A Risk Charge Calculation Based on Conditional Probability

Topic #1: Risk Evaluation

David Ruhm¹, The Hartford, USA
and

Donald Mango², American Re-Insurance (Munich Re), USA

2003 ASTIN Colloquium

Note: This paper and supporting Microsoft Excel files are available online at http://www.casact.org/coneduc/specsem/sp2003/papers/.

Abstract

In this paper, a method will be illustrated which begins at the aggregate (portfolio) level for evaluating risk, and ends by producing prices for the component individual risks, effectively allocating the total portfolio risk charge. The result is an internally consistent allocation of diversification benefits. The method effectively extends any risk-valuation theory used at the aggregate portfolio level to the individual risks comprising the portfolio. The resulting prices are additive, with each risk’s price reflecting the degree to which it contributes to total portfolio risk.

Keywords: risk charge, allocation, conditional probability, additivity.

1. Background and Introduction

There are several methods for assigning risk charges to individual risks within a portfolio. Among them are utility functions, risk-adjusted probabilities, risk-adjusted weights, etc. After applying any of these methods to price individual risks, the issue of covariance and diversification must then be dealt with, because the portfolio owner’s real exposure is to the aggregate portfolio result. In other words, there is no risk other than portfolio risk – risk is aggregate by its nature.

Accounting for aggregate portfolio effects in property-casualty insurance prices has historically created some difficult problems, including:

1) Additivity or sub-additivity of prices;
2) Measuring how much diversification efficiency actually exists;
3) Allocating the diversification benefits back to the individual risks; and
4) Order-dependence.

We begin with the following premise: Several separate but somewhat interdependent risk-bearing financial quantities are held as a risk portfolio over a specific time horizon. The type of value that is “at risk” can be selected in any reasonable way: liquidation value, book value, or the change over the specified time period in an alternative calculation of value. We assume that the following are given:

¹ E-mail: david.ruhm@thehartford.com. Contact: The Hartford, Corporate Research, Hartford Plaza, HO-GL-140, Hartford, CT, 06115, USA. Phone: (860)547-8815  Fax: (860)547-4639.
² E-mail: dmango@amre.com. Contact: 685 College Road East, Princeton, NJ, 08543, USA. Phone: (609)951-8233.  Fax: (609)419-8750.
• The joint distribution of outcomes for the risks at the time horizon’s end; and
• The relative values to the portfolio owner of the possible aggregate outcomes
(possibly reflecting risk-averse valuation).

In [1], Venter showed that covariance loadings can be used to produce additive,
arbitrage-free risk charges, and also showed that a covariance loading results from a risk-
adjusted distribution that is based on the conditional expectation of a “target” variable.
Mango, in Appendix B of [2], demonstrated a method of allocating an overall capital cost
charge to individual portfolio components using a similar concept. The ratio of price to
probability (the “pricing density” function) was described and analyzed in a paper by
Buhlmann [3]. Ruhm [4] analyzed arbitrage-free risk loads in terms of the
price/probability ratio (the “risk discount” function).

In this paper, a method will be illustrated which synthesizes some results from each of
these papers. The method begins at the aggregate level for evaluating risk, and ends by
producing prices for individual risks, effectively allocating the total portfolio risk charge.
The result is an internally consistent allocation of diversification benefits, avoiding the
difficulties listed above. The method effectively extends any risk-valuation theory used at
the aggregate portfolio level to the individual risks comprising the portfolio. The
resulting prices are additive, with each risk’s price reflecting the degree to which it
contributes to total portfolio risk.

2. An Illustrative Example
Before providing a formal, mathematical description of the method, an example will help
to illustrate the idea. (This example is summarized in Exhibit 2, which is a printout of the
Microsoft Excel workbook Bowles Ruhm-Mango Exhibit 2, posted on the CAS
website.) For clarity of presentation, the simplest possible case will be analyzed: a
portfolio of only two risks, Risk 1 and Risk 2, each of which has only two possible
outcomes, a loss of either 100 or 200. Net present value factors are omitted for simplicity,
although in practice they would be applied to obtain a final price.

