Market value margin via mean-variance hedging

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Abstract

We use mean-variance hedging in discrete time, in order to value a terminal insurance liability. The prediction of the liability is decomposed into claims development results, that is, yearly deteriorations in its conditional expected value. We assume the existence of a tradeable derivative with binary pay-off, written on the claims development result and available in each period. In simple scenarios, the resulting valuation formulas become very similar to regulatory cost-of-capital-based formulas. However, adoption of the mean-variance framework improves upon the regulatory approach, by allowing for potential calibration to observed market prices, inclusion of other tradeable assets, and consistent extension to multiple periods. Furthermore, it is shown that the hedging strategy can also lead to increased capital efficiency and consistency of market valuation with Euler-type capital allocations.

Keywords: Cost-of-capital, market consistent valuation, market value margin, mean-variance hedging, Solvency II.

1 Introduction

Market consistent valuation of insurance liabilities is a fundamental feature of the new regulatory landscape, as exemplified by the Swiss Solvency Test (Swiss Solvency Test [25]) and Solvency II (European Commission [12]). Broadly speaking, regulatory valuation techniques distinguish between liabilities that can be replicated in deep and liquid markets and liabilities for which this is not possible. For the former type of liability, following standard financial arguments, the market value equals the initial cost of the replicating portfolio. For the latter, the market value is postulated as the sum of the expected present value of the liabilities and a *Market Value Margin* that is set using cost-of-capital arguments. Thus, an explicit link is induced between capital assessment and valuation for regulatory purposes.

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The application of the above regulatory principle is not straightforward. First, the cost-of-capital rate used is a rather arbitrary, exogenously specified constant figure. Second, liabilities cannot be readily classified as perfectly replicable or completely non-replicable. It is usually the case that a liability can only be partly hedged and it is not entirely clear how the regulatory valuation approach should proceed in this case. A recent effort to reconcile cost-of-capital principles with replication arguments is given by Möhr [17], who obtains Solvency II valuation formulas as a special case. Third, given the long-term nature of many insurance liabilities, it is not clear what a multi-period extension of cost-of-capital valuation principles should be. Salzmann and Wüthrich [22] investigate alternative multi-period versions of cost-of-capital valuation and show that conceptually consistent approaches generally become computationally expensive.

In this paper, valuation via mean-variance hedging of liabilities in discrete time is proposed; in particular we use the solution in terms of sequential regression of Černý and Kallsen [4]. Mean-variance hedging identifies the self-financing trading strategy that minimises the quadratic deviation between the investment portfolio and the insurance liability at maturity. The general theory of mean-variance hedging is surveyed by Schweizer [23]. For more recent developments we refer the reader to Černý and Kallsen [3] and references therein. Insurance applications of mean-variance hedging have been more common in life insurance, where products demonstrate a higher dependence on instruments traded in financial markets, see Thomson [27], Dahl and Møller [6], and Delong [8]. Application to non-life insurance (see Delong and Gerrard [8]) is less frequent, due to the greater difficulty in identifying suitable tradeable instruments.

However, the development of markets in insurance-linked securities, such as cat-bonds, provides the possibility of at least partially hedging (non-life) insurance portfolios that are exposed to specific risks, such as those arising from natural catastrophes. We do not review the large literature on such securities and their markets here, but refer to Doherty [10] and Cummins [5]. Indicatively we mention the progress that has been made in understanding the statistical behaviour of cat-bond prices (Papachristou [20]) and attempts to derive reinsurance prices that are consistent with observed cat-bond prices (Haslip and Kaishev [14]).

Mean-variance hedging is related to other incomplete market pricing methods that have been applied to insurance. In particular, the discrete-time risk-minimization approach of Föllmer and Schweizer [13] determines locally optimal trading strategies that are not necessarily self-financing, as they allow the injection of capital at fixed times. For insurance applications of risk-minimizing hedging strategies in continuous time, see Møller [18, 19]. An alternative approach has been to derive, via indifference arguments, market-consistent versions of traditional actuarial premium calculations rules, such as the variance and standard deviation premium principles, see Schweizer [24].

We adopt a simple mean-variance hedging framework that considers a terminal liability. Motivated from stochastic reserving in non-life insurance, the liability is decomposed into its expected value and yearly *claims development results* representing yearly deteriorations (or im-

provements) in the liability's prediction using conditional expected values (Merz and Wüthrich [16]). The liability can be partially replicated by a tradeable instrument that pays 1 monetary unit in each period, in the case that the claims development result exceeds a given threshold. Thus, investing in the derivative is a form of buying protection, loosely equivalent to issuing simple 1-year bonds that are subordinated to the insurance liabilities. In addition, we allow for the possibility of investing in a risk-free bank account and in a number of stocks.

We find that in the simple case of a 1-period model and only the derivative being tradeable, the valuation formula derived is a convex combination of the liability's expected value and its Tail-Value-at-Risk (TVaR) measure with security level defined by the derivative's threshold. Thus, the formula obtained is structurally very similar to the regulatory valuation formulas of the Swiss Solvency Test and Solvency II. However, the mean-variance hedging framework allows for several extensions, addressing the shortcomings of regulatory approaches. So, the multiplier of TVaR used (resembling in function a cost-of-capital rate) is now dependent on potentially observable market prices. Moreover, the ability to trade in stocks that may be correlated to the claims development results, allows us to deal consistently with partial replication. Finally, the framework of Černý and Kallsen [4] facilitates development of transparent and conceptually consistent multi-period valuation formulas, akin to those of Föllmer and Schweizer [13].

Our analysis predicates on the assumption that a derivative as described above can be traded or indeed (its complement) issued by the liability holder. This assumption will not always hold, but we believe that our analysis retains its significance. On the one hand, we show that the valuation formulas can be seen as worst-case scenario valuations, over the set of similar derivatives written on risks that are only partially correlated with the liability at hand (eg index-triggered cat-bonds). On the other hand, the similarity between regulatory formulas and the ones presented here can be exploited to illustrate a potential set of alternative assumptions that underlies the regulatory cost-of-capital approach. Thus, if regulation imposes a rather ad hoc approach to valuing illiquid liabilities, we demonstrate what a possible conceptual foundation of that approach may be.

In Section 2 the simple single-period and single-asset case is introduced and the corresponding valuation formulas are derived. We also show that the hedging approach used may lead to a more efficient use of capital, which is a positive side effect of the replication strategy. Furthermore, the link between valuing sub-portfolios and performing Euler-type capital allocations (Tasche [26]) is explained. In Section 3, general results for the multi-period and multi-asset case are presented and valuation formulas are derived for specific cases. Simple numerical examples illustrate the analysis. Finally, brief conclusions are given in Section 4.

Throughout the paper we assume that the (conditional) second moments of all random variables considered exist.

2 1-period and 1-asset case

2.1 Preliminaries

We start with a toy model to illustrate the key ideas of the paper. A single-period model is considered, with two time points $t = 0$ and $t = 1$. There is an insurance liability $H \geq 0$ that has to be met at time $t = 1$. At time $t = 0$, the insurer of H invests a total amount of v monetary units in order to replicate this insurance liability H as closely as possible. All assets and liabilities are considered in discounted units and there are two tradeable assets: a risk-free asset with price 1 at time $t = 0$ and payoff 1 at time $t = 1$ and a risky asset with price S_0 at $t = 0$ and payoff S_1 at $t = 1$.

ϑ_1 risky assets at cost $\vartheta_1 S_0$ are purchased at time $t = 0$ and the remainder of the initial wealth $v - \vartheta_1 S_0$ is invested in the risk-free asset. This asset portfolio generates value at time 1 given by

$$(v - \vartheta_1 S_0)1 + \vartheta_1 S_1 = v + \vartheta_1 \Delta S_1,$$

where $\Delta S_1 = S_1 - S_0$.

The optimal, with respect to a quadratic loss function, initial wealth V_0 and investment ξ_1 in the risky asset are calculated by minimizing the quadratic deviation between the liability and the asset portfolio's pay-off. That is,

$$(V_0, \xi_1) = \arg\min_{(v,\vartheta_1)} \mathbb{E}\left((v + \vartheta_1 \Delta S_1 - H)^2\right). \tag{1}$$

Since V_0 corresponds to the initial cost of replicating H as closely as possible w.r.t. the quadratic loss, we will throughout this paper identify V_0 with the market consistent value of H at time 0.

Standard arguments yield the solution to optimisation problem (1):

$$\xi_1 = \frac{\mathrm{Cov}(\Delta S_1, H)}{\mathrm{Var}(\Delta S_1)}, \tag{2}$$

$$V_0 = \mathbb{E}(H) - \frac{\mathrm{Cov}(\Delta S_1, H)}{\mathrm{Var}(\Delta S_1)}\, \mathbb{E}(\Delta S_1). \tag{3}$$

In this simple setting, the value of V_0 reflects the capital asset pricing model (CAPM) price of H, where the risky asset with pay-off S_1 plays the role of the market portfolio. The optimal portfolio has quadratic loss

$$\mathbb{E}\left((V_0 + \xi_1 \Delta S_1 - H)^2\right) = \mathrm{Var}(H) - \frac{\mathrm{Cov}(\Delta S_1, H)^2}{\mathrm{Var}(\Delta S_1)} \leq \mathrm{Var}(H). \tag{4}$$

We observe that for non-zero correlation between ΔS_1 and H, the quadratic loss is reduced, since part of the risk H can be hedged by the allocation of ξ_1 in the risky asset.

We study this problem for a particular choice of the risky asset in the following sections.

In the sequel, the risk measures Value-at-Risk (VaR) and Tail-Value-at-Risk (TVaR) are used extensively. For a random variable Z with continuous distribution function and a security

level $\alpha \in (0,1)$ they are defined in the common way, see for instance McNeil et al. [15],

$$\mathrm{VaR}_\alpha(Z) = \inf\{z \in \mathbb{R} : \mathbb{P}(Z \leq z) \geq \alpha\}, \tag{5}$$

$$\mathrm{TVaR}_\alpha(Z) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\beta(Z)d\beta = \frac{1}{1-\alpha}\mathbb{E}\left(\mathbf{1}_{\{Z \geq \mathrm{VaR}_\alpha(Z)\}}Z\right). \tag{6}$$

For the latter identity to hold true we need to assume that Z has a continuous distribution function, see Lemma 2.16 in McNeil et al. [15]. Note that for VaR the security (confidence) level α corresponds to the default probability $1-\alpha$.

