Closed-form solutions for options in incomplete markets

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Abstract: This paper reconsiders the predictions of the standard option pricing models in the context of incomplete markets. We relax the completeness assumption of the Black-Scholes (1973) model and as an immediate consequence we can no longer construct a replicating portfolio to price the option. Instead, we use the good-deal bounds technique to arrive at closed-form solutions for the option price. We determine an upper and a lower bound for this price and find that, contrary to Black-Scholes (1973) options theory, increasing the volatility of the underlying asset does not necessarily increase the option value. In fact, the lower bound prices are always a decreasing function of the volatility of the underlying asset, which cannot be explained by a Black-Scholes (1973) type of argument. In contrast, this is consistent with the presence of unhedgeable risk in the incomplete market. Furthermore, in an incomplete market where the underlying asset of an option is either infrequently traded or non-traded, early exercise of an American call option becomes possible at the lower bound, because the economic agent wants to lock in value before it disappears as a result of increased unhedgeable risk.

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1. Introduction

The purpose of this paper is to show how call options written on infrequently traded or non-traded assets can be priced in the setting of incomplete markets. Specifically, we assume that the underlying asset of the option carries both hedgeable and unhedgeable risk and use the good-deal bounds technique to value it and bring new insights into the predictions of the standard option pricing models. The result is a modified Black-Scholes (1973) closed-form solution. We find that, contrary to standard option pricing theory, the good-deal bounds prices do not always display an increasing pattern when the volatility of the underlying asset increases. In fact, we show that the prices of call options can decrease in response to an increase in the volatility of the underlying asset, when the underlying is either infrequently traded or non-traded.

The main contribution of our paper is to highlight the existence of the inverse relationship between the option value and the volatility of the underlying asset and the potential for early exercise even for an American call option on a non-dividend paying asset. These features appear in an incomplete market as direct consequences of unhedgeable risk coming from an infrequently traded or even non-traded underlying asset. However, they are overlooked by complete market models, because such models deal only with hedgeable sources of risk.

An implication of the decreasing option value in response to increased volatility of the underlying asset is the existence of an implied negative dividend yield. The economic agent is willing to accept a negative return in order to exit the incomplete market setting and avoid dealing with the unhedgeable sources of risk.

Our results can improve the way we price long-dated cash-flows or real options for instance. These two examples fall under the incidence of incomplete markets. Long-dated cash-flows become difficult to discount beyond maturities of 25 or 30 years and real options are generally options written on illiquid assets. In either case, the construction of a riskless replicating portfolio in order to price an option is no longer possible. To understand why this is important, we should first discuss the standard option pricing models.

Standard option pricing models are complete market models, which assume that all sources of risk can be perfectly hedged against and that options can be priced based on replication arguments. The main prediction of such models is that option value increases with an increase in the volatility of the underlying asset. The problem is that, if the underlying asset of an option is an infrequently traded asset or even a non-traded one, the replication arguments

fall apart, because we can no longer construct a riskless replicating portfolio out of the option and the underlying asset to price the option as in the Black-Scholes (1973) model.

This problem has been considered before. There are three strands of literature which try to tackle the problem: utility indifference pricing, pricing via coherent risk measures and pricing via a Sharpe ratio criterion.

Utility indifference pricing assumes a utility function for a representative agent who maximizes his utility of wealth, where wealth is influenced by an investment in the option. Duffie et al. (1997) derive optimal consumption and portfolio allocations in the context of incomplete markets. Davis (2006) focuses on the optimal hedging strategy in an incomplete market where an option is written on a non-traded asset and shows that the difference between complete and incomplete market prices is substantial.

Henderson (2007) and Miao and Wang (2007) restrict their analyses to real estate projects and derive semi-closed form solutions for options written on real estate assets using the utility indifference pricing technique. Both Henderson (2007) and Miao and Wang (2007) show that market incompleteness, in particular the degree of risk aversion, can actually reduce the option value.

The utility indifference pricing approach might seem like a good candidate for a pricing mechanism. Unfortunately, the results go only as far as a partial differential equation that the option price must satisfy and only for utility functions of exponential form. Furthermore, it is impossible to price short call positions using exponential utility, because the prices converge to infinity (Henderson and Hobson, 2004).

Pricing via a coherent risk measure was first introduced by Artzner et al. (1999) and it reduces to a search for the infimum[3] of all risk measures over an acceptance set. Carr et al. (2001) refine the idea and introduce a generalized version of the coherent risk measure. They argue that economic agents will not only invest in any arbitrage opportunity, but also in any opportunity that seems acceptable given their level of risk aversion. The problem is that the concept of 'acceptable opportunity' is a subjective one and it cannot be easily generalised to a market, but it rather characterises a particular economic agent.

[3] The infimum (i.e. inf) returns the greatest element of N that is less than or equal to all elements of M, where M is a subset of N.

Hansen and Jagannathan (1991) and later Cochrane and Saa-Requejo (2000) put forward the idea of pricing via a Sharpe ratio criterion, by exploiting the fact that investors would always trade in assets with very high Sharpe ratios and pure arbitrage opportunities. The methodology we propose for pricing options is a slightly modified version of the good-deal bounds approach of Cochrane and Saa-Requejo (2000). First, we use a change of measure to calculate the option price instead of solving for the stochastic discount factor and then take the expectation of the stochastic discount factor multiplied by the option payoff. Second, inspired by Hansen and Sargent's (2001) parameter uncertainty approach, we allow for a larger set of possible stochastic discount factors to price the option. We express the process for the stochastic discount factor under the physical measure P, assuming that its total volatility is lower than a general k, and then we perform a change measure to a new measure Q_{GDB} in order to price the option. Instead, Cochrane and Saa-Requejo (2000) assume that the process for the stochastic discount factor under the new measure is known.

Whenever we want to price a general claim, we calculate the expectation of a stochastic discount factor times the payoff of that claim (Cochrane, 2005). This is straightforward in a complete market, where all assets are assumed to be traded which means that we can observe their market price of risk. The volatility term of the stochastic discount factor is nothing else than the Sharpe ratio of the asset we are trying to price (Cochrane, 2005). Unfortunately, in an incomplete market, we cannot observe the market price of risk, because here there are also infrequently traded or even non-traded assets. We can however distinguish between hedgeable and unhedgeable risk and express the market price of risk for the unhedgeable component in terms of what we already know: the Sharpe ratio of a traded asset. This Sharpe ratio is an essential tool in determining the expression for the overall volatility of the stochastic discount factor, such that we can restrict the set of all possible discount factors to obtain the option price.

