Dynamic optimal investment in Markov-modulated Lévy markets with risk of default and general utility function

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Abstract

Dynamic optimal portfolio selection is studied for an investor operating in a Markov-modulated Lévy market facing a risk of default (or bancruptcy)in a fixed finite time with a terminal utility function of general form. This setting is a quite coprehensive generalisation to previous particular results in literature.

The intuitive recursive algorithm suggested shows outstanding convengence rates and computation efficiency to problems otherwise unsolvable unless using alternative numerical methods such as viscosity or minimax solutions.

Keywords:Lévy process, Markov modulation, optimal portfolio, probability of default, dynamic programming, Chebyshev polynomials

1 Introduction

The present article treats the problem of dynamic optimal portfolio selection for an investor facing a risk of default (or bancruptcy). This scenario generalises the usual approaches introducing a solvency region for the total wealth of the investor. Should the total wealth exit the solvency region, the investor is assumed to go bankrupt and is not able to continue the investment strategy and a possible penalty, as a function of the total wealth at the moment of bancruptcy, is realised.

The optimal portfolio selection problem is considered in a fixed finite time with a terminal utility function of general form. This is a generalisation of settings where a particular form (power, logarithmic, exponential) is assumed (e.g. Bauerle and Rieder (2004), Pasin and Vargiolu (2010), Nutz (2011)).

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Regarding the investment environment, the investor operates in a Markov-modulated Lévy market. This means that the available bundles of assets follow a Lévy diffusion process contingent on a state-of-economy variable governed by a Markov process. The selection of the investment strategy is represented as a trade-off between drift, diffusion and jump (Lévy measure) coefficients that belong to a feasible set of choices. The decision is instantaneously adjusted based on the current state of the market and the state-of-economy variable.

The setting of Lévy diffusion environment with Markov-modulation (also refered to as regime-swithching) is very appealing and has been treated recently (Pasin and Vargiolu (2010), Valdez and Vargiolu (2012)). The reason is that the Lévy markets are capable of capturing the evolution of real-world prices much closer than the historiccaly omnipresent Wiener diffusions. Moreover, Markov-modulations are at least as strong modelling tool since they are dense in certain broad class of stochastic processes (Asmussen (2000)) thus offer a possibility to approximate any behaviour of the underlying market.

This article presents a method to approximate both the invesment strategies and value function of the dynamic optimisation problem in probably the most comprehensive market environment treated up to the moment. The underlying vehicle is the Erlangisation principle introduced by Carr in 1998 (Carr (1998)) that transforms an initially complex problem in time to a series of simpler ones using the advantange of dealing with recursive exponential horizons. The solution is indeed showed to converge to the true value function in probability and the optimal investment strategy is approximated in ϵ-optimal sense, that is the proposed strategy delivers a value function that is at most ϵ away form the true value function with probability converging to 1.

The classical optimization setting, through the Hamilton-Jacobi-Bellman(HJB) equation, for this quite comprehensive model is presented in section 2 along with some serious limitations in finding solutions. Theorem 1 in Section 3 proves how the time dependence in HJB can be safely removed using a erlangian random horizon leading to a recursive optimization scheme in exponential horizons explained in section 3, theorem 2. In section 4, piecewise constant controls are introduced, in the mentioned recursive scheme of exponential times, and proved to offer a dramatic simplification in the tractability of the optimization problem with guaranteed convergence: the optimization is focused on scalars rather than on functions . An illustration using Chebyshev polynomials is presented in section 5 showing outstanding convengence rates and computation efficiency to a problem otherwise unsolvable unless using alternative numerical methods such as viscosity or minimax solutions.

2 The problem formulation

The class of stochastic processes considered in this chapter are Lévy diffusions, a subclass of semi-martingale family. Let $U_t, t \geq 0$ be a real-valued stochastic

process that satisfies the stochastic differential equation

$$\begin{aligned} \mathrm{d}U_t &= a(U_t, Y_t, \sigma_t)\mathrm{d}t + b(U_t, Y_t, \sigma_t)\mathrm{d}W_t + \\ &\quad + \int_{\mathbb{R}^k} \gamma(U_{t^-}, Y_{t^-}, \sigma_{t^-}, z) N(\mathrm{d}t, \mathrm{d}z) \\ U_0 &= u \end{aligned} \tag{1}$$

where a, b and γ are known real valued functions, W_t is a standard Wiener process and $N(\mathrm{d}t, \mathrm{d}z)$ is a compensated Poisson random measure. See Bertoin (1998) for more details on Lévy processes. Observe that the process defined by (1) is time-homogeneous, therefore its evolution is invariant under time shifts.

The process σ_t is the control assumed to be adapted and càdlàg. The process Y_t is a Markov process with finite state space $\{\delta_1, \ldots, \delta_m\}$ and intensity matrix $Q = \{q_{ij}\}$ that modulates the coefficients of (1).