2.2 Valuation formulas

Now a particular choice for the risky asset is made. Define

$$S_1 = \mathbf{1}_{D_1} \qquad \text{with} \qquad D_1 = \{Z_1 \geq d_1\}. \tag{7}$$

Hence, D_1 is the event that a random variable Z_1 equals or exceeds a specified threshold d_1. Choose $p_1 = \mathbb{E}(S_1) = \mathbb{P}(D_1) \in (0,1)$ and assume $S_0 = q_1 \in (p_1, 1)$.

There are two ways of considering such pay-offs in an insurance market. First, S_1 may be the pay-off from an index-linked insurance derivative (such as a weather or longevity derivative, or an industry loss-triggered cat-bond), with Z_1 playing the role of the relevant index. The derivative considered will be such that Z_1 is a reasonable proxy for the liability H, hence we require that Z_1 is positively correlated with H. In particular, the risky asset S_1 pays a positive return on the event D_1 associated with a large loss. Probability p_1 then is the real-world probability of such an event and q_1 is its risk-neutral probability implied by market prices. The choice $q_1 > p_1$ is explained by S_1 playing the role of reinsurance for large losses.

Alternatively, consider the case that the holder of the liability H sponsors a catastrophe bond, with D_1 being the triggering event. The bond structure is such that the holder of H pays 1 monetary unit at time 1 if D_1^c takes place and 0 units if D_1 occurs. Let $1-q_1$ be the price of the bond. Then, if the sponsor issues ϑ_1 of these bonds, the gains from the trade are

$$-\vartheta_1\left(\mathbf{1}_{D_1^c} - (1-q_1)\right) = -\vartheta_1\left(-\mathbf{1}_{D_1} + q_1\right) = \vartheta_1 \Delta S_1. \tag{8}$$

The solution to problem (1) now gives the optimal amount of bonds that should be issued. The bond will be constructed so as to maximise the correlation between the trigger Z_1 and the liability H. In particular, if an indemnity trigger is used, perfect positive correlation between Z_1 and H can be achieved, see Papachristou [20], otherwise so-called basis-risk remains.

For the tradeable instrument asset (7), equation (3) and the identity $\mathrm{Var}(S_1) = p_1(1-p_1)$ yield

$$V_0 = \mathbb{E}(H) + \frac{q_1 - p_1}{1-p_1}\left[\mathbb{E}\left(H | Z_1 \geq d_1\right) - \mathbb{E}(H)\right]. \tag{9}$$

Formula (9) can be further refined by choosing the particular trigger $Z_1 = H - \mathbb{E}(H)$. Thus, $S_1 = \mathbf{1}_{D_1}$ pays one unit when the insurance liability H exceeds its expected value plus threshold

d_1, corresponding to the case of an indemnity-triggered derivative, written specifically on the liability H. Also note that the threshold d_1 can now be interpreted as $d_1 = \mathrm{VaR}_{1-p_1}(H) - \mathbb{E}(H)$. Assuming that H has a continuous distribution function, formula (9) becomes

$$V_0 = \mathbb{E}(H) + \frac{q_1 - p_1}{1 - p_1}\left[\mathrm{TVaR}_{1-p_1}(H) - \mathbb{E}(H)\right], \tag{10}$$

where TVaR_{1-p_1} is the TVaR risk measure at security level $1 - p_1$. Therefore valuation takes place according to a simple premium rule: "expected value plus a percentage of the excess of TVaR over the expected value".

Even when the derivative is not indemnity-triggered (i.e. H and Z_1 are not perfectly correlated), (10) can still be interpreted as a conservative upper bound on the value given in (9). To see this observe that the vectors $(H, \mathbf{1}_{\{H>\mathbb{E}(H)+d_1\}})$ and $(H, \mathbf{1}_{\{Z>d_1\}})$ have the same marginals, but the elements of the former are comonotonic. Hence $\mathbb{E}(H\mathbf{1}_{\{H>\mathbb{E}(H)+d_1\}}) \geq \mathbb{E}(H\mathbf{1}_{\{Z_1>d_1\}})$ implies $\mathrm{TVaR}_{1-p_1}(H) \geq \mathbb{E}\left(H|Z_1 \geq d_1\right)$, where the first inequality follows from Proposition 6.2.6 in Denuit et al. [7].

For the rest of this paper, we will always identify Z_1 with $H - \mathbb{E}(H)$.

Formula (10) bears a close resemblance to valuation formulas used under solvency regimes such as Solvency II and the Swiss Solvency Test, where the market consistent value of a liability is set equal to its expected value plus a risk loading deriving from a cost-of-capital charge, see European Commission [12], Swiss Solvency Test [25], Pelsser [21] and Salzmann and Wüthrich [22]. If the regulator prescribes a regulatory risk measure ρ to support adverse events, the market consistent value under the cost-of-capital method equals

$$V_0^{\mathrm{CoC}} = \mathbb{E}(H) + \lambda\left[\rho(H) - \mathbb{E}(H)\right], \tag{11}$$

where λ is an exogenously specified *cost-of-capital rate*. In Solvency II jargon, the quantity $V_0^{\mathrm{CoC}} - \mathbb{E}(H) = \lambda\left[\rho(H) - \mathbb{E}(H)\right]$ is termed *market value margin* or cost-of-capital loading.

However, the valuation formulas (10) and (11) are different in several significant ways. We consider the Solvency II case and the Swiss Solvency Test case separately. The risk measure ρ under Solvency II is VaR at security level $\alpha = 99.5\%$. Let $p_1 = 1 - \alpha = 0.5\%$, such that $d_1 + \mathbb{E}(H) = \rho(H) = \mathrm{VaR}_\alpha(H)$ in formula (7). Then, both (10) and (11) are evaluated using the same security level $1 - p_1 = \alpha$. However, the market value margin $V_0 - \mathbb{E}(H)$ in (10) is calculated under the more conservative TVaR risk measure, compared to the cost-of-capital method (11).

On the other hand, the risk measure ρ in the Swiss Solvency Test is TVaR at security level $\alpha = 99\%$. If we let $p_1 = 1 - \alpha = 1\%$, then $\rho(H) = \mathrm{TVaR}_\alpha(H) \geq d_1 + \mathbb{E}(H) = \mathrm{VaR}_\alpha(H)$. Thus the risk measures used to derive the market value margin are the same, but now the threshold d_1 no longer corresponds to the value of the regulatory risk measure $\rho(H)$.

An advantage of the approach proposed here is that it dispenses with the need to specify an exogenously given cost-of-capital rate λ. This rate is replaced by $\frac{q_1-p_1}{1-p_1}$, which is a potentially observable quantity. In particular, for a fixed large loss probability $p_1 = \mathbb{P}(D_1)$, $V_0 - \mathbb{E}(H_1)$

increases with q_1, the large loss probability implied by market prices. This allows, at least in principle, calibration of formula (10) to the market prices of tradeable instruments. To see this, consider the interpretation of buying the derivative S_1 as the holder of H sponsoring a bond, sold at a price of $1-q_1$ and paying one unit in the event D_1^c, such that the pay-off is $\mathbf{1}_{D_1^c} = 1-S_1$. The expected rate of return for an investor in the bond is given by $r_B = \frac{\mathbb{E}[\mathbf{1}_{D_1^c}]}{1-q_1} - 1 = \frac{1-p_1}{1-q_1} - 1 > 0$ and we can write $\frac{q_1-p_1}{1-p_1} = 1 - \frac{1}{1+r_B}$. (Note here that, since we only consider discounted values, r_B corresponds to the spread above the risk-free rate.) Thus, investors' demand for a higher return on the debt is reflected by a higher risk loading in valuation formula (10).

Example 1. To illustrate these arguments, consider Papachristou [20], who performed statistical analysis of catastrophe bond spreads at the time of issue. In particular the behaviour of the "multiple" is studied, that is, the ratio of the spread to the annualised expected loss, which in our simple model can be identified with the probability p_1. It is found that the multiple tends to decrease in p_1, reflecting a higher risk premium for protection against extreme events. Furthermore, it is shown how the multiple changes with time and responds to insurance events; eg a rise is observed after Hurricane Katrina in 2006. For the period of 2003-2008 studied and the sample of bonds considered, the multiple for bonds with $p_1 = 1\%$ has tended to fluctuate between about 4 and 8. This implies that the spread r_B ranges from 4% to 8% and, consequently, the quantity $\frac{q_1-p_1}{1-p_1}$ varies from 3.8% to 7.4%. Interestingly, the range contains the exogenous cost-of-capital rate $\lambda = 6\%$ favoured by regulators. Hence, a rough illustrative analysis shows that the approach we propose would not produce radically different numbers to, say, valuation under the Swiss Solvency Test, though we now have the additional feature of the value reacting to changes in market conditions. □

Associating regulatory valuation formulas such as (11) to a mean-variance valuation framework brings further benefits. To start with, the formulas can be generalised to deal with liabilities over multiple periods, making use of quadratic hedging via sequential regression, as in Černý and Kallsen [4]. In contrast, it is conceptually not clear what a multi-period valuation formula based on cost-of-capital arguments should look like. Salzmann and Wüthrich [22] proposed a conceptually consistent approach, which presents substantial computational problems. Furthermore, the cost-of-capital approach is applied when no deep and liquid market exists for hedging insurance liabilities. When liabilities can be partially but not exactly hedged by traded assets, it is again not clear how to proceed. The quadratic framework easily allows inclusion of further tradeable assets that may improve hedging of the liability H. Such extensions are considered in Section 3.

2.3 Hedging and capital efficiency

The arguments presented above focused on deriving a risk-sensitive valuation formula, as an alternative to what is proposed in current insurance regulation. However, we did not consider

the change in portfolio risk after investment in the derivative with pay-off $S_1 = \mathbf{1}_{D_1}$. This is an issue worth considering, since the buyer of such a derivative would be interested in reducing the risk on the book and thus freeing up economic capital.