The parameters which ultimately determine the value of the option are the volatility of the underlying asset, the restriction on the volatility of the stochastic discount factor, the correlation coefficient between the underlying and a traded risky asset and the expected return of the investment. Interestingly, unlike in the standard option pricing models, the good-deal bounds option prices do not always increase as the volatility of the underlying asset increases. In fact, the lower bound prices are decreasing with increasing volatility of the underlying. This is a reflection of the additional uncertainty coming from the presence of unhedgeable risk, a feature of an incomplete market but not of the Black-Scholes (1973) complete-market setting. Furthermore, it can be shown that, at the lower bound, the drift term of the stochastic process for the infrequently traded underlying asset is lower than the risk-

free interest rate. Essentially, the economic agent is willing to invest at a negative implicit dividend yield in order to exit the incomplete market setting and avoid dealing with the unhedgeable sources of risk.

The negative implicit dividend yield gives rise to another phenomenon: the early exercise of an American call option even for a non-dividend paying asset. At the lower bound, the option value is decreasing with increasing volatility of the underlying asset, forcing the economic agent to exercise his option early for fear that, if he continues to wait, he will lose the entire value of the claim.

The advantage of the good-deal bounds over other incomplete market techniques is that the resulting option prices do not depend on a risk aversion parameter. However, even though one need not make any assumptions about the utility function of a representative economic agent and implicitly about this agent's level of risk aversion, one is still required to impose a restriction on the total volatility of the stochastic discount factor. This restriction is an exogenous parameter in the good-deal bounds framework.

Our closed-form solutions are comparable to the Black-Scholes (1973) option price. For very low values of the volatility of the underlying, the only source of uncertainty comes from the traded asset and we are back in the Black-Scholes (1973) framework. Similarly, for very high values of the correlation coefficient ρ, the good-deal bounds prices not only approach the Black-Scholes (1973) price, but this happens at a speed of $\sqrt{1-\rho^2}$, meaning that there is a large gap between the prices on an almost complete market and the Black-Scholes price at $\rho = 1$. Even at a $\rho = 0.99$, the speed of adjustment is already 0.14. Furthermore, when the restriction on the volatility of the stochastic discount factor is exactly equal to the Sharpe ratio of the traded asset, we again exit the incomplete market setting and the good-deal bounds prices converge to the Black-Scholes (1973) price. In other words, we generalise the market setting to the incomplete market and bring it closer to real life, but, at the same time, maintain a reference point, which is the Black-Scholes (1973) result.

The paper is organised as follows: Section 2 presents the mathematics of deriving the good-deal bounds closed-form solutions for a European call option, Section 3 deals with the behaviour of these prices for different parameter values, Section 4 presents closed-form solutions for a perpetual American call option in incomplete markets, Section 5 presents various applications of the good-deal bounds option pricing methodology and Section 6 concludes.

2. Incomplete markets – a closer look

The Black-Scholes (1973) model relies on the following assumptions: the price process for the underlying asset follows a geometric Brownian motion, with constant drift and constant volatility, the underlying asset is traded continuously and it pays no dividends, short selling is allowed, there are no transaction costs or taxes, there are no riskless arbitrage opportunities and the risk-free interest rate is constant (Hull, 2012).

In reality though there are a series of frictions which render every market incomplete: transaction costs, the presence of non-traded assets on that market or portfolio constraints, like no short-selling or a predetermined allocation to a particular asset or asset group in the portfolio.

The continuous trading assumption is the one that makes our case. The underlying idea of the Black-Scholes (1973) model is that we can construct a riskless replicating portfolio, by selling the option and buying delta units of the underlying asset. To satisfy the no-arbitrage condition, we then equate the instantaneous return of this portfolio with the return of a riskless asset. However, this is only possible because we can continuously trade in the underlying asset of the option. But, in a market that is incomplete due to the presence of infrequently traded assets or even non-traded assets, if the underlying happens to be one of these problematic assets, then we can no longer perform the option valuation in the convenient manner of the Black-Scholes (1973) complete market model (i.e. based on replication arguments).

As Duffie (1987) shows, the problem with pricing in incomplete markets is that imposing the no-arbitrage condition is no longer sufficient to arrive at a unique price for a general contingent claim (i.e. for a financial derivative). Take for instance the example of a non-traded underlying asset. It is impossible to exactly replicate a claim on such an asset, so we can expect to be confronted with more than one price system for this claim which is consistent with absence of arbitrage. In fact, if we just impose the no-arbitrage condition, the price will be situated within the arbitrage bounds (i.e. the interval given by all the possible values for the option price that satisfy the no-arbitrage condition). For a call option, the lower arbitrage bound is zero and the upper arbitrage bound is the price of the underlying. Such an interval is not very informative, because it is too wide to be useful. The solution is to make additional assumptions about the choice of pricing kernel (Duffie et al., 1997).

The good-deal bounds (GDB) mechanism is an incomplete market pricing mechanism, which uses a restriction on the total volatility of the stochastic discount factor as an additional restriction to arrive at tighter and more informative bounds for the option price. Hansen and Jagannathan (1991) and later Cochrane and Saa-Requejo (2000) exploit the fact that investors would always trade in assets with very high Sharpe ratios and pure arbitrage opportunities. Consequently, such investments would immediately disappear from the market, so we should only be interested in a Sharpe ratio that is high enough to induce trade, but not too high to include the deals which are too good to be true. The good-deal bounds pricing mechanism is simply a tool to rule out these good deals and the arbitrage opportunities (which Björk and Slinko (2006) call "ridiculously good deals"), such that the result is an option price within a tight and informative interval. Hodges (1998), Černý (2003) and Björk and Slinko (2006) even extend the Cochrane and Saa-Requejo (2000) setting to generalized Sharpe ratios for pricing in incomplete markets.

With the good-deal bounds we are in the context of partly hedgeable partly unhedgeable risk. We assume that we can find on the market a traded risky asset, correlated with the illiquid underlying asset, with which we can hedge this illiquid asset at least partly. The advantage over a complete-market model is that the option price reflects both the hedgeable and the unhedgeable risk. What is more, it is possible to derive closed-form solutions for the option price, as Cochrane and Saa-Requejo (2000) show in their paper. The GDB prices resemble the Black-Scholes price, which makes it even easier to understand the effect of adding unhedgeable risk.

The solution we propose in the next sub-sections involves more accessible calculations than the ones presented by Cochrane and Saa-Requejo (2000). We discuss European and American-style call options and show that closed-form solutions are available in both of these cases. We focus on the previously overlooked negative relationship between call value and an infrequently traded or non-traded underlying asset and further emphasize the early exercise feature that this negative relationship adds to an American call option.

2.1. Market setting

We want to price a European call option which is written on an infrequently traded asset V and which has a constant strike price K. Under these circumstances, the market is incomplete, because there are more sources of risk than traded assets. Assume that in this market we can also find a traded risky asset S, correlated with V, and a riskless asset B (a

bond). Even if asset *V* cannot be continuously traded, asset *S* is assumed to be continuously traded and correlated with *V* with a correlation coefficient *ρ*.