Suppose that losses for the two risks are distributed jointly as follows:

<table>
<thead>
<tr>
<th>Joint Loss Distribution</th>
<th>Risk 2 Loss</th>
<th>Row Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk 1 Loss =</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>200</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>25%</td>
<td>25%</td>
</tr>
</tbody>
</table>

| Column Total | 60% | 40% | 100% |

Expected values are 150 for Risk 1 and 140 for Risk 2, with 20% correlation. The
possible aggregate outcomes and their probabilities are determined by this structure:
At this point, valuation for risk comes into play. If the valuation is risk-neutral, meaning that there is no pricing adjustment for risk, then the value of the portfolio is simply its expected value:

\[
\text{Expected loss} = 200 \times 35\% + 300 \times 40\% + 400 \times 25\% = 290
\]

Expected loss is a risk-neutral calculation; there are implicit outcome weights within the formula, all equal to 1.0:

\[
\text{Expected loss} = 200 \times 35\% \times 1.0 + 300 \times 40\% \times 1.0 + 400 \times 25\% \times 1.0 = 290
\]

One way to introduce a risk adjustment is by giving outcome-specific weights in the expected value calculation. To produce risk-averse valuation, the more severe (higher loss) outcomes would receive larger weights, and the less severe outcomes would receive lower weights; for example,

<table>
<thead>
<tr>
<th>Portfolio Outcome</th>
<th>Risk-Averse Outcome Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>0.500</td>
</tr>
<tr>
<td>300</td>
<td>1.000</td>
</tr>
<tr>
<td>400</td>
<td>1.250</td>
</tr>
</tbody>
</table>

The weights could come from a utility-based derivation, an options-formula method, or any other source (including judgment) – the technique presented here is independent of the particular portfolio risk adjustment theory, and will operate with any of them.

After normalizing these weights (scaling them so their expected value is one), the aggregate table is:

<table>
<thead>
<tr>
<th>Portfolio Outcome</th>
<th>Outcome Probability</th>
<th>Normalized Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>35%</td>
<td>0.563</td>
</tr>
<tr>
<td>300</td>
<td>40%</td>
<td>1.127</td>
</tr>
<tr>
<td>400</td>
<td>25%</td>
<td>1.408</td>
</tr>
<tr>
<td><strong>Expected Value</strong> = 290</td>
<td><strong>Total = 100%</strong></td>
<td><strong>Expected Value = 1.000</strong></td>
</tr>
</tbody>
</table>

The risk-adjusted price for the total portfolio can now be calculated as the expected weighted outcome:

\[
\text{Risk-adjusted expected loss} =
\]
200*35%*0.563 +
300*40%*1.127 +
400*25%*1.408
= 315

This price can also be produced by a set of risk-adjusted probabilities, which are the products of the actual probabilities and the normalized weights:

<table>
<thead>
<tr>
<th>Portfolio Outcome</th>
<th>Actual Probability</th>
<th>Risk-adjusted Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>35%</td>
<td>20%</td>
</tr>
<tr>
<td>300</td>
<td>40%</td>
<td>45%</td>
</tr>
<tr>
<td>400</td>
<td>25%</td>
<td>35%</td>
</tr>
</tbody>
</table>

Expected Value = 290
Total = 100%
Risk-adjusted Expected Value = 315

The risk-adjusted expected value shown is a risk-loaded price for the total portfolio. Thus, a risk charge of 25 (=315 – 290) is implied by the set of relative weights and the probability distribution of the aggregate portfolio outcomes.

The risk charge will now be allocated to the individual risks. This allocation is based on the conditional relationship between each risk’s outcomes and the portfolio’s possible outcomes, so that each risk receives a charge that represents how much it contributes to undesirable portfolio outcomes. This principle is the basis of the method. The resulting prices are additive, so that the price of any combination of risks is found by simply adding the individual prices. The major advantage of this approach, which will be explored further below, is that it can handle any underlying dependence structure between the component risks.

3. Application of the Conditional Structure to Calculate Individual Risk Prices
As shown above, the price of the total portfolio is found by calculating the weighted expected value of the outcomes, using a set of normalized risk-adjustment weights. The pricing calculation for individual risks proceeds in essentially the same way.

In the example, Risk 1 has two possible outcomes, 100 and 200. Each of these outcomes will be assigned a risk-adjustment weight, and the price for Risk 1 will be calculated as the weighted expected value.