Markov-modulated models are widely used to model phenomena where abrupt changes in otherwise stable behaviour of the system occur. Markov modulation provides a set of model parameters for each behaviour state and governs the switching among them. For example in finance, see e.g. Elliott et al. (2007), Markov modulating is shown to perform better in explaining the behaviour of financial assets than usual Gaussian models.

The stochastic control is considered in a fixed horizon T or until the process U_t exits a region $S \subseteq \mathbb{R}$. The performance criterion v to be maximised is

$$\begin{aligned} v^\sigma(t, T, u, \delta_i) \equiv \mathrm{E}\big[& P(U_T, Y_T)\mathbb{I}_{\{\tau \geq T\}} + \\ & + L(U_\tau, Y_\tau)\mathbb{I}_{\{t < \tau < T\}} | U_t = u, Y_t = \delta_i \big] \end{aligned} \tag{2}$$

where $\tau = \inf\{s \geq t : U_s \notin S\}$ is the exit time of the process U_s from the region S. Functions P and L are arbitrary but continuous and represent the utility realised upon termination of the horizon and at exit of the controlled process from the region S.

Let us denote $J(t, T, u, \delta_i)$ the optimal value of the maximisation problem

$$J(t, T, u, \delta_i) \equiv \max_{\sigma \in \Pi} v^\sigma(t, T, u, \delta_i). \tag{3}$$

where the set Π contains all the admissible controls, that is such σ_t adapted for which a strong solution to the equation (1) exists and is unique. The attention will be restricted to controls of the form $\sigma_t = \sigma(U_{t^-}, Y_{t^-})$ also called Markov controls. Øksendal (2003, Th. 11.2.3.) gives fairly week sufficient conditions under which the optimal value of the problem restricted to Markov controls equals the optimal value of the problem with arbitrary adapted control. Therefore narrowing the control space Π to Markov controls is not too restrictive.

Solving (3) directly is not feasible since an explicit expression for $v^\sigma(t, T, u, \delta_i)$ is not available in the most general case. Following the dynamic programming approach, one can write the Hamilton-Jacobi-Bellman equation that characterises the value function $J(t, T, u, \delta_i)$

$$\sup_{\sigma \in \Pi} \left\{ \mathcal{A}^\sigma J(t, T, u, \delta_i) + \frac{\partial J(t, T, u, \delta_i)}{\partial t} \right\} = 0 \tag{4}$$

where $\mathcal{A}^\sigma$ is the infinitesimal generator of the controlled process U_s, see e.g. Fleming and Soner (2006). $\mathcal{A}^\sigma$ can be given, see Øksendal and Sulem (2005, pg. 40), for each $i = 1, \ldots, m$ by

$$\begin{aligned}\mathcal{A}^\sigma J(t,T,u,\delta_i) = \frac{1}{2} b(u,\delta_i,\sigma(u,\delta_i)) \frac{\partial^2}{\partial u^2} J(t,T,u,\delta_i) + \\ + a(u,\delta_i,\sigma(u,\delta_i)) \frac{\partial}{\partial u} J(t,T,u,\delta_i) + \sum_{j=1}^{m} q_{ij} J_u(t,T,u,\delta_j) + \\ + \int_{\mathbb{R}} \{ J(t,T,u+\gamma(u,\delta_i,\sigma(u,\delta_i),z),\delta_i) \\ - J(t,T,u,\delta_i) - \gamma(u,\delta_i,\sigma(u,\delta_i),z) \frac{\partial}{\partial u} J(t,T,u,\delta_i) \Big\} \nu(\mathrm{d}z) \quad (5)\end{aligned}$$

where $\nu(\mathrm{d}z) = \mathrm{E}\left[N(1,\mathrm{d}z)\right]$ is the Lévy measure of the process U_s.

Moreover, if σ^* is such that

$$\mathcal{A}^{\sigma^*} J(t,T,u,\delta_i) + \frac{\partial J(t,T,u,\delta_i)}{\partial t} = 0 \quad (6)$$

then σ^* is the optimal control for (3) and $J(t,T,u,\delta_i) = v^{\sigma^*}(t,T,u,\delta_i)$.

In this case, the optimal control can be found as the maximiser of the expression under the supremum in (4) whence the following must hold

$$\frac{\partial}{\partial \sigma} \mathcal{A}^{\sigma^*} J(t,T,u,\delta_i) = 0. \quad (7)$$

The HJB equation (4) together with optimality condition (7), in the view of (5), form a system of non-linear second order partial integro-differential equations. The solution of such system is usually a very difficult task even using numerical procedures. Analytic solutions are seldom found due to:

- Optimal control condition (7) may not yield an explicit value for σ in a closed formula.
- The admissible controls $\sigma \in \Pi$ may not be continuous and only finite or infinite countable.
- The PDE contained in curly brackets in (4) lacks of an explicit solution in most of the practical cases, and most important
- even though the previous assumptions hold, the optimal control usually depend on the value function $J(t,T,u,\delta_i)$