The dynamics of the assets in the current market setting are as follows:

$$dS = \mu_S S dt + \sigma_S S dz \tag{1}$$

where: dz – Brownian motion

$$dV = \mu_V V dt + \sigma_V V\left(\rho dz + \sqrt{1-\rho^2} dw\right) \tag{2}$$

where: ρ – correlation coefficient between the assets V and S

dz, dw – independent Brownian motions

$$dB = rBdt \tag{3}$$

where: r – deterministic short interest rate

Like any other contingent claim, our call option can be priced as the expectation of a stochastic discount factor times the payoff of the option (Björk, 2009):

$$C_t = E_t\left[\frac{\Lambda_T}{\Lambda_t}\max(V_T - K, 0)\right] \tag{4}$$

where E_t *– the expectation at time t*

C – the price of the call option

Λ – the stochastic discount factor

V_T *– the value of the infrequently traded asset at maturity time T*

K – the strike price

The stochastic discount factor, also known as the continuous-time pricing kernel, is the product between a risk-free rate discount factor and a Radon-Nikodym derivative (Björk, 2009). If the interest rate is deterministic, then the risk-free rate discount factor comes out of the expectation as a constant. The Radon-Nikodym derivative rewrites the expectation of the payoff process under the real-world probability measure as an expectation under a new measure which is equivalent to the initial one. In the end, the arbitrage-free price of the option will simply be the discounted expected value of the payoff under this new probability measure.

2.2. Pricing with good-deal bounds

In a complete market, where we assume that the underlying asset of an option is continuously traded, we can observe the market price of risk *(μ - r)/σ* for this traded asset. Then, the process for the stochastic discount factor, which we denote *Λ*, is simply:

$$d\Lambda = -r\Lambda dt - \frac{\mu - r}{\sigma}\Lambda dz \tag{5}$$

The volatility term of the stochastic discount factor is actually a Sharpe ratio hence the justification for the good-deal bounds pricing mechanism. Restricting the volatility of the discount factor is thus equivalent to restricting the Sharpe ratio of a traded risky asset (Hansen and Jagannathan, 1991). The minus sign in front of the volatility component shows that the stochastic discount factor assigns more weight to the bad outcomes of the value of the underlying: whenever *z* decreases (determining a bad outcome for the underlying), the discount factor increases.

We can use the same logic for incomplete markets. Even though our underlying is not liquidly traded and we cannot observe its market price of risk, we can still express it in terms of what we know, for instance the market price of risk of a traded asset correlated with the infrequently traded underlying. This leads to a partial hedge of the focus derivative (using the traded asset *S* correlated with the illiquid asset *V*), so we are confronted with both hedgeable and unhedgeable sources risk.

Consequently, the stochastic discount factor in an incomplete market should look like:

$$d\Lambda = -r\Lambda dt - \kappa_1\Lambda dz - \kappa_2\Lambda dw \tag{6}$$

where: *dz – hedgeable component*

dw – unhedgeable component

Restricting the total volatility of the discount factor to be at most *k*, we can write:

$$\kappa_1^2 + \kappa_2^2 \leq k^2 \tag{7}$$

Fixing $\kappa_1 = (\mu - r)/\sigma$ as the market price of the hedgeable risk, κ_2 has the solution:

$$\kappa_2 \in \left[-\sqrt{k^2 - \left(\frac{\mu - r}{\sigma}\right)^2}, \sqrt{k^2 - \left(\frac{\mu - r}{\sigma}\right)^2}\right] \tag{8}$$

κ_2 can take any value in the interval above. To each κ_2 in this interval corresponds an option price, so, eventually, the option price will also be within an interval.

Remember the expression for the price of the call option in equation (4). Instead of calculating the expectation of the stochastic discount factor in equation (6) times the option payoff, the way Cochrane and Saa-Requejo (2000) do, we could equivalently perform a Girsanov transformation (Björk, 2009) on the process *V* and simply calculate the expectation of the resulting process, which will have a new Brownian motion and a new drift term.

To understand why this is the case, we should start with the physical (real-world) measure *P*:

$$C_0 = E^P\left[\frac{\Lambda_T}{\Lambda_0}\max(V_T - K, 0)\right] \tag{9}$$

We first search for any probability measure equivalent to the physical measure *P*, which we can generically call Q_{GDB}, in order to rewrite the expectation under this new probability measure. The change of measure can be performed through a Radon-Nikodym derivative and using the Girsanov Theorem, which says that we can modify the drift of a Brownian motion process by interpreting that process under a new probability distribution (Björk, 2009).

The stochastic discount factor (i.e. the continuous-time pricing kernel), Λ_T/Λ_0, is actually the product of a risk-free rate discount factor and a Radon-Nikodym derivative (Björk, 2009):

$$\frac{\Lambda_T}{\Lambda_0} = e^{-rT}\frac{dQ_{GDB}}{dP} \tag{10}$$

The Radon-Nikodym derivative of Q_{GDB} with respect to *P*, dQ_{GDB}/dP, performs the change we want, from the physical measure *P* to the new probability measure Q_{GDB}:

$$C_0 = E^P\left[e^{-rT}\frac{dQ_{GDB}}{dP}\max(V_T - K, 0)\right] \tag{11}$$

We can now evaluate the option payoff under any probability measure Q_{GDB}, equivalent to the initial probability measure *P*:

$$C_0 = e^{-rT} E^{Q_{GDB}}[\max(V_T - K, 0)] \tag{12}$$

What we want in fact is not any measure Q_{GDB}, but all the measures Q_{GDB} lower than or equal to k^2, because we are interested in placing an upper bound k on the total volatility of the stochastic discount factor ($\kappa_1^2 + \kappa_2^2 \leq k^2$). This translates into a minimization over the set of all Q_{GDB} risk measures, such that equation (12) can be re-written as:

$$C_0 = \min_{"Q_{GDB} \leq k^2"} e^{-rT} E^{Q_{GDB}}[(\mathrm{V_T} - \mathrm{K})^+] \tag{13}$$

In this case, the minimum of the payoff leads to the lower bound and the maximum (i.e. the minimum of the negative payoff) leads to the upper bound.

Under the notation in equation (13), the good-deal bounds are a coherent risk measure of Artzner et al. (1999), which searches for the infimum of all risk measures over an acceptance set. Delbaen (2002) further shows that any coherent risk measure can be expressed as a worst expected loss over a given set of probabilities and Jaschke and Küchler (2001) link the good-deal bounds to coherent risk measures by showing that the good-deal bounds are coherent valuation bounds.