If the Risk 1 = 100, there are only two possibilities for the portfolio’s total outcome: 200 (if Risk 2 also = 100) or 300 (if Risk 2 = 200). Given that Risk 1 = 100, the probabilities for Risk 2 are 70% and 30%, from Bayes’ Theorem:

\[ P(A | B) = \frac{P(A \text{ and } B)}{P(B)} \]

\[ P(\text{Risk 2 = 100} | \text{Risk 1 = 100}) = 35\% / 50\% = 70\% \]
\[ P(\text{Risk 2 = 200} | \text{Risk 1 = 100}) = 15\% / 50\% = 30\% \]
The weight for the situation (Risk 1 = 100) is then calculated as follows:

<table>
<thead>
<tr>
<th>Portfolio Outcome</th>
<th>Conditional Probability</th>
<th>Normalized Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>70%</td>
<td>0.563</td>
</tr>
<tr>
<td>300</td>
<td>30%</td>
<td>1.127</td>
</tr>
<tr>
<td>400</td>
<td>0%</td>
<td>1.408</td>
</tr>
<tr>
<td><strong>Total = 100%</strong></td>
<td><strong>Expected Value = 0.732</strong></td>
<td></td>
</tr>
</tbody>
</table>

By the same procedure, the weight for the (Risk 1 = 200) situation is calculated:

<table>
<thead>
<tr>
<th>Portfolio Outcome</th>
<th>Conditional Probability</th>
<th>Normalized Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>0%</td>
<td>0.563</td>
</tr>
<tr>
<td>300</td>
<td>50%</td>
<td>1.127</td>
</tr>
<tr>
<td>400</td>
<td>50%</td>
<td>1.408</td>
</tr>
<tr>
<td><strong>Total = 100%</strong></td>
<td><strong>Expected Value = 1.268</strong></td>
<td></td>
</tr>
</tbody>
</table>

Then, the price for Risk 1 is calculated as a weighted expected value, just as in the earlier calculation of the portfolio price:

<table>
<thead>
<tr>
<th>Risk 1 Outcome</th>
<th>Outcome Probability</th>
<th>Normalized Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>50%</td>
<td>0.732</td>
</tr>
<tr>
<td>200</td>
<td>50%</td>
<td>1.268</td>
</tr>
<tr>
<td><strong>Expected Value = 150</strong></td>
<td><strong>Total = 100%</strong></td>
<td><strong>Expected Value = 1.000</strong></td>
</tr>
</tbody>
</table>

Risk-adjusted
Expected Value = 163

Note that the weights for Risk 1’s outcomes have an expected value of exactly one. This means that the calculation can also be expressed in terms of risk-adjusted probabilities, which are the products of the actual probabilities and the weights:

<table>
<thead>
<tr>
<th>Risk 1 Outcome</th>
<th>Actual Probability</th>
<th>Risk-adjusted Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>50%</td>
<td>36.6%</td>
</tr>
<tr>
<td>200</td>
<td>50%</td>
<td>63.4%</td>
</tr>
<tr>
<td><strong>Expected Value = 150</strong></td>
<td><strong>Total = 100%</strong></td>
<td><strong>Total = 100%</strong></td>
</tr>
</tbody>
</table>

Risk-adjusted
Expected Value = 163

The tables for Risk 2, derived in an identical manner, are:

<table>
<thead>
<tr>
<th>Risk 2 Outcome</th>
<th>Outcome Probability</th>
<th>Normalized Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>60%</td>
<td>0.798</td>
</tr>
<tr>
<td>200</td>
<td>40%</td>
<td>1.303</td>
</tr>
<tr>
<td><strong>Expected Value = 150</strong></td>
<td><strong>Total = 100%</strong></td>
<td><strong>Expected Value = 1.000</strong></td>
</tr>
</tbody>
</table>

Risk-adjusted
Expected Value = \textbf{152}

<table>
<thead>
<tr>
<th>Risk 2 Outcome</th>
<th>Actual Probability</th>
<th>Risk-adjusted Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>60%</td>
<td>47.9%</td>
</tr>
<tr>
<td>200</td>
<td>40%</td>
<td>52.1%</td>
</tr>
</tbody>
</table>

Expected Value = \textbf{150}

Risk-adjusted Expected Value = \textbf{152}

Total = 100\%  
Total = 100\%

The prices for Risk 1 and Risk 2 add to the total portfolio price, as desired.