Regarding financial literature Capponi and Figueroa-Lopez (2011) and Capponi et al. (2011) address an optimisation problem of our type that includes regime-switching and a special set of investment possibilities obtaining explicit solutions for logarithmic utility function. Similarly, Pasin and Vargiolu (2010)

and Valdez and Vargiolu (2012) treat a regime-switching environment optimal portfolio selection problems and obtain solutions for CRRA utility functions. For example, Bauerle and Rieder (2004) solve similar stochastic control problem in cases when no jumps are present and the dependence of the coefficients on the control is linear. On the other hand, Øksendal and Sulem (2009) study this type of models in a pricing problem scenario but no explicit solutions are available due to complexity of the setting.

Our setting offers a generalised framework that includes most of the previously treated problems as a special case, the solution however, comes at a cost of lack of analytical tractability.

3 Erlangian horizon optimization

A method based on approximation of the valuation backwards time $T-t$ by a random horizon with Erlang distribution was presented by Carr (1998) in an article on the optimal execution of a put option time. Similar principle has later been used to find explicit expressions for the default risk problem with underlying fluid flow process by Asmussen et al. (2002). This approach is often called randomisation of the horizon.

Consider now the Erlang parametric model $\mathrm{Er}\,(\alpha, n)$ with pdf

$$f\,(x) = \frac{\alpha^n}{(n-1)!} x^{n-1} \mathrm{e}^{-\alpha x}, \quad x > 0, \alpha > 0$$

and integer valued n. An Erlang random variable is defined as the distribution of a sum of n independent exponential random variables with common parameter α. If $\xi \sim \mathrm{Er}\,(n/t, n)$ one can easily prove, for a fixed t

$$\mathrm{E}\,(\xi) = t, \quad \mathrm{E}\,(\xi - t)^2 = \frac{t^2}{n}$$

and thus ξ converges to t in L_2 and therefore in probability.

The maximisation problem (3) will be now considered in a hypothetical backwards $\mathrm{Er}\,(n/\,(T-t)\,, n)$ random time H_n, independent of (U_s, Y_s, W_s). Let Υ_n^σ be the performance criterion to be maximised in the mentioned random backwards horizon H_n with terminal time T

$$\begin{aligned}\Upsilon_n^\sigma(H_n, T, u, \delta_i) \equiv \mathrm{E}_{H_n}[v^\sigma(T - H_n, T, u, \delta_i)] \\ = \mathrm{E}_{H_n}[\mathrm{E}\left[P(U_T, Y_T)\mathbb{I}_{\{\tau \geq T\}} + \right. \\ \left. + L(U_\tau, Y_\tau)\mathbb{I}_{\{T-H_n<\tau<T\}} | U_{T-H_n} = u, Y_{T-H_n} = \delta_i\right]]\end{aligned}$$

where now $\tau = \inf\{s \geq T - H_n : U_s \notin S\}$ is the exit time of the process U_s from the region S.

The optimisation problem is now defined as

$$J_n(H_n, T, u, \delta_i) \equiv \max_{\sigma \in \Pi} \Upsilon_n^\sigma(H_n, T, u, \delta_i) \tag{8}$$

where J_n represents the value function.

The next theorem shows the convergence of the value function of the Erlangian horizon problem to the value function of the fixed horizon problem.

Theorem 1. *Let $J(t,T,u,y)$ be the value function of (3) and $J_n(H_n,T,u,y)$ be the value function of (8) then for each $u \geq 0$ and $y \in \{\delta_1,\ldots,\delta_m\}$ fixed*

$$\lim_{n\to\infty} J_n(H_n,T,u,y) = J(t,T,u,y).$$

Proof. (Following the proof of Theorem 2 in Liu and Loewenstein (2002).) Let σ be any feasible control in Π such that $v^\sigma(t,T,u,y)$ is continuous, then

$$\mathrm{E}_{H_n}\left[v^\sigma(T-H_n,T,u,\delta_i)\right] \leq J_n(H_n,T,u,y),$$

and taking the limit $n\to\infty$

$$v^\sigma(t,T,u,y) \leq \lim_{n\to\infty} J_n(H_n,T,u,y) \tag{9}$$

since $\lim_{n\to\infty} H_n = T-t$ and using the Helly-Bray Theorem (see Chow and Teicher (2003, corollary 8.1.6)). The maximum over all admissible controls on the left side of (9) yields

$$J(t,T,u,y) \leq \lim_{n\to\infty} J_n(H_n,T,u,y). \tag{10}$$

On the other hand, notice that

$$\begin{aligned}
J_n(H_n,T,u,y) \equiv \max_{\sigma\in\Pi} \Upsilon_n^\sigma(H_n,T,u,y) &= \max_{\sigma\in\Pi} \int_0^\infty v^\sigma(T-z,T,u,y)\,\frac{(n/(T-t))^n}{(n-1)!} z^{n-1}\mathrm{e}^{-\frac{n}{T-t}z}\mathrm{d}z \\
&\leq \int_0^\infty \max_{\sigma\in\Pi} v^\sigma(T-z,T,u,y)\,\frac{(n/(T-t))^n}{(n-1)!} z^{n-1}\mathrm{e}^{-\frac{n}{t}z}\mathrm{d}z \\
&= \int_0^\infty J(T-z,T,u,y)\;\frac{(n/(T-t))^n}{(n-1)!} z^{n-1}\mathrm{e}^{-\frac{n}{t}z}\mathrm{d}z
\end{aligned}$$