Let the solvency capital requirement be determined by a translation invariant risk measure ρ, such that $\rho(H - v) = \rho(H) - v$ for all $v \in \mathbb{R}$ and all random variables H under consideration (Axiom 6.1 in McNeil et al. [15]). Denote by G_1 the value of the optimal investment portfolio (V_0, ξ_1) at time 1, such that $G_1 = V_0 + \xi_1 \Delta S_1$. Then trading in the derivative frees up capital as long as the cost V_0 of the trading strategy and the capital requirement for the hedged loss $H - G_1$ add up to less than the capital requirement for the unhedged loss H:

$$V_0 + \rho(H - G_1) \leq \rho(H). \tag{12}$$

Using the translation invariance of ρ, we can rewrite the left-hand side of (12) as

$$V_0 + \rho(H - G_1) = \rho(H - (G_1 - V_0)) \leq \rho(H). \tag{13}$$

Noting that the portfolio that generates G_1 has initial price V_0, (13) states that investment in the portfolio can reduce the solvency capital requirement under the risk measure ρ. We implicitly make the assumption that the pair of assets we are allowed to invest in and the risk measure do not allow for acceptability arbitrage, see Artzner et al. [1], as otherwise any unacceptable position can be made acceptable using a sufficiently large short and long position.

Nonetheless, it is by no means obvious that inequality (12) will generally be satisfied, since the trading strategy is formulated to replicate the liability H as closely as possible in a quadratic norm and not specifically to minimise the capital requirement described by the risk measure ρ. It is now shown that for common tail risk measures used in regulation, the optimal investment strategy does indeed free up capital, if the risk measures used are focused on the extreme right tail and/or the derivative is not too expensive.

Proposition 1. *Assume H has a continuous and strictly increasing distribution function and let the risk measure ρ be either VaR_α or TVaR_α at security level $\alpha \in (0,1)$. Define $k = \frac{1}{1-p_1}\mathrm{TVaR}_{1-p_1}(H - \mathbb{E}(H))$. Then inequality (12) holds if and only if*

$$q_1 k \leq \rho(H) - \rho(H - \mathbf{1}_{D_1} k). \tag{14}$$

In particular, the following hold:

i) If $\alpha < 1 - p_1$ is small enough, such that $\mathrm{VaR}_\alpha(H) - \mathrm{VaR}_{1-p_1}(H) \leq -k$, then:

- *For $\rho \equiv \mathrm{VaR}_\alpha$, inequality (12) holds for no $q_1 \in (0,1)$.*
- *For $\rho \equiv \mathrm{TVaR}_\alpha$, inequality (12) holds for all $q_1 \in \left(\frac{p_1}{1-\alpha}, 1\right)$.*

ii) If $\alpha > 1 - p_1$ is large enough, such that $\mathrm{VaR}_\alpha(H) - \mathrm{VaR}_{1-p_1}(H) \geq k$, then, for either of $\rho \equiv \mathrm{VaR}_\alpha$ and $\rho \equiv \mathrm{TVaR}_\alpha$, inequality (12) holds for all $q_1 \in (p_1, 1)$ and the freed-up capital equals

$$\rho(H) - V_0 - \rho(H - G_1) = (1 - q_1)k. \tag{15}$$

Proof. Note that both VaR and TVaR are translation invariant and that $k > 0$ by the properties of TVaR (eg see Property 2.4.5 in Denuit et al. [7]). This allows to re-write (12) as (13). We have

$$G_1 - V_0 = \xi_1 \Delta S_1 = \frac{\mathrm{Cov}(\Delta S_1, H)}{\mathrm{Var}(\Delta S_1)} \Delta S_1 = \frac{\mathbf{1}_{D_1} - q_1}{1 - p_1} \mathrm{TVaR}_{1-p_1}(H - \mathbb{E}(H)) = (\mathbf{1}_{D_1} - q_1)k.$$

It follows that

$$\rho(H - (G_1 - V_0)) = \rho\left(H - (\mathbf{1}_{D_1} - q_1)k\right) = \rho\left(W\right) + q_1 k,$$

where $W = H - \mathbf{1}_{D_1} k$. Thus, the freed-up capital can be written as

$$\rho(H) - \rho(H - (G_1 - V_0)) = \rho(H) - \rho(W) - q_1 k,$$

so that to satisfy inequality (12) we need

$$q_1 k \leq \rho(H) - \rho(W). \tag{16}$$

Let $d = d_1 + \mathbb{E}(H)$ such that $D_1 \equiv \{H > d\}$ and $\mathrm{VaR}_{1-p_1}(H) = d$. Then we have for $w \in \mathbb{R}$

$$\begin{aligned}
\mathbb{P}\left(W \leq w\right) &= \mathbb{P}\left(W \leq w, D_1\right) + \mathbb{P}\left(W \leq w, D_1^c\right) \\
&= \mathbb{P}\left(H - k \leq w, H > d\right) + \mathbb{P}\left(H \leq w, H \leq d\right) \\
&= \mathbb{P}\left(d < H \leq w + k\right) + \mathbb{P}\left(H \leq \min\left\{w, d\right\}\right).
\end{aligned}$$

Let F be the distribution of H. It follows that the distribution of W is given by:

$$\mathbb{P}\left(W \leq w\right) = \begin{cases} F(w), & w \leq d - k \\ F(w) + F(w + k) - F(d), & d - k < w < d \\ F(w + k), & w \geq d. \end{cases}$$

It is easily seen that the distribution of W is continuous and strictly increasing. Moreover, when $\mathrm{VaR}_\alpha(H) \leq d - k$ (corresponding to case i) in the Proposition) it is $\mathrm{VaR}_\alpha(W) = \mathrm{VaR}_\alpha(H)$. On the other hand, when $\mathrm{VaR}_\alpha(H) \geq d + k$ (corresponding to case ii)), we have $\mathbb{P}(W \leq \mathrm{VaR}_\alpha(H) - k) = \mathbb{P}(H \leq \mathrm{VaR}_\alpha(H)) = \alpha$, such that $\mathrm{VaR}_\alpha(W) = \mathrm{VaR}_\alpha(H) - k$. We now deal with the two cases separately.

Case i) $\mathrm{VaR}_\alpha(H) \leq d - k$.

First let $\rho \equiv \mathrm{VaR}_\alpha$. Then $\mathrm{VaR}_\alpha(W) = \mathrm{VaR}_\alpha(H)$, such that condition (16) cannot be satisfied for any $q_1 > 0$.

Now let $\rho \equiv \mathrm{TVaR}_\alpha$. First observe that the vectors $(H, \mathbf{1}_{\{H>\mathrm{VaR}_\alpha(H)\}})$ and $(H, \mathbf{1}_{\{W>\mathrm{VaR}_\alpha(W)\}})$ have the same marginals, but the elements of the former are comonotonic. Hence

$$\mathbb{E}(H\mathbf{1}_{\{H>\mathrm{VaR}_\alpha(H)\}}) \geq \mathbb{E}(H\mathbf{1}_{\{W>\mathrm{VaR}_\alpha(W)\}}),$$

which follows from Proposition 6.2.6 in Denuit et al. [7]. Consider now

$$\begin{aligned}
\mathrm{TVaR}_\alpha(H) - \mathrm{TVaR}_\alpha(W) &= \frac{1}{1-\alpha}\left[\mathbb{E}(H\mathbf{1}_{\{H>\mathrm{VaR}_\alpha(H)\}})) - \mathbb{E}(W\mathbf{1}_{\{W>\mathrm{VaR}_\alpha(W)\}})\right] \\
&\geq \frac{1}{1-\alpha}\left[\mathbb{E}(H\mathbf{1}_{\{W>\mathrm{VaR}_\alpha(W)\}})) - \mathbb{E}(W\mathbf{1}_{\{W>\mathrm{VaR}_\alpha(W)\}})\right] \\
&= \frac{1}{1-\alpha}\mathbb{E}(\mathbf{1}_{D_1}k\mathbf{1}_{\{W>\mathrm{VaR}_\alpha(W)\}}) \\
&= \frac{1}{1-\alpha}k\mathbb{P}(H > d, W > \mathrm{VaR}_\alpha(W)) \\
&= \frac{1}{1-\alpha}k\mathbb{P}(H > \max\{d, k + \mathrm{VaR}_\alpha(W)\}) \\
&= \frac{1}{1-\alpha}k\mathbb{P}(H > \max\{d, k + \mathrm{VaR}_\alpha(H)\}) \\
&= \frac{1}{1-\alpha}k\mathbb{P}(H > d) = \frac{p_1}{1-\alpha}k.
\end{aligned}$$

Hence, by (16) it is sufficient to have $q_1 \leq \frac{p_1}{1-\alpha}$ for inequality (12) to hold.

Case ii) $\mathrm{VaR}_\alpha(H) \geq d + k$.

First let $\rho \equiv \mathrm{VaR}_\alpha$. Then $\mathrm{VaR}_\alpha(W) = \mathrm{VaR}_\alpha(H) - k$, such that the freed up capital equals

$$\mathrm{VaR}_\alpha(H) - \mathrm{VaR}_\alpha(W) - q_1 k = k - q_1 k,$$

which proves the stated result.

Now consider $\rho \equiv \mathrm{TVaR}_\alpha$. For $\beta \in [\alpha, 1)$ it is $\mathrm{VaR}_\beta(H) \geq \mathrm{VaR}_\alpha(H) \geq d + k$. Therefore $\mathrm{VaR}_\beta(W) = \mathrm{VaR}_\beta(H) - k$ and by the integral identity for TVaR given in (6) the freed up capital becomes

$$\begin{aligned}
\mathrm{TVaR}_\alpha(H) - \mathrm{TVaR}_\alpha(W) - q_1 k &= \mathrm{TVaR}_\alpha(H) - \frac{1}{1-\alpha}\int_\alpha^1 (\mathrm{VaR}_\beta(H) - k)d\beta - q_1 k \\
&= k - q_1 k,
\end{aligned}$$

which again is the stated result. □

Case i) of Proposition 1 refers to the case where the security level α is so low that the risk reduction effected by the derivative is not reflected in the VaR measure, due to that risk measure's insensitivity to the extreme tail of the distribution of H. Thus, investing any amount in the derivative incurs a cost, with no apparent benefit. When TVaR is used, the extreme tails are reflected in the risk measurement and the benefit from investing in the derivative is recognised, as long as the derivative is not too expensive (q_1 is not too high). On the other hand, case ii) refers to the situation where the security level α is very high, such that under all

scenarios considered by the risk measure, the derivative produces a pay-off of 1 monetary unit, which is always higher than the price $q_1 < 1$. Consequently, a capital saving is always produced. However, the freed-up capital, as seen in equation (15), depends on the price q_1. Thus, if q_1 is close to its lowest level p_1, there is no market risk premium for the derivative and the freed up capital is maximised. On the contrary, if q_1 is close to 1, the market considers the event of the derivative paying as nearly certain, such that the derivative becomes very expensive, and investing in it produces only a small capital reduction.