We proceed with the pricing of the European call option. The illiquid asset *V* follows the stochastic process in equation (2). Assume that the value of asset *V* at time *T* is a lump-sum. The traded risky asset *S* follows the stochastic process in equation (1) and is correlated with *V*, hence asset *S* can be used as a partial hedge. The stochastic discount factor is the one in equation (6).

We fix the volatility part corresponding to the hedgeable risk to be equal to the Sharpe ratio of the risky asset *S*:

$$\kappa_1 = \frac{\mu_S - r}{\sigma_S} \tag{14}$$

The stochastic discount factor in this context becomes:

$$d\Lambda = -r\Lambda dt - \frac{\mu_S - r}{\sigma_S}\Lambda dz - \kappa_2 \Lambda dw \tag{15}$$

Next, we apply the following Girsanov transformations on the process *V* for the Brownian motions z and w:

$$dz = dz^* - \frac{\mu_S - r}{\sigma_S} dt \tag{16}$$

$$dw = dw^* - \kappa_2 dt \tag{17}$$

The process for *V* becomes:

$$dV = (\mu_V - \sigma_V \rho \frac{\mu_S - r}{\sigma_S} - \sigma_V \sqrt{1-\rho^2}\kappa_2)Vdt + \sigma_V V\left(\rho dz^* + \sqrt{1-\rho^2}dw^*\right) \tag{18}$$

Notice that the Girsanov transformations only affect the drift term and the adjustment they make is equal to the volatility of the stochastic discount factor. The expected return is now lowered by the market price of each type of risk, but proportionately to how much can be hedged and how much is left unhedged (ρ and $\sqrt{1-\rho^2}$, respectively).

We can finally understand the intuition behind the chosen process for the stochastic discount factor. Remember our goal: pricing a European call option. For any such contract there exists a buyer and a seller, each with their reservation price. The buyer's reservation price shows the buyer's maximum valuation and the seller's reservation price, the seller's minimum valuation of the contract. For any price lower than his maximum valuation, the buyer will decide to buy. Similarly, for any price higher than his minimum valuation, the seller will agree to sell. Otherwise, the transaction will no longer take place.

The question is: how can we derive such prices? The answer lies in the payoff structure. The call price is positively determined by the value of the underlying at terminal time *T*, V_T. However, *V* is a stochastic process positively determined by the drift term μ_V. So, the call price will ultimately be positively determined by μ_V. The only way we can minimise the call price to arrive at the buyer's reservation price is if we take the lowest possible value of μ_V and that occurs only when $\kappa_2 = \sqrt{k^2 - \left(\frac{\mu_S - r}{\sigma_S}\right)^2}$. Similarly, the only way the call price is maximised to derive the seller's reservation price is if μ_V takes the highest value possible, meaning that $\kappa_2 = -\sqrt{k^2 - \left(\frac{\mu_S - r}{\sigma_S}\right)^2}$.

We are now dealing with a generic interval *[call$_{min}$, call$_{max}$]* for the call price, which, in terms of the good-deal bounds pricing, is given by the stochastic discount factors in the interval $\left[-\sqrt{k^2-\left(\frac{\mu_S-r}{\sigma_S}\right)^2}, \sqrt{k^2-\left(\frac{\mu_S-r}{\sigma_S}\right)^2}\right]$. *call$_{min}$* is the lower bound and *call$_{max}$* is the upper bound for the option price.

Bear in mind though that these are individual transactions. What we are modelling here is not a market for a homogeneous good for which there are numerous buyers and sellers bidding and asking prices at the same time (like a stock market), but a market for infrequently traded assets, where occasionally there exists an interested buyer or a seller. Under these conditions, we can only specify the likely interval for the price of the option. Eventually, by making additional assumptions about the type of market and about which counterparty we are (the buyer or the seller), we could uniquely determine the value of the option.

The process for asset *V* can be re-written as:

$$dV = (r-q_1)Vdt + \sigma_V V\left(\rho dz^* + \sqrt{1-\rho^2}dw^*\right) \tag{19}$$

where: $q_1 = r - \mu_V + \sigma_V \rho \frac{\mu_S - r}{\sigma_S} + \sigma_V \sqrt{1-\rho^2}\kappa_2$

If μ_V, σ_V and κ_2 are constants, then *V* follows a lognormal distribution. Remembering equation (12) and noticing that the process for *V* looks like the process for a stock paying a dividend yield equal to q_1, we can express the option price as the Black-Scholes price of a call option on a dividend paying stock:

$$C_0 = V_0 e^{-q_1 T} N(d_1) - K e^{-rT} N(d_2) \tag{20}$$

$$C_0 = e^{-rT}\left[V_0 e^{\left[\mu_V - \sigma_V \rho\left(\frac{\mu_S - r}{\sigma_S}\right) - \sigma_V\sqrt{1-\rho^2}\kappa_2\right]T} N(d_1) - K N(d_2)\right] \tag{21}$$

where: $N(.) - cumulative\ normal\ distribution\ function$

$$d_1 = \frac{ln\left(\frac{V_0}{K}\right) + [\mu_V - \sigma_V \rho\left(\frac{\mu_S - r}{\sigma_S}\right) - \sigma_V\sqrt{1-\rho^2}\kappa_2 + \frac{\sigma_V^2}{2}]T}{\sigma_V\sqrt{T}}$$

$$d_2 = d_1 - \sigma_V\sqrt{T}$$

V_0 – forward value

$$\kappa_2 = -\sqrt{k^2 - \left(\frac{\mu_S - r}{\sigma_S}\right)^2}\ ,\ \ for\ the\ upper\ bound$$

$$\kappa_2 = \sqrt{k^2 - \left(\frac{\mu_S - r}{\sigma_S}\right)^2}\ ,\ \ for\ the\ lower\ bound$$

The final option price is a modified version of the Black-Scholes option pricing formula. The modification reflects exactly the adjusted drift term that was used to describe the process *V* under the new probability measure Q_{GDB} and which appears in equation (18). The drift is adjusted downwards to reflect the higher degree of uncertainty which exists on an incomplete market compared to a complete market due to the part of the total risk which remains unhedged.

Notice that, unlike in the Black-Scholes (1973) model, in the incomplete market, the expected return μ_V is still present in the expression for the option price (see equation (21)). Even if we were able to find a traded asset *S* that is perfectly correlated with our underlying, making ρ equal to 1, the pricing formula would still depend on the expected return of the illiquid asset *V*. This is because the underlying and the risky asset correlated with it have different expected returns and both have to be taken into account in the pricing mechanism.

3. Implications of the good-deal bounds

3.1. Calibrating the upper and the lower bound

The difficulties of the good-deal bounds pricing mechanism are the choice and calibration of the volatility restriction *k*.