The inequality comes from the fact that the optimal control σ^* that maximises the whole integral in the first line is feasible in the maximisation problem under the integral in the second line for each z. Taking limit in both sides of the inequality and applying the Helly-Bray Theorem again the complementary inequality to (10) follows

$$\lim_{n\to\infty} J_n(H_n,T,u,y) \leq J(t,T,u,y)$$

completing the proof. □

4 The time independent recursive algorithm

Mind that an Erlang random variable is defined as the distribution of the sum of iid exponential random variables. The complication of treating a series of convergent random horizons is compensated by the advantage of the memoryless property of the individual exponential horizons. This simplifies the optimal control problem since the dependence on time within each horizon is eliminated. If the problem can be treated in exponential horizon, appending such horizons the Erlangian horizon is reproduced and the convergence argument applied to approximate the solution of a fixed horizon problem.

As a first step, the maximisation problem (3) will be considered in a hypothetical exponential backwards random time H_1 again independent of (U_s, Y_s, W_s) and $\mathrm{E}\left(H_1\right) = T - t$. Let now Υ_1^σ be the performance criterion to be maximised in the mentioned random backwards exponential horizon H_1 with terminal time T

$$\begin{aligned}\Upsilon_1^\sigma(H_1, T, u, \delta_i) \equiv \mathrm{E}_{H_1}\left[v^\sigma(T - H_1, T, u, \delta_i)\right] = \mathrm{E}_{H_1}[\mathrm{E}\left[P(U_T, Y_T)\mathbb{I}_{\{\tau \geq T\}} + \right.\\ \left. + L(U_\tau, Y_\tau)\mathbb{I}_{\{T-H_1<\tau<T\}} | U_{T-H_1} = u, Y_{T-H_1} = \delta_i\right]]\end{aligned}$$

Notice that the first expectation pertains to the random horizon H_1 and admits the so-called Laplace-Carson transform representation

$$\Upsilon_1^\sigma\left(H_1, T, u, \delta_i\right) = \int_0^\infty v^\sigma\left(T - z, T, u, \delta_i\right) \alpha \mathrm{e}^{-\alpha z} \mathrm{d}z$$

and this fact will be now exploited to approximate the solution of the problem (11).

The optimisation problem is now given by

$$J_1(H_1, T, u, \delta_i) \equiv \max_{\sigma \in \Pi} \Upsilon_1^\sigma\left(H_1, T, u, \delta_i\right). \tag{11}$$

One should notice at this stage that, using basic properties of the Laplace transform, the Hamilton-Jacobi-Bellman equation

$$\sup_{\sigma \in \Pi}\left\{\mathcal{A}^\sigma J_1(H_1, T, u, \delta_i) + \left(\frac{1}{T-t}\right) J_1(H_1, T, u, \delta_i) - \left(\frac{1}{T-t}\right) P\left(u, \delta_i\right)\right\} = 0 \tag{12}$$

actually embeds an ODE.

The next theorem presents the basis for the iterative procedure to actually evaluate the value function in the Erlangian horizon. In order to state the argument formally we need to introduce the following notation

$$\begin{aligned} v^\sigma(t, T, u, \delta_i, P) &\equiv v^\sigma(t, T, u, \delta_i) \\ J_n(H_n, T, u, \delta_i, P) &\equiv J_n(H_n, T, u, \delta_i) \\ \Upsilon_n^\sigma(H_n, T, u, \delta_i, P) &\equiv \Upsilon_n^\sigma(H_n, T, u, \delta_i) \end{aligned}$$

whenever the pay-off function P needs to be specified explicitly.

Theorem 2. *For integers $n \geq 2$*

$$J_n(H_n, T, u, \delta_i, P) = J_1(H_1, T, u, \delta_i, J_{n-1}) \tag{13}$$

Proof. It is assumed that the n iid exponential times composing H_n are all observable and also independent of (U_s, Y_s, W_s). Let $H_n = \vartheta + K_{n-1}$, where $\vartheta \sim \mathrm{Exp}(n/(T-t))$ and $K_{n-1} \sim \mathrm{Er}(n/(T-t), n-1)$ also assumed independent each other.