Of course, in many cases it will be $-k < \mathrm{VaR}_\alpha(H) - \mathrm{VaR}_{1-p_1}(H) < k$, a case not fully characterised in Proposition 1. The following numerical example shows that for realistic parameter choices, investment in the derivative will tend to be capital efficient.

Example 2. Let H be Log-normally distributed such that $\mathbb{E}(H) = 100,\ \mathrm{Var}(H) = 20^2$. We consider two cases of the derivative, with $p_1 = 0.01$ and $p_1 = 0.05$. For illustrative purposes, we follow again Papachristou [20], choosing for $p_1 = 0.01$ (resp. $p_1 = 0.05$) a multiple of 6 (resp. 4), leading to $q_1 = 0.066$ (resp. $q_1 = 0.208$).

Table 1: Valuation of a Log-normal liability with $\mathbb{E}(H) = 100,\ \mathrm{Var}(H) = 20^2$.

p_1	0.01	0.05
q_1	0.066	0.208
$\mathrm{TVaR}_{1-p_1}(H)$	166.56	147.95
$\frac{q_1-p_1}{1-p_1}$	0.057	0.167
V_0	103.77	107.99
k	67.23	50.47

Table 1 summarises the quantities needed for the valuation of the liability H according to (10), as well as the value $k = \frac{1}{1-p_1}\mathrm{TVaR}_{1-p_1}(H - \mathbb{E}(H))$ appearing in Proposition 1. While for $p_1 = 0.05$ the risk measure $\mathrm{TVaR}_{1-p_1}(H)$ is substantially lower, this is compensated by a higher ratio $\frac{q_1-p_1}{1-p_1}$, such that the liability H has a higher market value when a derivative with $p_1 = 0.05$ is available.

It is easy to check that for all security levels $\alpha \in [0.99, 0.999]$, a plausible range for regulatory risk measurement, it is $\mathrm{VaR}_\alpha(H) - \mathrm{VaR}_{1-p_1}(H) \in (0, k)$. From equation (14), the maximum value of the price q_1 that leads to freeing up capital is given by the relation $q_1 \leq \big(\rho(H) - \rho(H - \mathbf{1}_{D_1}k)\big)/k$, where $\rho \equiv \mathrm{VaR}_\alpha$ or $\rho \equiv \mathrm{TVaR}_\alpha$. The maximum such level of q_1 is plotted in Figure 1 against the security level α of the regulatory risk measure used, for $\alpha \in [0.99, 1)$.

It is seen that in each case the value of q_1 is well below the plotted curves, such that for the plausible range of security levels α, investment in the derivative indeed frees up capital. □

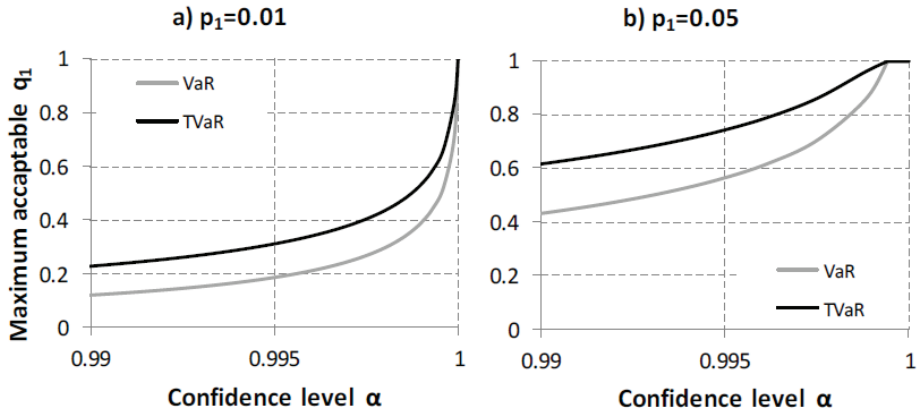

Figure 1: Maximum q_1 such that inequality (13) is satisfied.

2.4 Capital allocation and valuation of sub-portfolios

In general, the liability H will be a portfolio loss, consisting of m sub-portfolios $H^{(1)}, \ldots, H^{(m)}$, such that $H^{(1)} + \cdots + H^{(m)} = H$. It is then of interest to work out the market consistent value for each of the $H^{(i)}$s.

For this purpose, we first note that equation (3) can be written as

$$V_0 = \mathbb{E}^{\mathbb{Q}}(H), \quad \text{where} \quad \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\mathbb{E}(\Delta S_1^2) - \Delta S_1 \mathbb{E}(\Delta S_1)}{\mathrm{Var}(\Delta S_1)}. \tag{17}$$

$\mathbb{Q}$ is a (possibly signed) martingale measure; it is easy to check that $\mathbb{E}^{\mathbb{Q}}(\Delta S_1) = 0$ (for details see Černý and Kallsen [4]). In the particular case considered in this paper, where $\Delta S_1 = \mathbf{1}_{\{H \geq \mathbb{E}(H)+d_1\}} - q_1$, the change of measure is induced by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = 1 + \frac{q_1 - p_1}{1 - p_1} \left(\frac{1}{p_1} \mathbf{1}_{\{H \geq \mathbb{E}(H)+d_1\}} - 1 \right). \tag{18}$$

It is then natural to consider as the market value of the sub-portfolio $H^{(i)}$, the quantity $V_0^{(i)} = \mathbb{E}^{\mathbb{Q}}(H^{(i)})$. From (18) we obtain

$$V_0^{(i)} = \mathbb{E}(H^{(i)}) + \frac{q_1 - p_1}{1 - p_1} \left[\mathbb{E}(H^{(i)} | H \geq \mathbb{E}(H) + d_1) - \mathbb{E}(H^{(i)}) \right]. \tag{19}$$

From (19) it becomes apparent that the market value of $H^{(i)}$ is closely related to a TVaR-based capital allocation using the Euler method, see eg Tasche [26] or McNeil et al. [15], Section 6.3. Capital allocation is a process by which the total risk of a portfolio, as quantified by a risk measure, is allocated to its constituent parts. Assume again for simplicity that the distribution function of H is continuous and strictly increasing. The Euler allocation arises from a simple marginal cost argument, leading to an allocation of TVaR taking the form $\mathrm{TVaR}_{1-p_1}(H) = \sum_{i=1}^{m} \mathbb{E}(H^{(i)} | H \geq \mathrm{VaR}_{1-p_1}(H))$, where it is understood that $\mathrm{VaR}_{1-p_1}(H) = \mathbb{E}(H) + d_1$.

Specifically, the marginal cost argument works as follows. Denote $c(\epsilon) = \mathrm{TVaR}_{1-p_1}(H + \epsilon H^{(i)})$, for $\epsilon > 0$. Then, by equation (6.26) in McNeil et al. [15], the marginal risk contribution of $H^{(i)}$ is given by

$$\lim_{\epsilon \to 0} \frac{c(\epsilon) - c(0)}{\epsilon} = \mathbb{E}(H^{(i)} | H \geq \mathrm{VaR}_{1-p_1}(H)). \tag{20}$$

If valuation of the portfolio takes place using (11) under the $\rho \equiv \mathrm{TVaR}_{1-p_1}$ risk measure, then the Euler allocation of (20) gives a natural way to allocate the total capital $\mathrm{TVaR}_{1-p_1}(H)$ (and hence its cost) to sub-portfolios $H^{(1)}, \ldots, H^{(m)}$, such that the value of $H^{(i)}$ according to the cost-of-capital method is given by $\mathbb{E}(H^{(i)}) + \lambda \left[\mathbb{E}(H^{(i)} | H \geq \mathrm{VaR}_{1-p_1}(H)) - \mathbb{E}(H^{(i)})\right]$. The similarity to the market value $V_0^{(i)}$ obtained in (19) is again striking, reinforcing the previously observed parallels between our model and the regulatory cost-of-capital valuation approach.

3 The multi-period and multi-asset case

3.1 Preliminaries

We now extend the previous setup to a model with several assets traded over multiple time periods. We consider a finite time horizon $T \in \mathbb{N}$ and a finite set of trading dates $\mathcal{T} = \{0, 1, \ldots, T\}$. The filtered probability space is denoted by $(\Omega, \mathbb{P}, \mathcal{F}, \mathbb{F})$ with finite and discrete time filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}$ such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F} = \mathcal{F}_T$. The corresponding conditional expectations, variances, and covariances are denoted by $\mathbb{E}_t(X) = \mathbb{E}(X|\mathcal{F}_t)$, $\mathrm{Var}_t(X) = \mathbb{E}_t(X^2) - \mathbb{E}_t(X)^2$, $\mathrm{Cov}_t(X, Y) = \mathbb{E}_t(XY) - \mathbb{E}_t(X)\mathbb{E}_t(Y)$ for $t \in \mathcal{T}$.

The insurance liability considered is represented by a non-negative, $\mathcal{F}_T$-measurable, square-integrable random variable $H \in L^2(\mathbb{P}) = L^2(\Omega, \mathbb{P}, \mathcal{F}_T)$. Moreover, we assume that we have $n \in \mathbb{N}$ tradeable risky assets. Their price processes are represented by the n-dimensional, $\mathbb{F}$-adapted stochastic process $(S_t)_{t \in \mathcal{T}}$. Denote the elements of S_t by $S_t^{(i)}$, $i = 1, \ldots, n$, and the (one-period) returns in vector notation as $\Delta S_{t+1} = S_{t+1} - S_t$. It is assumed that all the conditional second moments of the asset price processes exist, that is, $\mathbb{E}_t(\Delta S_{t+1}^{(i)} \Delta S_{t+1}^{(j)}) < \infty$ for all i, j and $t < T$. Furthermore, the conditional returns of traded assets are linearly independent such that the matrix with elements $\left\{\mathbb{E}_t(\Delta S_{t+1}^{(i)} \Delta S_{t+1}^{(j)})\right\}_{1 \leq i,j \leq n}$ has full rank. For any vector $y \in \mathbb{R}^n$, y' denotes the transpose of y.