The selection of k: Once we have found a traded risky asset correlated with the non-traded underlying of the option, we can express the restriction on the volatility of the stochastic discount factor in terms of the Sharpe ratio of this traded asset. Mathematically, *k* must be at least equal to the Sharpe ratio of the risky traded asset *S* in order for $\kappa_2 = \left[-\sqrt{k^2 - \left(\frac{\mu_S - r}{\sigma_S}\right)^2}, \sqrt{k^2 - \left(\frac{\mu_S - r}{\sigma_S}\right)^2}\right]$ to be defined.

Cochrane and Saa-Requejo (2000) suggest that the bound *k* be set equal to twice the market price of risk on the stock market. In other words, we relate the unknown *k* to something that

we can find out, the Sharpe ratio of a traded asset. We have to realise though that the larger the difference between the restriction *k* and the Sharpe ratio of the traded risky asset is the wider the option price bounds become.

Interpretation of the good-deal bounds: Cochrane and Saa-Requejo (2000) point towards a nice interpretation of the bounds as a bid-ask spread. The lower bound would correspond to the bid price and the upper bound, to the ask price. The bid and ask prices also relate to our interpretation of the good-deal bounds prices in terms of reservation prices. The buyer's reservation price shows the buyer's maximum valuation of an asset and the seller's reservation price, the seller's minimum valuation of that asset. For any price lower than his maximum valuation, the buyer will agree to buy and, for any price higher than his minimum valuation, the seller will want to sell. Otherwise, no transaction occurs.

The bid-ask spread idea is reiterated by Carr et al. (2001). However, unlike Cochrane and Saa-Requejo (2000), the authors do not use a Sharpe ratio criterion to restrict the price interval for assets in incomplete markets, but a generalized version of the coherent risk measure first introduced by Artzner et al. (1999). Carr et al. (2001) argue that economic agents will not only invest in any arbitrage opportunity, but also invest in any acceptable opportunity. Acceptable opportunities are defined as claims for which the difference between their payoff and their hedge is not necessarily non-negative, but simply acceptable according to the level of risk aversion of the economic agent. As discussed previously in Section 2.2, the good-deal bounds are equivalent to a coherent measure of risk.

Inspired by, among others, Cochrane and Saa-Requejo (2000) and Carr et al. (2001), Cherny and Madan (2010) introduce the concepts of 'conic finance' and 'two price markets'. Essentially, they argue that every asset should be characterised by a bid and an ask price, not by one price, and derive closed-form solutions for both put and call options. The presence of the bid-ask spreads is motivated by the different levels of liquidity in the market. This is exactly the idea put forward by our paper: an option written on an illiquid asset will no longer be characterised by one price but by a price interval, which can have different properties than the one suggested by the one price models such as the Black-Scholes (1973).

3.2. Option price behaviour in incomplete markets

We continue with a sensitivity analysis. Standard (complete-market) option pricing theory predicts that an increase in the volatility of the underlying asset always leads to an increase in the value of the option. The good-deal bounds technique, which prices the assets starting from the assumption that the market is incomplete, shows that this is not always the case. This can best be seen graphically in *Figure 1*, where, all else equal, the volatility of the underlying asset increases from 1% to as much as 50%, but the option prices on the lower bound no longer follow an increasing pattern.

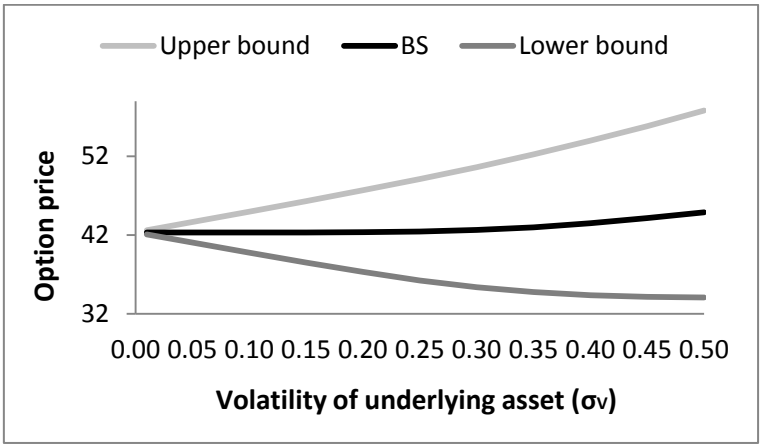

Figure 1 plots the sensitivity of the call price with respect to the volatility of the underlying asset (good-deal bounds prices vs. Black-Scholes prices). The parameter values are as follows: σ_S = 16%, μ_S = 8%, r = 4%, Sharpe ratio asset S = 0.25, k = 0.5, ρ = 0.8, V_0 = 100, K = 60, T = 1 year and $\mu_V = r + \rho \frac{\sigma_V}{\sigma_S}(\mu_S - r)$.

By fixing the market price of risk and all other parameters except for the volatility of the underlying asset, we see that, at an increase in the volatility of the underlying, both the Black-Scholes (1973) option price and the upper bound prices are increasing. However, the lower bound prices display a decreasing pattern, instead of an increasing one as we would expect. This means that we are willing to pay less and less for an asset as uncertainty increases (the lower bound prices are the buyer's reservation prices). This is exactly the feature that cannot be explained by the complete-market models which take into account only the hedgeable sources of risk, not the unhedgeable ones as well. However, the results are consistent with the findings of Henderson (2007) and Miao and Wang (2007), who also conclude that market incompleteness can decrease the option value.

For very low values of σ_V, the prices converge, because, if we eliminate all the uncertainty in the underlying asset (the infrequently traded asset), the only source of uncertainty left comes from the traded asset and we are back in the Black-Scholes framework. As σ_V increases though, the effect of the unhedged risk also increases and that is reflected in the steady widening of the bounds.

Note that, for arbitrage reasons, the following CAPM-type of relationship must hold: $\mu_V = r + \rho \frac{\sigma_V}{\sigma_S}(\mu_S - r)$ (see Davis, 2006).

The fact that the option value decreases in response to an increase in unhedgeable risk can be explained in terms of an implicit negative dividend yield. At the lower bound, the adjusted drift term of the stochastic process for asset *V* is $\mu_V - \sigma_V \rho \frac{\mu_S - r}{\sigma_S} - \sigma_V \sqrt{1-\rho^2}\kappa_2$, which can be shown to be lower than the risk-free interest rate *r*. Essentially, the economic agent is willing to invest at a negative implicit dividend yield in order to exit the incomplete market setting and avoid dealing with unhedgeable sources of risk.