Following Venter [1], the conditional method can be conveniently expressed as a covariance risk load formula that can be applied to any risk, including any derivative of a portfolio component (such as an excess loss layer):

\[ \text{Risk Load} = \text{Cov}(Z, R), \]

where \( Z \) represents the normalized weight (as a function of the aggregate portfolio outcome) and \( R \) represents the individual risk’s outcome. The reader can verify by inspection, using the definition of covariance, that all the risk loads derived in the example above are produced by this formula.

In summary, the key points just demonstrated are:

1. The total portfolio risk charge is determined by risk assessment at the aggregate level;
2. This is split to the individual risks based on the conditional relationship between the risks’ outcomes and the aggregate results for the portfolio.
3. All prices are completely determined by the portfolio-level weights (which can be interpreted as risk relativities) and the probability structure, so that no other information is required.
4. Correlations between risks (and between each risk and the portfolio) are included in the prices in full detail, via the conditional probabilities.
5. Prices produced by this method are additive.
6. Being based on risk-adjusted probabilities, the prices are arbitrage-free within the context of the portfolio and its specified risk valuation structure (i.e., the specified set of weights).
7. The method can be summarized as a covariance risk load formula, where the reference variable is the set of normalized risk relativities.
4. The State-Price Structure Underlying the Example

An implicit state-price structure underlies the prices calculated by this method, where the states are defined as the possible combinations of the risks’ outcomes:

<table>
<thead>
<tr>
<th>State Aggregate</th>
<th>Outcome</th>
<th>Weight</th>
<th>Probability</th>
<th>State Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>(100, 100)</td>
<td>200</td>
<td>0.563</td>
<td>35%</td>
<td>0.197</td>
</tr>
<tr>
<td>(100, 200)</td>
<td>300</td>
<td>1.127</td>
<td>15%</td>
<td>0.169</td>
</tr>
<tr>
<td>(200, 100)</td>
<td>300</td>
<td>1.127</td>
<td>25%</td>
<td>0.282</td>
</tr>
<tr>
<td>(200, 200)</td>
<td>400</td>
<td>1.408</td>
<td>25%</td>
<td>0.352</td>
</tr>
</tbody>
</table>

Weighted Expected = 315

Each “state price” is the product of the normalized weight and the state’s probability. The state prices add to exactly one. They are the risk-adjusted probabilities underlying the risks’ prices.

Any one of these state prices can be interpreted as the (undiscounted) value of an instrument (a derivative instrument of the two risks) that pays one dollar if the specified state occurs, and zero otherwise. Since exactly one of the states must occur, the states’ prices should add to one, because a portfolio holding exactly one of each derivative will produce one dollar with certainty. (This is what was meant by the phrase “internally consistent in the arbitrage-free sense,” as used above.) Normalizing the weights causes the state prices to add up to exactly one.

In assuming this two-risk portfolio, the portfolio-holder has effectively taken a short position in 200 of the (100,100) instruments, 300 of the (100,200) instruments, 300 of the (200,100) instruments, and 400 of the (200,200) instruments. One can multiply these amounts by their respective state prices and verify that the total price of this combination equals the total portfolio price of 315.

5. A More Detailed Example

Exhibit 1 shows summarized results of applying this method to underwriting results from the Bohra/Weist paper [7] submitted to the CAS 2001 DFA Call for Papers on “DFA Insurance Company.” The Microsoft Excel workbook Bowles Ruhm-Mango Exhibit 1 demonstrating this will be posted on the CAS website.
The valuation formula used to determine the risk-averse outcome weighting is a two-sided utility transform of total underwriting income $U_{IT}$ to risk-adjusted underwriting income $R_{UIT}$ via the following formula:

If $U_{IT} \geq 0$

\[
R_{UIT} = U_{IT} \times \left[ 1 + \left( \frac{U_{IT}}{1M} \right)^2 \right]
\]

Else

\[
R_{UIT} = U_{IT} \times \left[ 1 + \left( -\frac{U_{IT}}{100K} \right)^{0.5} \right]
\]

Section (2) on Exhibit 1 shows these parameters and curve forms, which were selected to calibrate to a desired overall implied portfolio risk premium, calculated as follows:

(1) $E[U_{IT}] = ($96.9M)

(2) Risk Adjustment Curve Parameters

| Upside Scale | 1,000,000 |
| Upside Shape | 200.00% |
| Downside Scale | 100,000 |
| Downside Shape | 50.00% |