Conditioning on times ϑ and K_{n-1}

$$\begin{aligned}
&\max_{\sigma_1,\ \sigma_{n-1}\in\Pi} \mathrm{E}_\vartheta \mathrm{E}_{K_{n-1}}\ \mathrm{E}\left[v^{\sigma_{n-1}}(T-K_{n-1}, T, U_{T-K_{n-1}}, Y_{T-K_{n-1}}, P)\mathbb{I}_{\{T-K_{n-1}\leq\tau\}}\right.\\
&\qquad \left. + L(U_\tau, T_\tau)\mathbb{I}_{\{\tau<T-K_{n-1}\}} \mid U_{\vartheta+K_{n-1}} = u, Y_{\vartheta+K_{n-1}} = \delta_i\right]\\
&= \max_{\sigma_1,\ \sigma_{n-1}\in\Pi}\ \mathrm{E}_\vartheta\ \mathrm{E}_{K_{n-1}} v^{\sigma_1}(T-(\vartheta+K_{n-1}), T-K_{n-1}, u, \delta_i, v^{\sigma_{n-1}}(T-K_{n-1}, T, U_{T-K_{n-1}}, Y_{T-K_{n-1}}, P))\\
&= \max_{\sigma_1\ \in\Pi} \mathrm{E}_{K_{n-1}} \Upsilon^{\sigma_1}(T-K_{n-1}, T-K_{n-1}, u, \delta_i, v^{\sigma^*_{n-1}}(T-K_{n-1}, T, U_{T-K_{n-1}}, Y_{T-K_{n-1}}, P))\\
&= \mathrm{E}_{K_{n-1}} \Upsilon^{\sigma^*_1}(T-K_{n-1}, T-K_{n-1}, u, \delta_i, v^{\sigma^*_{n-1}}(T-K_{n-1}, T, U_{T-K_{n-1}}, Y_{T-K_{n-1}}, P))\\
&= J_1(H_1, T, u, \delta_i, J_{n-1})
\end{aligned}$$

using optimal controls σ^*_1 and σ^*_{n-1}. Notice that

$$\begin{aligned}
&J_n(H_n, T, u, \delta_i, P)\\
=\ &\mathrm{E}_{K_{n-1}} \Upsilon^{\sigma^*_n}(T-K_{n-1}, T-K_{n-1}, u, \delta_i, v^{\sigma^*_n}(T-K_{n-1}, T, U_{T-K_{n-1}}, Y_{T-K_{n-1}}, P))\\
\leq\ &\mathrm{E}_{K_{n-1}} \Upsilon^{\sigma^*_1}(T-K_{n-1}, T-K_{n-1}, u, \delta_i, v^{\sigma^*_{n-1}}(T-K_{n-1}, T, U_{T-K_{n-1}}, Y_{T-K_{n-1}}, P))
\end{aligned}$$

and also since Markov controls are used

$$\begin{aligned}
&\mathrm{E}_{K_{n-1}} \Upsilon^{\sigma^*_1}(T-(\vartheta+K_{n-1}), T-K_{n-1}, u, \delta_i, v^{\sigma^*_{n-1}}(T-K_{n-1}, T, U_{T-K_{n-1}}, Y_{T-K_{n-1}}, P))\\
=\ &\mathrm{E}_{K_{n-1}} \Upsilon^{\sigma^*_1}(T-(\vartheta+K_{n-1}), T-K_{n-1}, u, \delta_i, v^{\sigma^*_n}(T-K_{n-1}, T, U_{T-K_{n-1}}, Y_{T-K_{n-1}}, P))\\
=\ &\Upsilon^{\sigma'}_n(H_n, T, u, \delta_i, P) \leq J_n(H_n, T, u, \delta_i, P)
\end{aligned}$$

where

$$\sigma' = \begin{Bmatrix} \sigma^*_1 & t \in (T-H_n, T-K_{n-1}) \\ \sigma^*_n & t \geq T-K_{n-1} \end{Bmatrix}$$

Finally, by a time shift and the time homogeneous nature of the process

$$J_n(H_n, T, u, \delta_i, P) = J_1(H_1, T, u, \delta_i, J_{n-1})$$

because first the optimal expected penalty-reward is calculated at $(T-K_{n-1}) - \vartheta = T - H_n$ and the subsequent reward function involves an initial time at $T-K_{n-1}$ and termination at T. □

Theorem (13) shows that once the stochastic control problem can be treated in an exponential horizon, the Erlangian horizon (as an approximation to the fixed horizon) can be recovered by iteratively updating the utility function P_n involved in the performance criterion.

Notice again that the optimal controls may be now obtained using the simplified HJB in (12) with the clear advantage of dealing with value functions that satisfy an ODE instead of a PDE. But in despite of this theoretical simplificacion, the practical use of this result to solve optimization problems is still argueable.

An alternative approach is presented in the next section to direct numerical approximation of the initial problem and, at the same time, avoiding the mathematical complexity of the analytical approach.