If we use the same CAPM-type of formulation for the expected return of the infrequently traded asset *V*, $\mu_V = r + \rho \frac{\sigma_V}{\sigma_S}(\mu_S - r)$, then the drift term of the process *V* becomes equal to $r\sigma_S - \sigma_S \sigma_V \sqrt{1-\rho^2}\kappa_2$, a value obviously lower than $r\sigma_S$. Because σ_S is non-negative, we can easily divide by it on both sides and finally write:

$$r - \sigma_V \kappa_2 \sqrt{1-\rho^2} < r \tag{22}$$

This is consistent with the results of Henderson (2007). Using a utility indifference approach, she also finds that, on an incomplete market, investment can occur at a negative implicit dividend yield given a set of specific parameter values.

The inverse relationship between the option value and the volatility of the underlying is not independent of the moneyness of the option though. In fact, as *Figure 2* shows, the more in-the-money the call option is the more pronounced this inverse relationship is. As soon as the option is at-the-money, the negative effect of volatility on option value disappears.

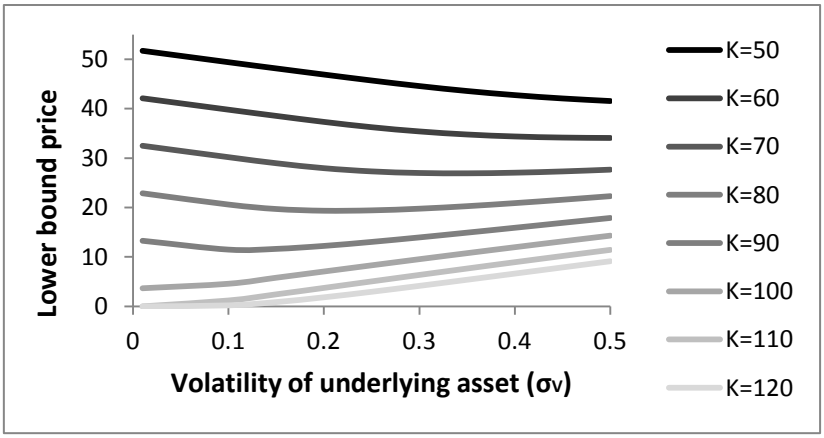

Figure 2 plots the lower bound prices for different values of the strike price and of the volatility of the underlying asset. The parameter values are as follows: V_0 = 100, T = 1 year, r = 4%, σ_V = 15%, σ_S = 16%, μ_S = 8%, ρ = 0.8, Sharpe ratio S = 0.25. The drift term is still given by: $\mu_V = r + \rho \frac{\sigma_V}{\sigma_S}(\mu_S - r)$.

A very important parameter for the option price is the restriction *k* on the total volatility of the stochastic discount factor. We know that the stochastic discount factor is the product of a risk-free rate discount factor and a Radon-Nikodym derivative (Björk, 2009). Its expected value is a constant and it is equal to the risk-free rate discount factor. It is the variance of the stochastic discount factor that changes and that we restrict via *k*. The variance of the stochastic discount factor shows the "distance" between the physical (real world) probability measure *P* and the new probability measure Q_{GDB}. When these two probability measures are very close to each other, the variance of the stochastic discount factor is low and the good-deal bounds are tight. The farther the probability measures are from each other, the higher the variance of the stochastic discount factor is and the wider the bounds become.

The effect of the volatility restriction *k* on the option price is presented in *Figure 3*. When the restriction is exactly equals to the Sharpe ratio of the traded asset, we exit the incomplete market setting and the GDB prices converge to the Black-Scholes price. Afterwards, as *k* increases, the bounds widen. The prices on the lower bound decrease rapidly and the ones on the upper bound experience a sharp increase.

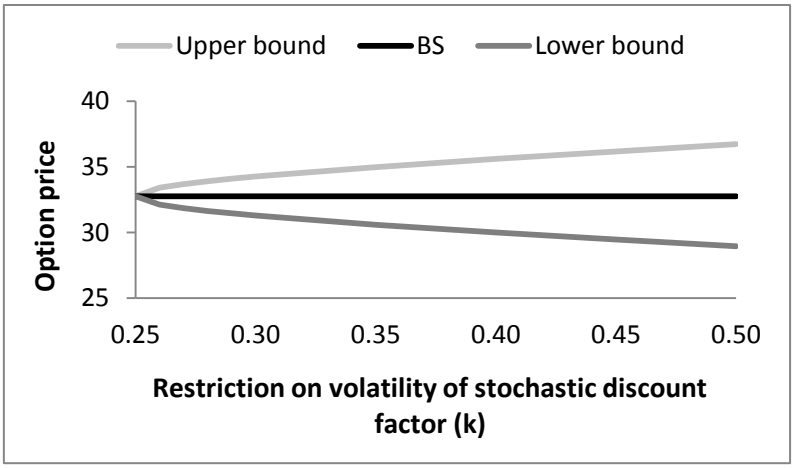

Figure 3: The sensitivity of the call price with respect to the restriction on the volatility of the stochastic discount factor. The parameter values are as follows: V_0 = 100, K = 70, T = 1 year, r = 4%, σ_V = 15%, σ_S = 16%, μ_S = 8%, ρ = 0.8, Sharpe ratio S = 0.25 and $\mu_V = r + \rho \frac{\sigma_V}{\sigma_S}(\mu_S - r)$.

The correlation coefficient *ρ* between the underlying asset and the traded risky asset also plays a role in determining the behaviour of the option prices. *Figure 4* shows that as the correlation coefficient increases and the partial hedge improves the GDB prices approach the Black-Scholes price more and more. In fact, for a perfect (negative or positive) correlation, the two types of prices equalise. The largest price difference can be observed at the other extreme, when *ρ* = 0, because here we cannot hedge any part of the risk and we deal only with unhedgeable sources of risk.

The interesting part about *Figure 4* is the fact that there is a large gap between the prices on an almost complete market and the Black-Scholes price at *ρ* = 1. In fact, the GDB prices approach the Black-Scholes price at the speed of $\sqrt{1-\rho^2}$. This is consistent with the results of Davis (2006), via a utility indifference approach.

Following Davis (2006), we make the notation:

$$\varepsilon = \sqrt{1-\rho^2} \tag{23}$$

in order to perform a Taylor expansion around *ε*=0 (⇔ *ρ*=1).

