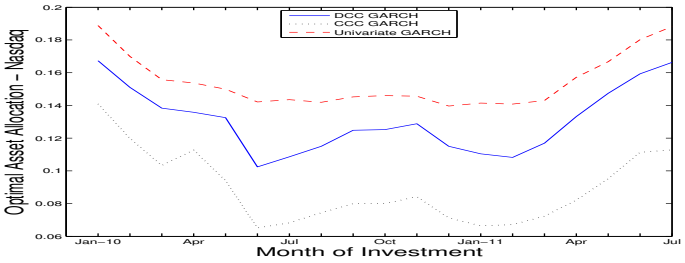
(c) *CVaR* and single rolling window



(d) *CVaR* and double rolling window



(e) *EPD* and single rolling window



(f) *EPD* and double rolling window

Out-of-Sample Performance

Using optimal solutions and time series of observed returns over the out-of-sample period, we test the performance of the DCC, CCC and UNI MGARCH models relative to two criteria:

1. Solvency performance:
   - Average total assets invested (premium + optimal capital).
   - Average solvency value.
   - Maximum solvency value.
2. Portfolio performance:
   - Average adjusted ROC.
   - Standard deviation adjusted ROC.
   - Sharpe Ratio.
   - Portfolio Turnover.

- Average solvency values:

$$\widehat{RP} = \frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} \Phi\left(d_{t+k\tau}\right),$$

$$\widehat{CVaR} = \frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} \left(\frac{E[Y_{t+k\tau}]}{1-\beta} \Phi\left(\sigma_{t+k\tau} - \Phi^{-1}(\beta)\right) - \mathbf{R}_{t+k\tau}^T \mathbf{z}_{t+(k-1)\tau}^* \right),$$

$$\widehat{EPD} = \frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} \left[E[Y_{t+k\tau}] \Phi\left(d_{t+k\tau} + \sigma_{t+k\tau}^2\right) - \mathbf{R}_{t+k\tau}^T \mathbf{z}_{t+(k-1)\tau}^* \Phi\left(d_{t+k\tau}\right) \right].$$

$$\mathbf{z}_{t+(k-1)\tau}^* = (p_{t+(k-1)\tau} + c_{t+(k-1)\tau}^*) \mathbf{x}_{t+(k-1)\tau}^*,$$

$$d_{t+k\tau} = \frac{-\log \mathbf{R}_{t+k\tau}^T \mathbf{z}_{t+(k-1)\tau}^* + \mu_{t+k\tau}}{\sigma_{t+k\tau}},$$

Table: Out-of-sample solvency performance under the single rolling window exercise.

	Solvency Performance		
	Avg. Assets invested	Avg. Solvency Value	Max. Solvency Value
Problem 1. Ruin Constraint		Avg. Ruin Probability (%)	Max. Ruin Prob (%)
Covariance Model			
DCC	2581.30	0.497	0.543
CCC	2580.77	0.498	0.550
UNI	2581.13	0.497	0.585
Problem 2. CVaR Constraint		Avg. CVaR	Max. CVaR
Covariance Model			
DCC	2782.24	-3.665	46.718
CCC	2782.06	-3.236	53.702
UNI	2781.69	-5.220	87.822
Problem 3. EPD Constraint		Avg. EPD Ratio (%)	Max. EPD Ratio (%)
Covariance Model			
DCC	2982.54	0.248	0.272
CCC	2982.33	0.249	0.275
UNI	2981.90	0.248	0.292

Table: Out-of-sample solvency performance under the double rolling window exercise.

	Solvency Performance		
	Avg. Assets invested	Avg. Solvency Value	Max. Solvency Value
Problem 1. Ruin Constraint		Avg. Ruin Probability (%)	Max. Ruin Prob (%)
Covariance Model			
DCC	2129.17	0.497	0.546
CCC	2129.06	0.497	0.553
UNI	2128.81	0.496	0.589
Problem 2. CVaR Constraint		Avg. CVaR	Max. CVaR
Covariance Model			
DCC	2258.85	-1.773	28.645
CCC	2258.75	-1.319	32.857
UNI	2258.48	-2.114	58.813
Problem 3. EPD Constraint		Avg. EPD Ratio (%)	Max. EPD Ratio (%)
Covariance Model			
DCC	2222.25	0.248	0.273
CCC	2222.15	0.249	0.276
UNI	2221.86	0.248	0.294

- Portfolio performance indicators:

$$\begin{aligned}
\hat{\mu}_{AROC} &= \frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} AROC_{t+(k-1)\tau, t+k\tau}, \\
\hat{\sigma}_{AROC} &= \sqrt{\frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} (AROC_{t+(k-1)\tau, t+k\tau} - \hat{\mu}_{AROC})^2}, \\
\widehat{SR}_{AROC} &= \frac{\hat{\mu}_{AROC}}{\hat{\sigma}_{AROC}}, \\
Turnover &= \frac{1}{l_{B'} - 1} \sum_{k=1}^{l_{B'}-1} \sum_{i=1}^{n} |x^*_{i,t+k\tau} - x^*_{i,t+(k-1)\tau}|.
\end{aligned}$$

- These quantities are computed for the adjusted return on capital (AROC):

$$AROC_{t,t+\tau} = \frac{(p_t + c^*_t)\mathbf{R}^T_{t+\tau}\mathbf{x}^*_t - E[Y_{t+\tau}]}{c_t} - 1.$$

Table: Out-of-sample portfolio performance under the single rolling window exercise.

	Portfolio Performance			
	Avg. AROC (%)	Std. AROC	Sharpe Ratio	Turnover
Problem 1. Ruin Constraint				
Covariance Model				
DCC	3.67	1.15	3.18	0.013
CCC	3.66	1.25	2.92	0.018
UNI	3.78	2.07	1.83	0.005
Problem 2. CVaR Constraint				
Covariance Model				
DCC	3.32	1.07	3.11	0.013
CCC	3.31	1.16	2.85	0.018
UNI	3.42	1.92	1.79	0.005
Problem 3. EPD Constraint				
Covariance Model				
DCC	3.06	1.13	2.71	0.014
CCC	3.05	1.22	2.49	0.019
UNI	3.17	2.02	1.57	0.005

Table: Out-of-sample portfolio performance under the double rolling window exercise.

	Portfolio Performance			
	Avg. AROC (%)	Std. AROC	Sharpe Ratio	Turnover
Problem 1. Ruin Constraint				
Covariance Model				
DCC	5.08	1.46	3.46	0.019
CCC	5.06	1.56	3.24	0.021
UNI	5.17	2.25	2.29	0.020
Problem 2. CVaR Constraint				
Covariance Model				
DCC	4.67	1.38	3.38	0.018
CCC	4.65	1.47	3.16	0.020
UNI	4.75	2.11	2.25	0.018
Problem 3. . EPD Constraint				
Covariance Model				
DCC	4.92	1.67	2.95	0.019
CCC	4.91	1.77	2.77	0.022
UNI	5.03	2.46	2.05	0.019

Conclusions

- We propose three problems to jointly solve for the optimal capital requirement and its optimal portfolio allocation.
- We provide a novel semiparametric approach for solving these problems.
- Asset correlation plays an important role in the behaviour of the optimal capital required and the portfolio structure.
- Optimal required capital is very stable when the liability parameters remain constant over the rolling horizon; however, the variation is substantial when the liability is re-estimated at each step. The differences between the optimal portfolio weights are not as pronounced for the single versus double rolling exercises.
- The out-of-sample exercise indicates that the DCC model outperforms the CCC and No-Correlation model relative to both the solvency and portfolio performance criteria.