5 Erlangian recursion using piecewise exponential controls

The Laplace-Carson transform of the horizon $T-t$ introduced in the previous section turned the problem of optimisation in a fixed horizon into the series of optimisation problems in exponential horizons. This is arguably easier to solve in scenarios where integral transform provides explicit solutions to treated problems.

In this section the approximation of the value function will be taken further with the aim to simplify even more the optimisation problems to be solved in each step of the recursive algorithm. As before, $T-t$ is aproximated by a random horizon

$$H_n \sim \mathrm{Er}\left(n/\left(T-t\right), n\right)$$

that converges to $T-t$ with increasing n.

This results in elimination of the time dependence of the optimisation problem. Now, moreover, the control that in principle is an adapted process evolving in time, will be restricted to a piecewise constant process (constant on each exponential horizon composing H_n). By intuition, since the length of each exponential interval is infinitesimal with probability1 as n increases, the optimisation on restricted set of controls will converge to the unrestricted one, therefore the convergence of the procedure outlined earlier is not compromised. Theorem 3 in this section proves this idea formally.

We will introduce some necessary notation. Let $\overline{J}_1$ be the value function of the problem in exponential horizon restricted to a piecewise constant control

$$\overline{J}_1(H_1, T, u, \delta_i) \equiv \max_{\sigma} \Upsilon^{\sigma}\left(H_1, T, u, \delta_i\right) \tag{14}$$

Notice that the optimal (constant) control will still depend on the values u, δ_i at the beginning of the exponential horizon and on the intensity α. In a strict sense one should write $\sigma_{\alpha,u,\delta_i}$, however, subscripts are removed for the sake of simplicity in the notation.

Assuming that controls are restricted to be constant on each exponential interval composing the Erlangian horizon H_n, the value function of the restricted problem is denoted $\overline{J}_n$. The set of all piecewise constant controls is $\overline{\Pi}$ and

$$\overline{J}_n(H_n, T, u, \delta_i) \equiv \max_{\sigma \in \overline{\Pi}} \Upsilon_n^{\sigma}(H_n, T, u, \delta_i). \tag{15}$$

Theorem 3. *Let $J(t,T,u,y)$ be the value function (3) and $\overline{J}_n(H_n,T,u,\delta_i)$ the value function (15) then for each $u \geq 0$ and δ_i*

$$\lim_{n\to\infty} \overline{J}_n(H_n,T,u,\delta_i) = J(t,T,u,\delta_i)$$

Proof. Let V_s^n be a Poisson process, independent of (U_s, Y_s) with intensity $\dfrac{n}{T-t}$, let $T_1,\ldots,T_n$ be the first n jump times of V_s^n and define process $(\overline{U}_s^n, \overline{Y}_s^n) = (U_{T_k}, Y_{T_k})$ where $k = \max\{i : T_i < s\}$. That is $(\overline{U}_s, \overline{Y}_s)$ is a process that remains constant in exponential intervals $[T_i, T_{i+1})$ and equal to the value of the process (U_s, Y_s) at the beginning of each interval.

Let us consider the process U_s^n with constant controls in exponential intervals

$$\begin{aligned} \mathrm{d}U_s^n &= a(U_s^n, Y_s, \sigma_t(\overline{U}_s^n, \overline{Y}_s^n))\mathrm{d}s + b(U_s^n, Y_s, \sigma_s(\overline{U}_s^n, \overline{Y}_s^n))\mathrm{d}W_s \\ &\quad + \int_{\mathbb{R}^k} \gamma(U_{s^-}^n, Y_{s^-}, \sigma_{s^-}(\overline{U}_s^n, \overline{Y}_s^n), z)N(\mathrm{d}s, \mathrm{d}z) \\ U_0^n &= u. \end{aligned} \tag{16}$$

and applying theorem 6.9 from Jacod and Shiryaev (2002, pg. 578) yields $U_s^n \to U_s$ in law.

To verify the assumptions of the theorem one needs to check that $\overline{U}_s^n \to U_s$ and $\overline{Y}_s^n \to Y_s$ in law. For that purpose notice that $\dfrac{V_s^n}{n}$ converges in law to $\dfrac{s}{T-t}$ and the equation

$$\overline{U}_s^n = U_0 + \frac{n}{T-t}\int_0^s (U_{s-}^n - \overline{U}_{s-}^n)\mathrm{d}s$$

has the solution

$$\overline{U}_s^n = \mathrm{e}^{\frac{ns}{T-t}} U_s^n - \mathrm{e}^{-\frac{ns}{T-t}} \int_0^s \mathrm{e}^{\frac{nr}{T-t}} U_r^n \mathrm{d}r \to U_s^n \quad \text{as} \quad n \to \infty.$$

To complete the proof, observe that $\overline{J}_n$ and J are continuous functionals of U_s^n and U_s respectively, what yields the convergence (see i.e. Proposition 3.8 Jacod and Shiryaev (2002, pg.348)). The same reasoning yields the convergence of $\overline{Y}_s^n$. □

The next corollary provides an iterative scheme to actually evaluate the value function $\overline{J}_n$ based on $\overline{J}_1$.