An $\mathcal{F}_0$-measurable initial endowment v is given. A trading strategy $\vartheta = (\vartheta_t)_{t \in \mathcal{T} \setminus \{0\}}$ is an n-dimensional, $\mathbb{F}$-previsible process, i.e. $\vartheta_t \in \mathbb{R}^n$ is $\mathcal{F}_{t-1}$ measurable for all $t > 0$. The value at time $t > 0$ of an investment portfolio with initial endowment v and trading strategy ϑ is given by

$$G_t^{v,\vartheta} = v + \sum_{k=1}^{t} \vartheta_k' \Delta S_k. \tag{21}$$

Only strategies such that $G_T^{v,\vartheta} \in L^2(\mathbb{P})$ are admitted; for a detailed technical discussion of admissibility see Černý and Kallsen [4].

Directly extending the discussion in Section 2, the aim is to derive the optimal (w.r.t. the quadratic loss) initial endowment and trading strategy, such that the quadratic deviation between the portfolio value $G_T^{v,\vartheta}$ and the liability H is minimised. In other words, we need to solve:

$$\arg\min_{(v,\vartheta)} \mathbb{E}\big((G_T^{v,\vartheta} - H)^2\big). \tag{22}$$

The solution to problem (22) is provided by Theorem 8.7 in Černý and Kallsen [4]:

Theorem 2. *The process given by the recursion $L_T = 1$ and for $t < T$*

$$L_t = \mathbb{E}_t(L_{t+1}) - \mathbb{E}_t(L_{t+1}\Delta S'_{t+1}) \left(\mathbb{E}_t(L_{t+1}\Delta S_{t+1}\Delta S'_{t+1})\right)^{-1} \mathbb{E}_t(L_{t+1}\Delta S_{t+1})$$

is $(0,1]$-valued and the probability measure $\mathbb{P}^*$, with

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \prod_{t=1}^{T} \frac{L_t}{\mathbb{E}_{t-1}(L_t)}$$

is well defined. Let $\mathbb{E}^*_{t-1}(\cdot)$ denote conditional expectations under $\mathbb{P}^*$. The following processes are well defined:

$$\begin{aligned}
a_t^* &= \mathbb{E}^*_{t-1}(\Delta S'_t)\ \mathbb{E}^*_{t-1}(\Delta S_t \Delta S'_t)^{-1}, \\
b_t^* &= a_t^*\ \mathbb{E}^*_{t-1}(\Delta S_t), \\
V_{t-1}^* &= \mathbb{E}^*_{t-1}\left(\frac{1 - a_t^*\Delta S_t}{1 - b_t^*} V_t^*\right), \quad V_T^* = H, \\
\xi_t^* &= \mathbb{E}^*_{t-1}((V_t^* - V_{t-1}^*)\Delta S_t)'\ \mathbb{E}^*_{t-1}(\Delta S_t \Delta S'_t)^{-1}.
\end{aligned}$$

For initial endowment v define the trading strategy $\phi(v)$ iteratively by

$$\phi_t(v) = \xi_t^* + a_t^*\left(V_{t-1}^* - G_{t-1}^{v,\phi(v)}\right).$$

Then, the pair $(V_0^*, \phi(V_0^*))$ solves the optimisation problem (22).

The probability measure $\mathbb{P}^*$ is termed the *opportunity-neutral measure*. The opportunity-neutral measure $\mathbb{P}^*$ is not a martingale measure. Switching to $\mathbb{P}^*$ is necessary in the case that asset returns are not independent, in order to compensate for one-period Sharpe ratios at a given time not being the same in all states (see Černý and Kallsen [4]). In the case that $\mathbb{P}^* = \mathbb{P}$, we can keep the same notation as in Theorem 2, after dropping the superscript * from all variables. In particular, it can be shown that if the quantity $b_t = \mathbb{E}_{t-1}(\Delta S'_t)\mathbb{E}_{t-1}(\Delta S_t \Delta S'_t)^{-1}\mathbb{E}_{t-1}(\Delta S_t)$ is state-independent, i.e. $\mathcal{F}_0$-measurable, then the process L_t becomes deterministic and as a result $\mathbb{P}^* = \mathbb{P}$.

3.2 Valuation of an insurance liability

We now work towards deriving multi-period valuation formulas, generalizing those of Section 2. First, we decompose the $\mathcal{F}_T$-measurable liability $H \in L^2(\mathbb{P})$ as

$$H = \mathbb{E}(H) + Y_1 + \cdots + Y_T \qquad \text{with} \qquad Y_t = \mathbb{E}_t(H) - \mathbb{E}_{t-1}(H), \tag{23}$$

where Y_t is termed the *claims development result*, see Merz and Wüthrich [16]. The notion of the claims development result is based on the understanding that the insurance companies need to close their books after every period. At time t they will book the so-called *best-estimate liability* $\mathbb{E}_t(H)$, updating the previous prediction $\mathbb{E}_{t-1}(H)$. The resulting adjustment of the best estimate produces a claims development result of Y_t in period t, which may be a gain or a loss. Essentially, Y_t corresponds to the single-period risk exposure of the holder of H and the regulator asks for a risk measure to support possible shortfalls in Y_t in period t. Since the time series $Y_1, \ldots, Y_T$ is formed by the innovations of a martingale, its elements are uncorrelated and have zero mean.

Under some simplifying assumptions, explicit valuation formulas for H can be derived.

Proposition 3. *Assume that $a_t = a_t^*$ and $b_t = b_t^*$ are state-independent and that*

$$\mathbb{E}_{t-1}(\Delta S_t \Delta S_{t+i} Y_{t+i}) = \mathbb{E}_{t-1}(\Delta S_t)\mathbb{E}_{t-1}(\Delta S_{t+i} Y_{t+i}),$$

for $0 < t < t+i \leq T$. Then the optimal initial endowment of Theorem 2 becomes:

$$V_0 = \mathbb{E}(H) + \sum_{t=1}^{T} \mathbb{E}\left(\frac{1 - a_t \Delta S_t}{1 - b_t} Y_t\right) = \mathbb{E}(H) - \sum_{t=1}^{T}\sum_{i=1}^{n} \frac{a_t^{(i)} \mathrm{Cov}(\Delta S_t^{(i)}, Y_t)}{1 - b_t}.$$

Proof. Since b_t^* is state-independent we have $\mathbb{P}^* = \mathbb{P}$. From Theorem 2 then it is $V_T = H$ (we drop the superscript * everywhere) and

$$\begin{aligned} V_{T-1} = \mathbb{E}_{T-1}\left(\frac{1 - a_T \Delta S_T}{1 - b_T} H\right) &= \mathbb{E}_{T-1}\left(\frac{1 - a_T \Delta S_T}{1 - b_T}\left(\mathbb{E}(H) + \sum_{s=1}^{T} Y_s\right)\right) \\ &= \mathbb{E}(H) + \sum_{s=1}^{T-1} Y_s + \mathbb{E}_{T-1}\left(\frac{1 - a_T \Delta S_T}{1 - b_T} Y_T\right), \end{aligned}$$

where the normalization $\mathbb{E}_{T-1}\left(\frac{1 - a_T \Delta S_T}{1 - b_T}\right) = 1$ is used. Now evaluate V_{T-2}:

$$\begin{aligned} V_{T-2} =& \mathbb{E}_{T-2}\left(\frac{1 - a_{T-1} \Delta S_{T-1}}{1 - b_{T-1}} V_{T-1}\right) \\ =& \mathbb{E}(H) + \sum_{s=1}^{T-2} Y_s + \mathbb{E}_{T-2}\left(\frac{1 - a_{T-1} \Delta S_{T-1}}{1 - b_{T-1}} Y_{T-1}\right) \\ &+ \mathbb{E}_{T-2}\left(\frac{1 - a_{T-1} \Delta S_{T-1}}{1 - b_{T-1}} \frac{1 - a_T \Delta S_T}{1 - b_T} Y_T\right), \end{aligned}$$

where to the last term we have applied the tower property of conditional expectations. From the assumption of uncorrelatedness across time periods it follows for the last term

$$\mathbb{E}_{T-2}\left(\frac{1-a_{T-1}\Delta S_{T-1}}{1-b_{T-1}}\frac{1-a_T\Delta S_T}{1-b_T}Y_T\right)=\mathbb{E}_{T-2}\left(\frac{1-a_T\Delta S_T}{1-b_T}Y_T\right).$$

This process can now be iterated to evaluate $V_{T-3},\dots,V_0$ which provides the first claim of Proposition 3. The second claim follows by a rearrangement of terms and the fact that Y_t has mean zero. □

The conditions of Proposition 3 correspond, loosely speaking, to the assumption that the conditional expected performance of assets over each time period is already known at time $t=0$ and that assets and liabilities are uncorrelated across time periods. Then, valuation reduces to a multi-period and multi-asset version of CAPM, where in each period t the claims development result Y_t is valued relatively to the vector of assets ΔS_t.

Now assume again that an insurance derivative is traded, which we identify with the first traded risky asset. The derivative is written at each time $t-1$ and pays 1 unit at time t, if the claims development result Y_t exceeds a given high threshold d_t. Specifically, we assume

$$\Delta S_t^{(1)}=\mathbf{1}_{D_t}-q_t, \tag{24}$$

where $D_t=\{Y_t\geq d_t\}$, $\mathbb{E}_{t-1}(\mathbf{1}_{D_t})=p_t$, and q_t is the $\mathcal{F}_{t-1}$-measurable price at time $t-1$ with $p_t<q_t<1$. (In fact, much of the following analysis remains unchanged if we assume, similarly to Section 2, that the event $D_t=\{Z_t\geq d_t\}$, where Z_t is an (index) variable closely correlated to Y_t. For the sake of simplicity, we do not pursue this route here). Moreover, let p_t,q_t be state-independent, so that the assumptions of Proposition 3 are satisfied, and assume that the Y_t's have continuous distribution functions. Proposition 3 then gives:

$$\begin{aligned}V_0=&\mathbb{E}(H)-\sum_{t=1}^{T}\frac{a_t^{(1)}\mathrm{Cov}(\Delta S_t^{(1)},Y_t)}{1-b_t}-\sum_{t=1}^{T}\sum_{i=2}^{n}\frac{a_t^{(i)}\mathrm{Cov}(\Delta S_t^{(i)},Y_t)}{1-b_t}\\=&\mathbb{E}(H)-\sum_{t=1}^{T}\frac{a_t^{(1)}p_t}{1-b_t}\mathrm{TVaR}_{1-p_t}(Y_t)-\sum_{t=1}^{T}\sum_{i=2}^{n}\frac{a_t^{(i)}\mathrm{Cov}(\Delta S_t^{(i)},Y_t)}{1-b_t},\end{aligned}$$

such that the value consists of a weighted sum of the $\mathrm{TVaR}_{1-p_t}(Y_t)$ and the covariances between Y_t and other traded assets.