(3) Risk-Adjusted Expected U/W Income

$R_{UIT} = ($244,714)

(4) Portfolio Risk Premium

$147,762 = (1) - (3)$

<table>
<thead>
<tr>
<th>LOB</th>
<th>Expected Loss</th>
<th>Expected U/W Income</th>
<th>Expected Risk-adjusted U/W Income</th>
<th>Allocated Risk Premium</th>
<th>Risk Premium as % of E[L]</th>
</tr>
</thead>
<tbody>
<tr>
<td>CA</td>
<td>115,995</td>
<td>(10,946)</td>
<td>(23,014)</td>
<td>12,068</td>
<td>10.4%</td>
</tr>
<tr>
<td>CMP</td>
<td>221,025</td>
<td>(7,910)</td>
<td>(23,152)</td>
<td>15,242</td>
<td>6.9%</td>
</tr>
<tr>
<td>HO</td>
<td>220,787</td>
<td>(19,460)</td>
<td>(67,474)</td>
<td>48,013</td>
<td>21.7%</td>
</tr>
<tr>
<td>PPA</td>
<td>437,352</td>
<td>(54,963)</td>
<td>(117,554)</td>
<td>62,591</td>
<td>14.3%</td>
</tr>
<tr>
<td>WC</td>
<td>145,131</td>
<td>(3,673)</td>
<td>(13,520)</td>
<td>9,847</td>
<td>6.8%</td>
</tr>
<tr>
<td>TOTAL</td>
<td>1,140,291</td>
<td>(96,952)</td>
<td>(244,714)</td>
<td>147,762</td>
<td>13.0%</td>
</tr>
</tbody>
</table>

The valuation formula used to determine the risk-averse outcome weighting is a two-sided utility transform of total underwriting income $U_{IT}$ to risk-adjusted underwriting income $R_{UIT}$ via the following formula:

The expected value of both the unadjusted and risk-adjusted underwriting income results for each LOB are shown in columns (7) and (8) of Exhibit 1. The Allocated Risk Premium by LOB equals the expected unadjusted U/W income minus the expected risk-adjusted U/W income—see Column (9). Column (10) displays these values as percentages of expected loss, putting them in a common format for inclusion in any premium-loading formula.
6. Derivation of Conditional Risk Charge Formulas (Discrete Case)

Assume a portfolio containing n risks with common time horizon T.

Definitions
- \( R_i \) = the outcome of the \( i^{th} \) risk at time T.
- \( w = \{R_1, \ldots, R_n\} \) = the state at time T, as defined by the portfolio.
- \( N = N(w) = \sum R_i \) = the aggregate portfolio result.
- \( V(N) \) = the valuation function that maps the aggregate portfolio result to its value.
- \( Z(N) = V(N)/N \) = the valuation weighting function. \( V(N) \) is scaled so that \( E[Z] = 1 \).
- \( p(\cdot) \) denotes the probability operator, and \( E[\cdot] \) denotes the expectation operator.
- \( v \) = the risk-free present value factor corresponding to the time horizon T.
- \( P = vE[V] = \text{the total value of the portfolio.} \) The additive definition of the portfolio value is consistent with arbitrage-free valuation, and is based on the implicit assumption that \( V(N) \) completely represents the values of the possible aggregate portfolio outcomes, with no additional modification necessary.

Conclusion 1: \( P = v\sum E[ZR_i] \).

Proof: \( P = vE[V] = vE[ZN] = vE[\sum R_i] = vE[\sum ZR_i] = v\sum E[ZR_i] \).

Additional Definitions
For fixed \( i \), define the following variables:
- \( P_i = vE[ZR_i] = v\{E[R_i] + \text{Cov}(Z, R_i)\} \) (By Lemma 1, \( P = \sum P_i \).
- \( X(i) = \{\text{possible values taken by } R_i\} \)
- \( N(r) = \{\text{possible values of } N \mid R_i = r\} \)

Conclusion 2: \( P_i = vE[rE[Z \mid R_i = r]] \).

Proof: \( P_i = vE[ZR_i] = vE[E[Zr \mid R_i = r]] = vE[rE[Z \mid R_i = r]] \).

Corollary: \( P = v\sum E[rE[Z \mid R_i = r]] \).