Corollary 1. *Let P be a reward function, u the initial condition, $\alpha > 0$ a real parameter. For every natural $k \geq 2$*

$$\overline{J}_k(H_k, T, u, \delta_i, P) = \overline{J}_1(H_1, T, u, \delta_i, \overline{J}_{k-1}) \tag{17}$$

Proof. Notice that the optimisation problem for $\overline{J}_k$ is the same as for J_k only with respect to different set of admissible controls. Since the set of controls is not relevant in the proof of the previous theorem the result follows. □

Corollary 1 shows that in order to evaluate the value function $\overline{J}_n$ of the optimisation problem in an Erlangian horizon it is sufficient to be able to evaluate the value function of the problem in exponential horizon $\overline{J}_1$ restricted to constant control. This calculation is iterated n times while in each step the value function of the previous step becomes the reward function of the next one.

The value function $\overline{J}_1$ is the basic building block of the method. Since in each exponential horizon the control is constant, it can be treated as a parameter and $\Upsilon_1^\sigma\left(H_1, T, u, \delta_i\right)$ can be obtained from the Laplace-Carson transform of the usual Fokker-Planck equation (see i.e. Risken (1996))

$$\mathcal{A}^\sigma \Upsilon^\sigma\left(H_1, T, u, \delta_i\right) + \alpha \Upsilon^\sigma(H_1, T, u, \delta_i) - \alpha P(u, \delta_i) = 0. \tag{18}$$

Posterior maximisation with respect to σ yields $\overline{J}_1$. Notice that the maximisation is a standard optimisation problem on real numbers.

The simplicity of the equation (18) to be solved, compared to system (6) and (7), is the main advantage of the method that together with Corollary (17) provides a semi-analytic treatment of the stochastic control problem presented.

6 Numerical illustration

In this section we will illustrate the application of the theorems proved above. The risk process considered follows

$$\mathrm{d}R_t = c\mathrm{d}t + \rho \mathrm{d}W_t - \mathrm{d}X_t, \qquad R_0 = u \tag{19}$$

where X_t is a compound Poisson process with intensity $\lambda = \frac{1}{3}$ and lognormal claim size distribution $\mathcal{LN}(1, 2)$. This process represents claims collected at a constant rate $c = 3$ perturbed by a diffusion with volatility $\rho^2 = 0.25$ that can be interpreted as aggregate small claims and claims collection accruals. The Poisson process then represents catastrophic claims (with average occurrence once every 3 periods) with lognormal (heavy-tail) severity distribution. The investment opportunities will be represented by a riskless asset $\mathrm{d}S_t^{(1)} = r\mathrm{d}t$ and a risky asset $\mathrm{d}S_t^{(2)} = \nu \mathrm{d}t + \xi \mathrm{d}W_t$. The proportion invested into a risky asset will be denoted as π. No short-selling is allowed, therefore $\pi \in [0, 1]$. The parameters of available assets are taken as follows $r = 1, \nu = 2, \xi = 1$. Altogether, the reserve process, including investment, can be written as

$$\mathrm{d}U_t = \left[c + (r + \pi(\nu - r))U_t\right] \mathrm{d}t + \sqrt{\rho^2 + \pi^2 \xi^2 U_t^2} \mathrm{d}W_t - \mathrm{d}X_t \qquad U_0 = u \tag{20}$$

This implies the following linear relationship between the volatility $\sigma = \pi\xi$ and expected return on investment $\mu\sigma$

$$\mu(\sigma) = r + \pi(\nu - r) = r + \sigma \frac{\nu - r}{\xi}. \tag{21}$$

Notice that no Markov modulation is considered in this example (what is equivalent to taking Y_t = constant) as no further insight would be added besides

more complex notation. The penalty-reward function considered is $P(u) = 1$, $L(u) = 0$

$$v^{\sigma}(T, u) = \mathrm{E}\left[\mathbb{I}_{\{\tau \geq T\}} | U_0 = u\right] = \mathrm{P}\left[\tau \geq T | U_0 = u\right] \equiv \varphi(u, T) \tag{22}$$

what represents the survival probability. The optimisation problem (3) then turns to maximisation of survival probability in a fixed horizon T. Similar problems have been treated in Hipp and Plum (2003) and others but no closed form solution exist. Following the development presented above, in order to be able to apply the iterative scheme from Theorem 2, the fixed horizon T will be approximated by a series of n exponential horizons with parameter $\frac{n}{T}$

$$\varphi^*(u, \alpha) = \int_0^{\infty} \varphi(u, T) \alpha \mathrm{e}^{-\alpha T} \mathrm{d}t. \tag{23}$$