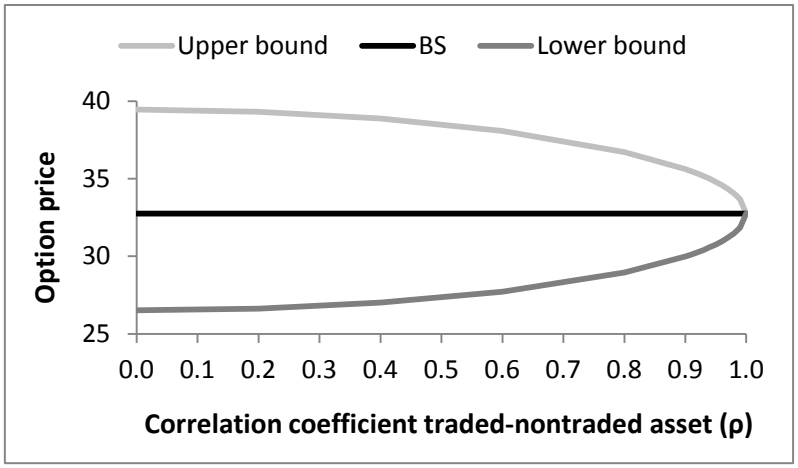

Figure 4 plots the sensitivity of the call price with respect to the correlation coefficient between the underlying asset and the correlated traded risky asset. The parameter values are as follows: V_0 = 100, K = 70, T = 1 year, r = 4%, σ_V = 15%, σ_S = 16%, μ_S = 8%, Sharpe ratio S = 0.25 and $\mu_V = r + \rho\frac{\sigma_V}{\sigma_S}(\mu_S - r)$.

Knowing that $\mu_V = r + \rho\frac{\sigma_V}{\sigma_S}(\mu_S - r) = r + \frac{\sigma_V}{\sigma_S}(\mu_S - r)\sqrt{1-\varepsilon^2}$, the good-deal bounds price in equation (21) becomes:

$$\begin{aligned} C_0 &= V_0 N(d_1) - Ke^{-rT}N(d_2) - \varepsilon N(d_1)V_0\sigma_V\kappa_2 T + O(\varepsilon^2) \\ &= BS\,price - \varepsilon N(d_1)V_0\sigma_V\kappa_2 T + O(\varepsilon^2) \end{aligned} \tag{24}$$

where: O(.) – higher order terms

The good-deal bounds option price converges to the BS price of a call option on a non-dividend paying asset at the speed of ε, where ε is defined in equation (23). Even at a ρ = 0.99, ε is already equal to 0.14 and it has a substantial impact on the BS price. Furthermore, since the good-deal bounds prices are a modified version of the Black-Scholes price, the second term of the Taylor expansion in equation (24) incorporates a 'delta-effect': $N(d_1)$, the delta measure of a call option written on a non-dividend paying asset, multiplied by additional terms to account for market incompleteness.

4. A perpetual American call option

So far, the analysis has focused on European call options. The advantage of deriving prices for American type of options is that we are also able to time an investment, not only calculate its value. In a complete market, the price of an American call option coincides with the prices of a European call option, because there is no incentive to exercise the option early as long as the underlying asset does not pay any dividends. However, we will now show that in an incomplete market, where the underlying asset of an option is either infrequently traded or non-traded, the economic agent does have an incentive to exercise an American call option early when he is faced with increasing unhedgeable risk which erodes the option value. Furthermore, Merton (1973) has shown that, in a complete market setting, the prices of perpetual American options are closed-form solutions. We will also conduct our analysis for a perpetual American option, such that we can arrive at analytical formulas comparable to the ones obtained in Section 2.

Assume that *F(V)* is the price of a perpetual American call option written on *V*, where *V* is the infrequently traded asset described by the stochastic process in equation (18). For ease of calculations, we make a notation for the drift term of the stochastic process in equation (18), $\tilde{\mu} = \mu_V - \sigma_V \rho \frac{\mu_S - r}{\sigma_S} - \sigma_V \sqrt{1-\rho^2}\kappa_2$. Using the Feynman-Kac formula, the derivative *F* must satisfy the following PDE:

$$F_t + F_V \tilde{\mu} V + \frac{1}{2} F_{VV} \sigma_V^2 V^2 - rF = 0 \tag{25}$$

where: r – deterministic short interest rate

Merton (1973) has also shown that the PDE in equation (25) reduces to an ODE, due to the fact that the perpetual option has infinite maturity (i.e. $t \to \infty$). The derivative of *F* with respect to time drops out and equation (25) reduces to:

$$F_V \tilde{\mu} V + \frac{1}{2} F_{VV} \sigma_V^2 V^2 - rF = 0 \tag{26}$$

s.t. $F(0) = 0$ (boundary condition)

$F(V_*) = V_* - K$ (value-matching condition)

$F'(V_*) = 1$ (smooth-pasting condition)

Assume that *F(V)* is of the form $F(V) = V^{\lambda}$. Then:

$$\lambda^2 \frac{1}{2}\sigma_V^2 + \lambda\left(\tilde{\mu} - \frac{1}{2}\sigma_V^2\right) - r = 0 \tag{27}$$

For $V^{\lambda} > 0$, the solutions to equation (27) are:

$$\lambda_1 = -\frac{\sqrt{\sigma_V^4 + (8r - 4\tilde{\mu})\sigma_V^2 + 4\tilde{\mu}^2} - \sigma_V^2 + 2\tilde{\mu}}{2\sigma_V^2} \text{ and } \lambda_2 = \frac{\sqrt{\sigma_V^4 + (8r - 4\tilde{\mu})\sigma_V^2 + 4\tilde{\mu}^2} + \sigma_V^2 - 2\tilde{\mu}}{2\sigma_V^2} \tag{28}$$

For arbitrary constants C_1 and C_2, the general solution to equation (27) is:

$$F(V) = C_1 V^{\lambda_1} + C_2 V^{\lambda_2} \tag{29}$$

s.t. $F(0) = 0$

$V_* - K = C_1 V_*^{\lambda_1} + C_2 V_*^{\lambda_2}$

$1 = C_1 \lambda_1 V_*^{\lambda_1 - 1} + C_2 \lambda_2 V_*^{\lambda_2 - 1}$

To satisfy the boundary condition *F(0) = 0*, C_1 must be zero, otherwise *F(V)* converges to infinity when *V* goes to zero and λ_1 is negative. *F(V)* is then simply $F(V) = C_2 V^{\lambda_2}$.

Via the value-matching and smooth-pasting conditions, we are able to determine that the constant C_2 is given by $C_2 = \frac{V_*^{1-\lambda_2}}{\lambda_2}$ and that the analytical solutions for the optimal investment threshold and the option value are respectively:

$$V_* = \frac{K\lambda_2}{\lambda_2 - 1}, with\ \lambda_2 > 1\ s.t.\ V_* > K \tag{30}$$

$$F(V_*) = V_* - K = \frac{K}{\lambda_2 - 1} \tag{31}$$

The early exercise of the perpetual American option can happen once the value *V* is at least as large as the threshold V_*, where V_* is higher than the strike price *K*.

Similar to the analytical solution for the European call option presented in equation (21), the option value for the perpetual American call is dependent on the drift term of the underlying asset *V*, μ_V, and on the restriction on the volatility of the stochastic discount factor, *k*. Furthermore, the optimal investment threshold and the option value differ for the lower and the upper bound prices, leading to potentially different investment decisions depending on

whether we have a long or a short position in the underlying asset (i.e. whether we are on the lower bound or on the upper bound).