In practice, the calculation of \( P_i \) can be performed by taking the inner expectation across values of \( N \) (since \( Z \) is determined by \( N \)), and taking the outer expectation across values of \( R_i \):

\[
P_i = vE_{r \in X(i)}[rE_{n \in N(i)}[Z(n) \mid R_i = r]].
\]

This formula encapsulates the method shown above and in the exhibits.
7. A Connection to CAPM Pricing
The Capital Asset Pricing Model (“CAPM”) specifies expected returns for individual securities in terms of the total market return, under certain idealized conditions [6]:

\[ \text{E}[R_i] = r_f + \beta (\text{E}[R_m] - r_f) \]

By definition, expected return translates to price, provided the expected future value is known:

\[ \text{Price} = \frac{\text{E}[\text{Future Value}]}{(1 + \text{E}[\text{Return}])} \]

The CAPM formula can therefore be viewed as a pricing formula, given the expected future value of the security. Also, the formula is similar to the conditional risk charge method, in that the portfolio-level risk premium (the spread above risk-free, \((\text{E}[R_m] - r_f)\), which corresponds to a risk charge) is taken as an input, and is used to calculate risk premia for the individual component securities which comprise the market portfolio.

If we view the market as a portfolio, we can apply the conditional risk charge method to the idealized CAPM scenario. Since the CAPM theory already generates the prices that must occur in such a market, the question that naturally occurs is, “Would the conditional risk charge method produce correct prices for the individual securities in the CAPM world?”

One would expect the answer to be “yes”, since the conditional method produces a covariance risk load, and the CAPM also produces covariance risk premia. The connection is shown as follows:

Let \( M \) represent the future value of a portfolio that is comprised of all stocks in the same proportion as in the total market (the “market portfolio”), and let \( P \) represent the current price of the market portfolio. Suppose there exists a weighting function on market return, \( Z(R_m) \), such that:

\[ \text{E}[Z] = 1 \]
\[ P = \frac{\text{E}[ZM]}{(1 + r_f)} \]

This is the characterization of portfolio risk charge that is the basis for the conditional method. (The existence of \( Z \) will be demonstrated below by construction.) The second condition is equivalent to \( \text{E}[ZR_m] = r_f \):

\[ \text{E}[ZM] = P(1 + r_f) \]
\[ \text{E}[ZM/P] = 1 + r_f \]

Using \( M/P = (1 + R_m) \),

\[ \text{E}[Z(1 + R_m)] = 1 + r_f \]
\[ \text{E}[Z] + \text{E}[ZR_m] = 1 + r_f \]
\[ E[ZR_M] = r_f \]

For any stock, define \( \varepsilon_i \) by:

\[ R_i = r_f + \beta (R_M - r_f) + \varepsilon_i \]

By taking expectations and covariances with respect to \( R_M \) on both sides, we obtain:

\[ E[\varepsilon_i] = 0 \]
\[ \text{Cov}[R_M, \varepsilon_i] = 0 \]

Multiplying by \( Z \) and taking expectations yields:

\[ E[Z R_i] = r_f E[Z] + \beta (E[Z R_M] - r_f E[Z]) + E[\varepsilon_i] E[Z], \]

using the fact that \( Z \) is a function of \( R_M \) and the independence of \( R_M \) and \( \varepsilon_i \). Then,

\[ E[Z R_i] = r_f (1) + \beta ( r_f - r_f (1)) + 0 \]
\[ E[Z R_i] = r_f \]
\[ E[Z(1 + R_i)] = 1 + r_f \]

Letting \( P_i \) and \( S \) represent the price and future value of the stock, respectively:

\[ \frac{1+R_i}{P_i} = S \]
\[ E[Z \frac{S}{P_i}] = 1 + r_f \]
\[ P_i = \frac{E[Z S]}{(1 + r_f)} \]

The last equation is the conditional risk charge formula, with the present value factor made explicit. Thus the price implied by the CAPM formula is the conditional method’s price.