The fixed horizon T will be approximated by a series of n exponential horizons with parameter $\frac{n}{T}$, whereas in each of the horizons the problem to be solved is

$$J_i(\frac{n}{T}, u) \equiv \max_{\pi \in \Pi} \Upsilon^{\sigma}(\alpha, u, J_{i-1}). \tag{24}$$

with $J_0 = P = 1$. As proved in the Theorem 3, to achieve convergence, it is sufficient to consider strategies π constant on each exponential interval. The function Υ for a constant π satisfies satisfies the following integro-differential equation

$$\begin{aligned} \frac{1}{2}(\rho^2 + \pi^2 \xi^2 u^2) \frac{\partial^2}{\partial u^2} \Upsilon + (c + (r + \pi(\nu - r))u) \frac{\partial}{\partial u} \Upsilon \\ - (\lambda + \alpha)\Upsilon + \lambda \int_0^u \Upsilon(\alpha, u - x) f(x) \mathrm{d}x + \alpha J_{i-1}(u) = 0. \end{aligned} \tag{25}$$

as derived in Diko and Usábel (2011) an approximation method by chebyshev polynomials is used to calculate the solution to this problem. Since feasible strategies are bounded it is possible to evaluate Υ for a grid of possible values of $\pi \in [0, 1]$ and take the maximum value as an approximation to the solution of (24). In this example we took equidistant grid of granularity 0.1. The table 1 shows the results of approximated value function $J(u, T)$ for various values of initial reserves u and number of exponential intervals n that approximate the fixed horizon $T = 10$. The convergence is achieved to up to 3 decimal places for as few as 100 intervals.

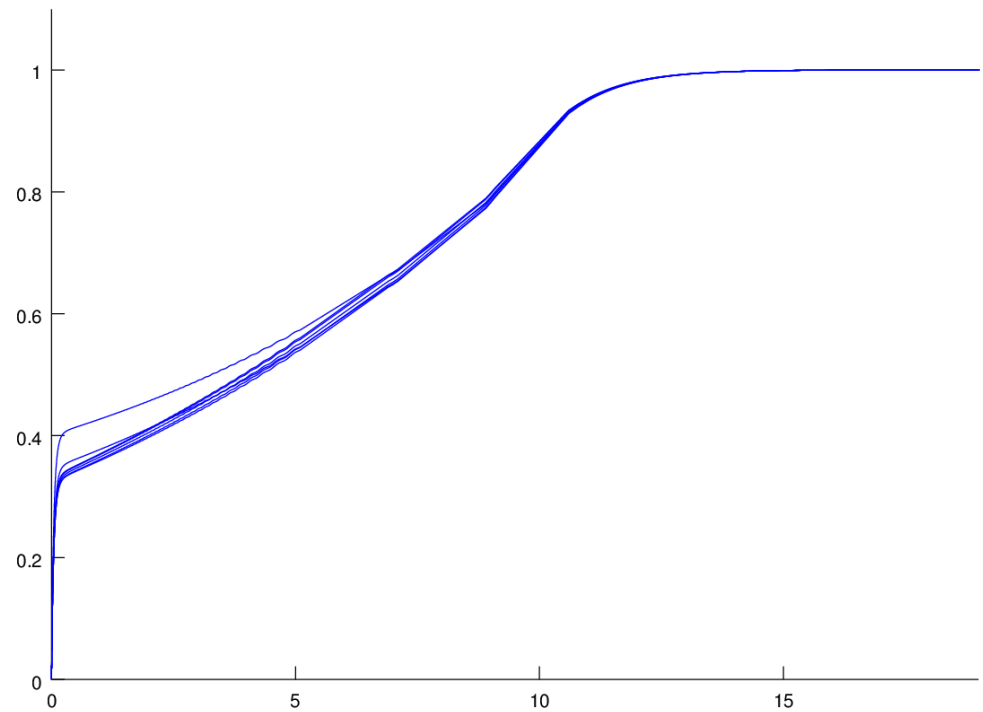

Figure 1: Convergence of the maximal survival probability in the horizon $T = 10$ as a function of the initial reserve.

n – number of intervals

u	1	2	5	10	20	50	100
0.1	0.354370	0.306438	0.287529	0.288241	0.291537	0.295415	0.297638
0.5	0.413865	0.361477	0.341409	0.341409	0.346224	0.349630	0.350752
1	0.427487	0.377567	0.359446	0.361308	0.365365	0.369168	0.370406
2	0.456898	0.412100	0.397678	0.400614	0.405446	0.410055	0.411589
5	0.570571	0.542250	0.536840	0.541189	0.547234	0.554050	0.556663
10	0.886121	0.882775	0.883860	0.885951	0.888517	0.891156	0.891655
15	0.999024	0.998997	0.999008	0.999027	0.999049	0.999072	0.999074

Figure 1 depicts $J(u, T)$ (maximal survival probability in horizon $T = 10$) as a function of u for $n = 1, 2, 5, 10, 20, 50, 100$. The optimal strategy that leads to the value function can be recovered using the relationship between the value function and the optimal strategy.

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