A further simplification arises when we do not consider any tradeable assets except the derivatives on Y_t. Then,

$$\begin{aligned}a_t=&\frac{\mathbb{E}_{t-1}(\mathbf{1}_{D_t}-q_t)}{\mathbb{E}_{t-1}((\mathbf{1}_{D_t}-q_t)^2)}=\frac{p_t-q_t}{(p_t-q_t)^2+p_t(1-p_t)},\\b_t=&\frac{\mathbb{E}_{t-1}(\mathbf{1}_{D_t}-q_t)^2}{\mathbb{E}_{t-1}((\mathbf{1}_{D_t}-q_t)^2)}=\frac{(p_t-q_t)^2}{(p_t-q_t)^2+p_t(1-p_t)}.\end{aligned}$$

Therefore the single-asset and multi-period valuation formula becomes

$$V_0 = \mathbb{E}(H) + \sum_{t=1}^{T} \frac{q_t - p_t}{1 - p_t} \mathrm{TVaR}_{1-p_t}(Y_t), \tag{25}$$

which directly generalises the one-period setup (10). In (25), the market consistent value V_0 is given as a weighted sum of TVaR risk measures, applied to the claims development results Y_t and evaluated with respect to the information at time 0. Insofar, the resulting valuation method bears some formal resemblance to a simple multi-period generalization of the cost-of-capital formula (11), termed the *split of total uncertainty approach* in Salzmann and Wüthrich [22] or *expected risk margin* in Möhr [17].

Before stating some caveats for the approach presented in this section, a numerical example illustrating the ideas discussed above is presented.

Example 3. In this example we consider a long-tail $\mathcal{F}_T$-measurable liability H, $E(H) = 100$, with $T = 10$ years, and two tradeable assets in each period. These are a derivative with price at time $t-1$ of q_t and pay-off at time t of $\mathbf{1}_{\{Y_t \geq d_t\}}$, and a stock with price process $S_t^{(2)}$, such that

$$\Delta S_t^{(1)} = \mathbf{1}_{\{Y_t \geq d_t\}} - q_t, \quad \Delta S_t^{(2)} = S_t^{(2)} - S_{t-1}^{(2)}.$$

We assume that the claims development results $Y_1, \ldots, Y_T$ are mutually independent and so are the stock returns $\Delta S_1^{(1)}/S_0, \ldots, \Delta S_T^{(1)}/S_{T-1}$. Moreover, the pair $(Y_t,\ \Delta S_t^{(2)}/S_{t-1}^{(2)})$ is defined via a bivariate Log-normal model (conditionally on $S_{t-1}^{(2)}$), such that

$$\begin{aligned} Y_t &= \exp\left(\mu_t + \sigma_t Z_t^{(1)}\right) - \exp\left(\mu_t + \sigma_t^2/2\right), \\ \Delta S_t^{(2)}/S_{t-1}^{(2)} &= \exp\left(m + s Z_t^{(2)}\right) - 1, \end{aligned}$$

where $(Z_t^{(1)},\ Z_t^{(2)})$ follow a bivariate standard normal distribution with correlation r. This implies that we can write $Z_t^{(2)} = r Z_t^{(1)} + \sqrt{1-r^2} W_t$, where $(Z_t^{(1)},\ W_t)$ are independent standard normal variables. Also note that, as required, $\mathbb{E}_{t-1}(Y_t) = 0$. The model for Y_t used here is illustrative, as in a more realistic application one would need to derive the dynamics of Y_t from a stochastic reserving model.

The model parameters are as follows:

$p_t = 0.01$ and $q_t = 0.066$ for all $t = 0, \ldots, T$, implying that the threshold d_t is always set at the 99th percentile of Y_t and that we are certain of the derivative price in future periods;

$\mu_t = 0.4586(T - t + 1)$ and $\sigma_t = 0.198$ for all $t = 0, \ldots, T$, implying that the standard deviation of the claims development results Y_t reduces over time, reflecting payment of claims and increase in information;

$m = 0.15$ and $s = 0.2$; implying i.i.d. returns for the stock; and

r is allowed to vary in the range $(-1, 1)$, such that both a positive correlation (stock prices tend to increase at times of high claims development) and a negative correlation (stock prices tend to decrease at times of large claims development) are considered.

We proceed by applying Proposition 3. The necessary calculations are somewhat tedious and are documented in the Appendix.

We consider the correlation parameter values $r \in \{-0.5,\ 0,\ 0.5\}$, leading to the market values of H equaling $V_0 = V_0(r) = \{111.90,\ 109.64,\ 107.69\}$ respectively. In Figure 2, we plot the market risk margin applied for each year of the liability's run-off, that is, the quantities

$$\sum_{i=1}^{2} \frac{a_t^{(i)} \mathrm{Cov}(\Delta S_t^{(i)}, Y_t)}{1 - b_t}, \quad t = 1, \ldots, T.$$

The following observations can be made:

The case $r = 0$ is equivalent to the absence of the stock, such that V_0 is given by the expression (25).

When $r = 0.5 > 0$, long positions in the stock produce a natural hedging effect for the liability; investment returns pay for claims development results. This situation, which is desirable for the holder of the liability, *decreases* V_0 in relation to the scenario $r = 0$.

When $r = -0.5 < 0$, short positions in the stock are taken. Thus, in order to hedge the liability, negative expected stock returns are incurred. This adverse situation, analogous to the liability being subject to systemic risk, *increases* V_0 in relation to the scenario $r = 0$.

It can be seen from Figure 2 that the annual contributions to the market value of H decrease with time. This is explained by the decay of the standard deviation of Y_t in our model as t increases.

□

The simplicity of (25) is appealing but comes at the cost of making some strong assumptions. The first thing to note is that, while reinsurance for the claims development results Y_t can be purchased (under the term *adverse development cover*), derivatives with pay-off $\mathbf{1}_{D_t}$ are typically not traded. This makes calibration of valuation formulas such as (25) problematic. It is worth noting however, that with the advances in quantifying the risks Y_t via stochastic reserving methods, practitioners are increasingly expressing an interest in the creation of such a market, see Gesmann [11].

Furthermore, the assumption that the asset returns are uncorrelated across time periods, made in Proposition 3, is not justifiable in an obvious way. While the random variables $Y_1, \ldots, Y_T$ are uncorrelated, they are not necessarily independent. It follows that the random variables $\mathbf{1}_{D_1}, \ldots, \mathbf{1}_{D_T}$ are in general *not* uncorrelated. But, even if the random variables $Y_1, \ldots, Y_T$

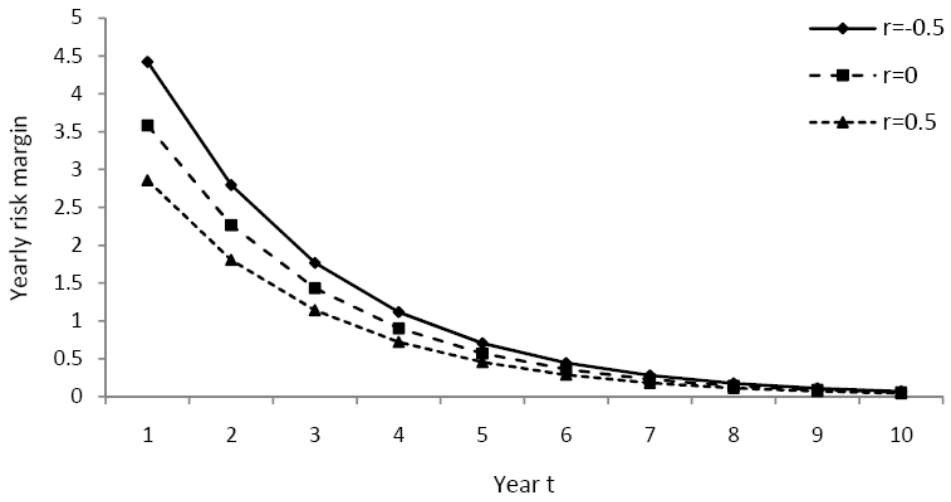

Figure 2: Yearly risk margins for different levels of correlation parameter r.

happen to be independent (as in a Gaussian model where uncorellatedness and independence are equivalent), the state-independence of the prices q_t is again a strong assumption, as it implies that information on the outcome of Y_{t-1} has no bearing on the market value of $\mathbf{1}_{\{Y_t \geq d_t\}}$. However it is conceivable that, even if the random variables Y_t are independent of each other, markets take a different view, such that a high level of Y_{t-1} is associated with a high market price q_t for the payoff $\mathbf{1}_{\{Y_t \geq d_t\}}$. The following numerical example illustrates the effect that dependence of market prices on past performance of the derivative $\mathbf{1}_{\{Y_t \geq d_t\}}$ may have on the market value of the terminal liability H.

Example 4. To avoid computational issues, we now consider a shorter term liability H, with $T = 2$ and $\mathbb{E}(H) = 100$. In this example there is no stock correlated with claims development results, such that the only tradeable asset is the derivative with price at time $t-1$ of q_t and pay-off at time t of $\mathbf{1}_{\{Y_t \geq d_t\}}$. Again, we assume that the claims development results are mutually independent and follow a shifted Log-normal distribution:

$$Y_t = \exp\left(\mu_t + \sigma_t Z_t\right) - \exp\left(\mu_t + \sigma_t^2/2\right), \quad t = 1, 2,$$

where Z_1, Z_2 are independent standard normals. The parameters of the claims development results are $\mu_1 = 4.586,\ \mu_2 = 4.127,\ \sigma_1 = \sigma_2 = 0.198$.