An example of \( Z(R_M) \) can be explicitly constructed. Define \( Z(R_M) \) by:

\[ Z(R_M) = f(R_M + E[R_M] - r_f) / f(R_M), \]

where \( f() \) is the probability density function for \( R_M \). Then, \( Z \) satisfies the two conditions:

\[ E[Z R_M] = \int R_M f(R_M + E[R_M] - r_f) \, dR_M \]

Substituting \( u = R_M + E[R_M] - r_f \),

\[ E[Z R_M] = \int (u - E[R_M] + r_f) \, f(u) \, du \]
\[ E[Z R_M] = E[R_M] - E[R_M] + r_f \]
\[ E[Z R_M] = r_f \]

Also,
\[
E[Z] = \int f(R_M + E[R_M] - r_f) \, dR_M \\
E[Z] = 1
\]

Under CAPM, \( f() \) is normal, and this \( Z(R_M) \) function is derived by applying the Wang transform to the distribution of \( R_M \) [5].

Thus, the same mathematics can be used to derive the market price for a security in the CAPM model and an agent’s price for a risk in the agent’s portfolio. The only differences are the conditional probability structure and the relative risk weights specific to each situation. In this model, market pricing and agent pricing can be viewed as parallel calculations with different parameters.

8. All complete, additive pricing systems are represented by the covariance formula

To this point, we have shown that it is possible to obtain additive prices by using the conditional pricing method. Surprisingly, any set of additive prices must follow the conditional pricing formula:

\[
\text{Price} = W \left( E[R] + \text{Cov}[R, Z] \right),
\]

as long as the set of prices is “complete” (i.e., any derivative of the risks has a unique price under the pricing system). Thus, this formula characterizes all complete, additive pricing systems, and any such set of prices is fully described by its underlying \( Z \)-function and its “wealth transfer factor” \( W \). (See Venter [1] for a related result concerning risk-adjusted probability distributions.)

This is proven as follows: For a collection of \( n \) risks with outcomes \( R_1, \ldots, R_n \), let \( \Omega \) represent the state-space of possible combinations of outcomes, and define the random variable \( \omega \in \Omega \) as the realized outcome state (\( \omega \) corresponds to the \( n \)-tuple of actual outcomes \( (R_1, \ldots, R_n) \)). For each \( x \in \Omega \), define \( I_x \) as the indicator payoff function for the state \( x \):

\[
I_x(\omega) = 1 \text{ if } \omega = x, \ 0 \text{ otherwise}
\]

\( I_x(\omega) \) is the payoff function for the derivative that pays one dollar if state \( x \) occurs, and zero otherwise. Since the pricing system is complete, each such derivative has a price, which we will denote by \( \pi(x) \). Define \( Z^*(x) = \pi(x)/p(\omega = x) \), the ratio of price to probability for the state \( x \). Then,

\[
\text{Cov}[I_x(\omega), Z^*(\omega)] = E[I_x(\omega)Z^*(\omega)] - E[I_x]E[Z^*]
\]

\[
= \sum_{\omega \in \Omega} p(\omega = x) I_x(\omega)Z^*(\omega) - E[I_x]E[Z^*]
\]

\[
= p(\omega = x)Z^*(x) - E[I_x]E[Z^*]
\]

\[
= \pi(x) - E[I_x]E[Z^*].
\]

Let \( W = E[Z^*] \) and let \( Z = Z^* / W \). Then \( E[Z] = 1 \), and:
\[ \text{Cov}[I_x, Z] = \text{Cov}[I_x, Z*/W] = (1/W)\text{Cov}[I_x, Z*] = (1/W)(\pi(x) - E[I_x]E[Z^*]) \]
\[ \text{Cov}[I_x, Z] = \pi(x)/W - E[I_x] \]
\[ \pi(x) = W (E[I_x] + \text{Cov}[I_x, Z]) \]

This proves the formula for the derivative corresponding to \( I_x \). Since any combination of the risks (or their derivatives) is equivalent to a linear combination of the \( I_x \)-derivatives, the result follows from additivity of prices, expectations and covariances.

A portfolio containing exactly one \( I_x \)-derivative for each \( x \in \Omega \) would pay $1 with certainty. This means that \( \sum_{x \in \Omega} \pi(x) \) represents the price for $1 certain under the pricing system, which is what the factor “\( W \)” represents:

\[ W = E[Z^*] = \sum_{x \in \Omega} p(\omega=x)Z^*(\omega) = \sum_{x \in \Omega} \pi(x) \]

If \( W \) differs from the risk-free discounted value of $1, the pricing system implicitly includes a wealth transfer factor:

\[ \text{Wealth Transfer Factor} = W(1+r) \]

In the case of a market, such as the insurance market, a conservative pricing system might rely on the availability of implicit wealth transfer from the market, which could be expected to disappear if and when market efficiency increases.