In a complete market, it is never optimal to exercise an American call option early if the underlying asset is a non-dividend paying asset. In an incomplete market, where the underlying asset of an option is either infrequently traded or non-traded such that we can no longer construct a replicating portfolio to price the option, things change. As *Figure 5* shows, at the lower bound, the early exercise is triggered by the increase in the volatility of the underlying asset. The option value and the optimal investment threshold for the lower bound prices are both decreasing in the volatility of the underlying. At the upper bound, early exercise is never optimal, because the optimal investment threshold V_* is always lower than the strike price *K*.

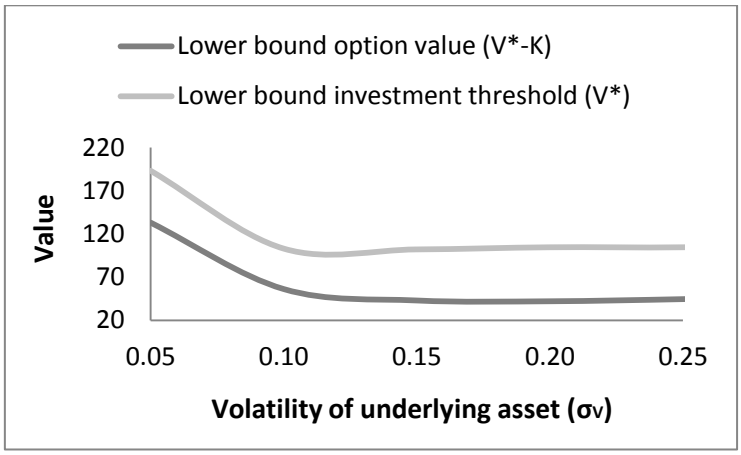

Figure 5 plots the lower bound optimal investment threshold and option value for different values of the volatility of the underlying asset. The parameter values are as follows: σ_S = 16%, μ_S = 8%, *r* = 4%, Sharpe ratio asset *S* = 0.25, *k* = 0.5, *ρ* = 0.8, V_0 = 100, *K* = 60 and $\mu_V = r + \rho\frac{\sigma_V}{\sigma_S}(\mu_S - r)$.

An increase in the volatility of the infrequently traded underlying asset means an increase in the unhedgeable sources of risk. Consequently, the economic agent is willing to pay less and less for the option, which explains the decreasing value, and exercises the option early to lock in value as long as it still exists.

This result contradicts the standard options theory, but it is supported by the work of Henderson (2007) and Miao and Wang (2007), who also show via utility indifference pricing that early exercise is possible for an American call option in an incomplete market. However, utility indifference pricing involves more difficult calculations than the ones presented in this

paper and, even though Henderson (2007) reaches closed-form solutions, the results of Miao and Wang (2007) are based on numerical computations.

5. Applications of good-deal bounds pricing

There are several potential applications of the good-deal bounds pricing mechanism described in this paper, like the valuation of long-dated contracts offered by life insurance companies and pension funds or the valuation of real options.

Life insurance companies and pension funds are confronted with two main types of unhedgeable risks: interest rate risk when valuing long-dated cash flows from premia and either longevity or mortality risk depending on the perspective. The problem with long-dated cash flows occurs for maturities beyond 30 years for which we do not have bonds traded in the market anymore. For longevity and mortality risk, the problem is even more serious, because we do not have an organized market to trade such risks on. In this situation, there are more sources of risk than traded assets and, as a result, the market becomes incomplete and we no longer have the possibility to price the assets based on the classical replication arguments. An appropriate solution could be good-deal bounds pricing, which deals with the impossibility to construct a replicating portfolio when pricing a contingent claim.

Real options are options written on non-financial assets. They are investments in real assets like land, buildings, even oil concessions or mines (Hull, 2012). These real assets are infrequently traded assets, meaning that it is difficult to observe prices for these assets let alone their market price of risk, meaning that their valuation can only be done in the context of incomplete markets.

Perhaps the most obvious example of an incomplete market is the real estate market. Real-estate assets are illiquid assets. Even if at some point in time we observe a trade for a particular property, the same property might never be traded again or, at best, traded at large intervals of time. Furthermore, real estate assets are heterogeneous: each property is unique and it is therefore impossible to trade multiple homogeneous units the way we do with liquid assets like stocks.

We can think of land as a real option. If we follow Titman (1985) and price vacant land as a European call option on a building that could potentially be built on that land, the underlying asset of such an option is the building, the strike price is the construction cost and the

exercise time is at the start of the development. Titman's (1985) main assumptions are that the market on which the real option exists is frictionless and that the price of the option can be calculated by means of replicating arguments. But, given that buildings are infrequently traded assets, the market setting is incomplete. Within this framework we can at best obtain a partial hedge for the underlying, without the possibility to construct a replicating portfolio and price the real option with the Black-Scholes (1973) option pricing formula. However, the good-deal bounds valuation technique can price contingent claims even in the presence of non-traded or infrequently traded underlying assets, provided that there exists a traded asset correlated with the underlying. For instance, in the last example mentioned, land as a call option on a building, the asset correlated with the price process of the infrequently traded building could be a REIT.

6. Conclusion

We price European and perpetual American call options in incomplete markets using the good-deal bounds pricing technique and derive closed-form solutions in order to gain further insights into the implications of option prices.

We find that, in an incomplete market, an increase in the volatility of the underlying asset does not always lead to an increase in the option price as Black-Scholes (1973) model would predict. This is due to the increase in the unhedgeable sources of risk additional to the hedgeable risk that we find on a complete market. Specifically, the lower bound prices (the buyer's prices) decrease as the volatility of the underlying asset increases, meaning that, when uncertainty increases, the buyer is willing to pay less and less for the option. Furthermore, in an incomplete market where the underlying asset of an option is either infrequently traded or non-traded, early exercise of an American call option becomes possible at the lower bound. The option value and the optimal investment threshold for the lower bound prices are both decreasing in the volatility of the underlying. The economic agent exercises the option early to lock in value before it disappears as a result of increased unhedgeable risk.

The advantage of the good-deal bounds methodology over other incomplete market techniques is that it arrives at closed-form solutions for the option prices and, more importantly, comparable to the Black-Scholes (1973) price. Furthermore, the technique we present in this paper is accessible and easy-to-implement and it opens the door to the pricing of assets such as long-dated contracts or real options.

The difficulties of this approach remain the choice and calibration of k, which is the restriction on the total volatility of the stochastic discount factor. The only guideline so far is to find a traded risky asset on the market, correlated with the infrequently traded underlying asset of the call option, and use the Sharpe ratio of this asset to set the restriction.

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