We now consider a derivative with a higher probability of a pay-off than in the previous example, such that $p_1 = p_2 = 0.05$ and $q_1 = 0.21$. However q_2 is no more known at time $t = 0$, but is instead dependent on Y_1. If the derivative produces a pay-off the market price of the derivative increases in the next period (and vice versa). Specifically, we define q_2 by:

$$q_2 = \begin{cases} \underline{q} \leq q_1, & \text{if } Y_1 < d_1, \\ \overline{q} \geq q_1, & \text{if } Y_1 \geq d_1. \end{cases}$$

To aid comparisons, we let $\mathbb{E}(q_2) = (1-p_1)\underline{q}+p_1\overline{q} = q_1$. The sensitivity of q_2 on past performance of the derivative is studied by considering three cases: $\overline{q}/\underline{q} = 1$, giving $\underline{q} = \overline{q} = 0.21$; $\overline{q}/\underline{q} = 2$, giving $\underline{q} = 0.2$, $\overline{q} = 0.4$; and $\overline{q}/\underline{q} = 4$, giving $\underline{q} = 0.183$, $\overline{q} = 0.730$.

In order to calculate the market value V_0^*, we make use of Theorem 2. In particular we have

$$\begin{aligned}
L_2 &= 1, \\
a_2^* &= a_2 = \frac{\mathbb{E}_1(\Delta S_2)}{\mathbb{E}_1(\Delta S_2^2)} = \frac{p_2 - q_2}{p_2 + q_2^2 - 2p_2q_2}, \\
b_2^* &= a_2 = \frac{\mathbb{E}_1(\Delta S_2)^2}{\mathbb{E}_1(\Delta S_2^2)} = \frac{(p_2 - q_2)^2}{p_2 + q_2^2 - 2p_2q_2}, \\
V_1 &= \mathbb{E}_1\left(\frac{1 - a_2\Delta S_2}{1 - b_2}H\right) \\
&= \mathbb{E}(H) + Y_1 + \mathbb{E}_1\left(\frac{1 - a_2\Delta S_2}{1 - b_2}Y\right) \\
&= \mathbb{E}(H) + Y_1 + \frac{q_2 - p_2}{1 - p_2}\mathrm{TVaR}_{1-p_2}(Y_2).
\end{aligned}$$

Hence V_1 can be explicitly calculated as a function of q_2, which is in turn a function of Y_1. To derive V_0, we need to calculate:

$$\begin{aligned}
L_1 &= 1 - b_2, \\
a_1^* &= \frac{\mathbb{E}(L_1\Delta S_1)}{\mathbb{E}(L_1\Delta S_1^2)}, \\
b_1^* &= \frac{\mathbb{E}(L_1\Delta S_1)^2}{\mathbb{E}(\Delta S_1^2)E(L_1)}, \\
V_0^* &= \frac{1}{\mathbb{E}(L_1)}\mathbb{E}\left(L_1\frac{1 - a_1^*\Delta S_1}{1 - b_1^*}V_1\right).
\end{aligned}$$

These calculations of the market value V_0^* can be easily done by Monte-Carlo simulation. Using a simulated sample of $5 \cdot 10^6$ from Y_1, we obtain the following results:

For $\overline{q}/\underline{q} = 1$ it is V_0^*=113.2;

For $\overline{q}/\underline{q} = 2$ it is V_0^*=114.2;

For $\overline{q}/\underline{q} = 2$ it is V_0^*=116.0.

Hence, it is seen that with increasing sensitivity of q_2 to the outcome of Y_1, the market value of the liability also increases. We envisage this case to be more realistic in comparison to a scenario where derivative prices are unaffected by observed losses, that is, where q_2 does not depend on the outcome of Y_1. However, at least for this short-tail example, the increase is not particularly dramatic. □

3.3 Hedging and capital allocation

In Sections 2.3 and 2.4 respectively, the issues of capital efficiency and capital allocation were discussed in relation to the single-period model. Here we briefly comment on the way that these ideas transfer to the multi-period setting.

3.3.1 Hedging and capital efficiency

The relation between hedging and capital efficiency becomes rather convoluted in the multi-period case. The reason for this is structural. While capital requirements in insurance are typically calculated with respect to a 1-year horizon, the optimal investment strategy is formulated to minimise a quadratic error calculated at the time horizon T. In particular, the trading strategy in each period will also reflect the performance of the portfolio to-date, which introduces path-dependency.

Consider the simplest possible case, where $Y_1, \ldots, Y_T$ are independent, a_t, b_t are state-independent, and the only traded asset is the derivative on Y_t. Then, from Theorem 2 it is seen that the optimal trading strategy for initial endowment V_0 is given by

$$\phi_t(V_0) = \xi_t + a_t \left(V_{t-1} - G_{t-1}^{V_0,\phi(V_0)} \right), \text{ where } \xi_t = \frac{\mathbb{E}_{t-1}((V_t - V_{t-1})\Delta S_t)}{\mathbb{E}_{t-1}(\Delta S_t^2)}.$$

The expression for ξ_t can be simplified. From the proof of Proposition 3, we find that

$$V_t = \mathbb{E}(H) + \sum_{s=1}^{t} Y_s + \sum_{r=t+1}^{T} \mathbb{E}_t \left(\frac{1 - a_r \Delta S_r}{1 - b_r} Y_r \right),$$

which given our assumptions implies

$$V_t - V_{t-1} = Y_t - \mathbb{E}_{t-1} \left(\frac{1 - a_t \Delta S_t}{1 - b_t} Y_t \right).$$

Straightforward but tedious manipulations then yield

$$\xi_t = \frac{\mathrm{Cov}_{t-1}(\Delta S_t, Y_t)}{\mathrm{Var}_{t-1}(\Delta S_t)}. \tag{26}$$

Hence the trading strategy ϕ_t consists of two parts: ξ_t, the values of which in this simple setting are known at time 0, and $a_t \left(V_{t-1} - G_{t-1}^{V_0,\phi(V_0)} \right)$, which reflects the value of the investment portfolio at time t. Note that ξ_t in (26) is essentially identical to the ξ_1 obtained in (2). Let $\delta_t = V_{t-1} - G_{t-1}^{V_0,\phi(V_0)}$ represent the difference between the value of the liability and the value of the investment portfolio at time t_1. Then, since $E_{t-1}(\Delta S_t) \leq 0 \implies a_t \leq 0$, in the multi-period case we adjust the trading strategy such that, if the shortfall is $\delta_t > 0$, less is invested in the risky asset and vice versa.

Analogously to what was discussed in Section 2.3, a plausible re-formulation of the capital efficiency condition (13) at time $t-1$ is

$$\rho_{t-1}(Y_t - (G_t^{V_0,\phi(V_0)} - G_{t-1}^{V_0,\phi(V_0)})) \leq \rho_{t-1}(Y_t), \tag{27}$$

where ρ_{t-1} is the regulatory risk measure evaluated given the information available at time $t-1$. The principle here is that the liability in respect to which capital needs to be held during $(t-1,t]$ is the corresponding claims development result Y_t. Inequality (27) represents the condition that the capital required to support Y_t minus the net gains from trading over the same interval is less than the capital required to support Y_t, assuming that all funds are invested in the risk free asset.

The left hand side of inequality (27) can be written as

$$\rho_{t-1}(Y_t - \phi_t(V_0)\Delta S_t) = \rho_{t-1}\left(Y_t - \xi_t \Delta S_t - a_t \delta_t \Delta S_t\right).$$

Define

$$\tilde{k}_t = a_t\delta_t + \frac{1}{1-p_t}\mathrm{TVaR}_{t-1,1-p_t}(Y_t).$$

Then, retracing the first steps in the proof of Proposition 1, it follows that the condition for inequality (27) to hold is, analogously to (14),

$$q_t\tilde{k}_t \leq \rho_{t-1}(Y_t) - \rho_{t-1}(Y_t - \tilde{k}_t \mathbf{1}_{D_t}). \tag{28}$$

For a special scenario, where it happens that $\{V_{t-1} = G_{t-1}^{V_0,\phi(V_0)}\} \equiv \{\delta_t = 0\} \equiv \{\phi_t(V_0) = \xi_t\}$, it is easily seen that the whole of Proposition 1 could be reformulated in the multi-period setting.

Finally, we remark that ξ_t corresponds exactly to the investment in the stock under the (non-self-financing) local risk-minimizing hedging strategy of Föllmer and Schweizer [13]. Therefore, under such a trading strategy with explicit one-period optimisation targets, the present discussion of capital efficiency would be much simplified.

3.3.2 Capital allocation and valuation

For simple cases, the arguments of Section 2.4 transfer easily to multiple time periods. Consider the conditions of Proposition 3 satisfied and also that only one tradeable asset is available, the derivative on Y_t. Similarly as before, let $H = H^{(1)} + \cdots + H^{(m)}$ and also define the claims development results for each subportfolio as $Y_t^{(i)} = \mathbb{E}_t(H^{(i)}) - \mathbb{E}_{t-1}(H^{(i)}),\ i = 1,\ldots,m$.

From Proposition 3 we have that the market value of the whole portfolio is

$$V_0 = \mathbb{E}(H) - \sum_{t=1}^{T} \frac{a_t \sum_{i=1}^{m} \mathrm{Cov}_{t-1}(\Delta S_t, Y_t^{(i)})}{1-b_t}.$$

We can thus deduce the market consistent value at time $t=0$ of the subportfolio $H^{(i)}$ as

$$V_0^{(i)} = \mathbb{E}(H^{(i)}) - \sum_{t=1}^{T} \frac{a_t \mathrm{Cov}_{t-1}(\Delta S_t, Y_t^{(i)})}{1-b_t} = \mathbb{E}(H^{(i)}) + \sum_{t=1}^{T} \frac{q_t - p_t}{1-p_t}\mathbb{E}_{t-1}\left(Y_t^{(i)} \middle| Y_t \geq d_t\right),$$

which again demonstrates the parallels between market valuation and capital allocation. It can be verified that this is the same as

$$V_0^{(i)} = \mathbb{E}^{\mathbb{Q}}(H^{(i)}),$$

where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{t=1}^{T} \frac{1 - a_t \Delta S_t}{1 - b_t} = \prod_{t=1}^{T} \left(1 + \frac{q_t - p_t}{1 - p_t} \left(\frac{1}{p_t} \mathbf{1}_{\{Y_t \geq d_t\}} - 1 \right) \right),$$

and $\mathbb{Q}$ is a martingale measure (see Černý and Kallsen [4]).