In summary, one can construct a complete, additive pricing structure by defining what constitutes risk (e.g., portfolio aggregate loss), assigning relative risk-weights, normalizing them, and selecting a wealth transfer factor. The main covariance pricing formula would then be applied to price any risk or derivative (e.g., risk layer or aggregate layer). Any additive, complete set of prices has an underlying set of normalized risk relativities (the \( Z \) function), and a wealth transfer scalar (\( W \)), and can be written as:

\[ \text{Price} = W (E[R] + \text{Cov}[R, Z]) \]

9. Conclusion
The conditional risk charge method described in this paper can be used to extend a portfolio risk measure down to the level of individual risks and their derivatives, such as excess loss layers. The risk load can be expressed conveniently as covariance with portfolio risk relativity. The resulting prices are additive, and reflect complex dependence relationships between the risks. In this way, the price for a risk is representative of the extent to which it contributes to each potential aggregate outcome and the relative values those outcomes have to the portfolio holder.
References


### Non-Conditional Probabilities for Each Risk

<table>
<thead>
<tr>
<th>Risk</th>
<th>Loss: 100</th>
<th>200</th>
<th>150</th>
<th><strong>P[Loss]</strong>: 50%</th>
<th>50%</th>
<th>100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk 1</td>
<td><strong>P[Loss]</strong>: 50%</td>
<td>50%</td>
<td>100%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk 2</td>
<td><strong>P[Loss]</strong>: 60%</td>
<td>40%</td>
<td>100%</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Correlation Matrix of Risks

<table>
<thead>
<tr>
<th>Risk 2</th>
<th>Risk 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>35%</td>
</tr>
<tr>
<td>200</td>
<td>25%</td>
</tr>
</tbody>
</table>

### Values of Possible Portfolio States

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Weighting</th>
<th>Weight Z[w]</th>
<th>Risk-Adjusted Probability</th>
<th>Utility</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>State w</td>
<td>p[w]</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>35%</td>
<td>0.500</td>
<td>0.563</td>
<td>20%</td>
<td>113</td>
</tr>
<tr>
<td>300</td>
<td>40%</td>
<td>1.000</td>
<td>1.127</td>
<td>45%</td>
<td>338</td>
</tr>
<tr>
<td>400</td>
<td>25%</td>
<td>1.250</td>
<td>1.408</td>
<td>35%</td>
<td>563</td>
</tr>
</tbody>
</table>

Total / Exp'd: 290.00 100% 0.888 1.000 100% 315.49

Risk Load: 25.49

### Decomposition of Z[w] ---> Z[Risk] by Conditional Analysis

| w    | Z[w] | P[w|R1=100] | P[w|R1=200] | P[w|R2=100] | P[w|R2=200] |
|------|------|-------------|-------------|-------------|-------------|
| 200  | 0.563| 70.00%      | 0.00%       | 58.33%      | 0.00%       |
| 300  | 1.127| 30.00%      | 50.00%      | 41.67%      | 37.50%      |
| 400  | 1.408| 0.00%       | 50.00%      | 0.00%       | 62.50%      |

Total: 100.00% 100.00% 100.00% 100.00%

E[Z | Rx=y ]: 0.732 1.268 0.798 1.303

### Risk Loaded Pricing for Each Risk

<table>
<thead>
<tr>
<th>Risk</th>
<th>Loss: 100</th>
<th>200</th>
<th>150</th>
<th><strong>Z[Loss]</strong>: 0.732</th>
<th>1.268</th>
<th>163.38</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk 1</td>
<td><strong>Z[Loss]</strong>: 0.732</td>
<td>1.268</td>
<td>163.38</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk 2</td>
<td><strong>Z[Loss]</strong>: 0.798</td>
<td>1.303</td>
<td>152.11</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Risk Load: 13.38

### Risk Loaded Pricing for Each Risk

<table>
<thead>
<tr>
<th>Risk</th>
<th>Loss: 100</th>
<th>200</th>
<th>140</th>
<th><strong>Z[Loss]</strong>: 0.798</th>
<th>1.303</th>
<th>152.11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk 2</td>
<td><strong>Z[Loss]</strong>: 0.798</td>
<td>1.303</td>
<td>152.11</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk Load: 12.11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>