4 Concluding remarks

We discussed the problem of valuing insurance liabilities in discrete time through mean-variance hedging. Key features of the proposed approach are the decomposition of the terminal liability into claims development results and the presence of a derivative on the claims development result in each period. In simple cases, the resulting valuation formulas become structurally very similar to regulatory cost-of-capital based formulas. However, adoption of the mean-variance framework improves upon regulatory formulas, by allowing for the potential calibration to observed market prices, the inclusion of other tradeable assets, and the consistent extension to multiple periods.

The similarity between the formulas derived here and the ones used in regulation should not obscure the very different interpretations underlying them. In our approach the market value margin obtained (difference between market consistent and expected values) does *not* correspond to the cost-of-capital, but reflects the cost of a replication portfolio. Hence, it is conceivable that a cost-of-capital loading may be *added* to the market consistent value that we obtain, since investors need to be compensated for the frictional costs that holding capital incurs (see eg the discussions in Zanjani [29] and Venter [28]). The analysis of Section 2.3 shows that the mean-variance hedging approach may also deliver some reduction in capital costs.

It is then useful to distinguish between the possible constituent parts of the value of a liability. Thus, if a cost-of-capital loading is added to the (partial) replication cost that our valuation formulas reflect, this should only represent frictional capital costs. In particular, it should not be further increased to act as a proxy for replication costs, as current regulatory valuation approaches maybe implicitly do. Finally, besides the cost of replication and the frictional cost of capital, it is plausible that an additional risk load is applied via a performance measure, purely to reward investors for risk taking. This need not be related to a tail risk measure like VaR or TVaR; for example, mean-variance hedging approaches can be adjusted to deliver a pre-specified minimal level of Sharpe ratio (Černý [2], Section 13.2).

Appendix

Calculations in Example 3

To determine V_0 we first need to calculate

$$a_t \Delta S_t = \mathbb{E}_{t-1}(\Delta S_t')\ \mathbb{E}_{t-1}(\Delta S_t \Delta S_t')^{-1} \Delta S_t,$$
$$b_t = E_{t-1}(\Delta S_t')\ E_{t-1}(\Delta S_t \Delta S_t')^{-1}\ \mathbb{E}_{t-1}(\Delta S_t).$$

In fact, it can be shown that these quantities are identical to

$$a_t \Delta S_t = \mathbb{E}_{t-1}(X_t')\ \mathbb{E}_{t-1}(X_t X_t')^{-1} X_t,$$
$$b_t = \mathbb{E}_{t-1}(X_t')\ \mathbb{E}_{t-1}(X_t X_t')^{-1}\ \mathbb{E}_{t-1}(X_t),$$

where X_t is the vector of returns with elements

$$X_t^{(i)} = \frac{S_t^{(i)}}{S_{t-1}^{(i)}} - 1,\ i = 1, \ldots, n.$$

More generally, in the definitions of a_t^*, b_t^* and V_t^* appearing in Theorem 2, the price process increment vector ΔS_t can be replaced by the return vector X_t. The validity of the substitution follows by observing that the value of the investment portfolio in (21) can be simply re-written as $v + \sum_{k=1}^{t} \tilde{\vartheta}_k' X_k$, where $\tilde{\vartheta}_k^{(i)} = \vartheta_k^{(i)} S_{k-1}^{(i)}$. Essentially this implies that if each asset is re-issued in each period with a price of 1, the mathematical structure of the optimisation problem (22) does not change.

This representation is practical, due to the way that the stock model is formulated. In particular, we have

$$Y_t = \exp(\mu_t + \sigma_t Z_t^{(1)}) - \exp\left(\mu_t + \sigma_t^2/2\right),$$
$$X_t^{(1)} = \frac{1}{q_t} \mathbf{1}_{\{Y_t > d_t\}} - 1,$$
$$X_t^{(2)} = \exp\left(m + s Z_t^{(2)}\right) - 1,$$

where $Z_t^{(2)} = r Z_t^{(1)} + \sqrt{1 - r^2} W_t$ and $(Z_t^{(1)},\ W_t)$ are independent standard normals. Therefore, to apply Proposition 3, we need to calculate the first and second moments of X_t as well as the covariances $\mathrm{Cov}_{t-1}\left(X_t^{(i)}, Y_t\right)$ for $i = 1, 2$.

The first moments of X_t are:

$$\mathbb{E}_{t-1}\left(X_t^{(1)}\right) = \frac{p_t}{q_t} - 1 \quad \text{and} \quad \mathbb{E}_{t-1}\left(X_t^{(2)}\right) = \exp\left(m + s^2/2\right) - 1.$$

The second moments of X_t are:

$$\mathbb{E}_{t-1}\left((X_t^{(1)})^2\right) = \left(\frac{p_t}{q_t} - 1\right)^2 + \frac{1}{q_t^2} p_t (1 - p_t),$$
$$\mathbb{E}_{t-1}\left((X_t^{(2)})^2\right) = 1 - 2\exp\left(m + s^2/2\right) + \exp\left(2m + 2s^2\right),$$
$$\mathbb{E}_{t-1}\left(X_t^{(1)} X_t^{(2)}\right) = 1 - \exp\left(m + s^2/2\right) - \frac{p_t}{q_t} + \mathbb{E}_{t-1}\left(\frac{1}{q_t} \mathbf{1}_{\{Y_t > d_t\}} \exp\left(m + s Z_t^{(2)}\right)\right).$$

Let $\tilde{d}_t = d_t + \exp\left(\mu_t + \sigma_t^2/2\right) = \exp\left(\mu_t + \sigma_t \Phi^{-1}(1-p_t)\right)$, where Φ is the standard normal distribution. Then $\mathbf{1}_{\{Y_t > d_t\}} = \mathbf{1}_{\{\exp(\mu_t + \sigma_t Z_t^{(1)}) > \tilde{d}_t\}}$, such that

$$\begin{aligned}
\mathbb{E}_{t-1}\left(\frac{1}{q_t}\mathbf{1}_{\{Y_t>d_t\}}\exp(m + sZ_t^{(2)})\right) &= \frac{1}{q_t}\mathbb{E}_{t-1}\Big(\mathbf{1}_{\{\exp(\mu_t+\sigma_t Z_t^{(1)})>\tilde{d}_t\}} \\
&\quad \cdot \exp\left(m + s(rZ_t^{(1)} + \sqrt{1-r^2}W_t)\right)\Big) \\
&= \frac{1}{q_t}\mathbb{E}_{t-1}\left(\exp\left(m + s\sqrt{1-r^2}W_t)\right)\right) \\
&\quad \cdot \mathbb{E}_{t-1}\left(\mathbf{1}_{\{Z_t^{(1)}>(\log\tilde{d}_t-\mu_t)/\sigma_t\}}\exp\left(srZ_t^{(1)}\right)\right) \\
&= \frac{1}{q_t}(\exp\left(m + s^2(1-r^2)\right)\cdot g(r),
\end{aligned}$$

where $g(r) = \mathbb{E}_{t-1}\left(\mathbf{1}_{\{Z_t^{(1)}>(\log\tilde{d}_t-\mu_t)/\sigma_t\}}\exp\left(srZ_t^{(1)}\right)\right)$. From the definition of $\tilde{d}_t$ it is

$$g(0) = p_t.$$

If $r > 0$, we can denote $k = (\log\tilde{d}_t - \mu_t)/\sigma_t$ and write

$$g(r) = \mathbb{E}_{t-1}\left(\mathbf{1}_{\{\exp(srZ_t^{(1)})>\exp(srk)\}}\exp\left(srZ_t^{(1)}\right)\right) = \exp(s^2r^2/2)\Phi\left(\frac{s^2r^2 - srk}{sr}\right),$$

from the properties of the Log-normal distribution. Finally, when $r < 0$ it is

$$\begin{aligned}
g(r) &= \mathbb{E}_{t-1}\left(\mathbf{1}_{\{\exp(s(-r)(-Z_t^{(1)}))<\exp(srk)\}}\exp\left(s(-r)(-Z_t^{(1)})\right)\right) \\
&= \exp(s^2r^2/2)\left[1 - \Phi\left(\frac{s^2r^2 - srk}{-sr}\right)\right].
\end{aligned}$$

We now move to the calculation of the covariances:

$$\begin{aligned}
\mathrm{Cov}_{t-1}(X_t^{(1)}, Y_t) &= \mathbb{E}_{t-1}\left(\frac{1}{q_t}\mathbf{1}_{\{Y_t>d_t\}}Y_t\right) \\
&= \mathbb{E}_{t-1}\left(\frac{1}{q_t}\mathbf{1}_{\{\exp(\mu_t+\sigma_t Z_t^{(1)})>\tilde{d}_t\}}\exp\left(\mu_t + \sigma_t Z_t^{(1)}\right)\right) - \frac{p_t}{q_t}\exp\left(\mu_t + \sigma_t^2/2\right) \\
&= \frac{1}{q_t}\exp\left(\mu_t + \sigma_t^2/2\right)\Phi\left(\frac{\mu_t + \sigma_t^2 - \log\tilde{d}_t}{\sigma_t}\right) - \frac{p_t}{q_t}\exp\left(\mu_t + \sigma_t^2/2\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Cov}_{t-1}(X_t^{(2)}, Y_t) &= \mathbb{E}_{t-1}\left(\exp\left(m + sZ_t^{(2)}\right)\exp\left(\mu_t + \sigma_t Z_t^{(1)}\right)\right) \\
&\quad - \mathbb{E}_{t-1}\left(\exp\left(m + sZ_t^{(2)}\right)\right)\mathbb{E}_{t-1}\left(\exp\left(\mu_t + \sigma_t Z_t^{(1)}\right)\right) \\
&= \mathbb{E}_{t-1}\left(\exp\left(m + \mu_t + (sr + \sigma_t)Z_t^{(1)} + s\sqrt{1-r^2}W_t\right)\right) \\
&\quad - \exp\left(m + \mu_t + (s^2 + \sigma_t^2)/2\right) \\
&= \exp\left(m + \mu_t + (sr + \sigma_t)^2/2 + s^2(1-r^2)/2\right) \\
&\quad - \exp\left(m + \mu_t + (s^2 + \sigma_t^2)/2\right).
\end{aligned}$$

This completes the required calculations. □

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