Tail distribution and dependence measures

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Résumé : L'ensemble des familles de copulas usuelles peuvent être distinguées suivant la structure de dépendance qu'elles engendrent, certaines familles allouant plus de masse sur la queue supérieure (*grands sinistres*) ou sur la queue inférieure (*petits sinistres*), par exemple. Gary Venter mentionnait, dans *Tails of Copulas*, qu'en assurance dommage les copulas accentuant la dépendance au niveau de la queue supérieure étaient d'un grand intérêt puisqu'ils permettaient de prendre en compte la corrélation entre les risques extrêmes. En fait, compte tenu du poids dans la distribution de la charge totale des grands sinistres, il convient de modéliser correctement la dépendance au sein de cette queue supérieure, et éviter surtout de sous estimer cette forme de dépendance. Nous allons ici voir comment étudier cette queue supérieure (*ou de façon analogue la queue inférieure*), du point de vue de la distribution tout d'abord, puis sous la forme d'une mesure de dépendance dans un second temps.

Ce papier se propose tout d'abord d'étudier le copula conditionel, ou copula tronqué, correspondant au copula du copula du couple (X,Y) étant donné que X et Y excèdent tous deux un seuil (*défini comme un quantile, à probabilité donné, i.e.* $X > F_X^{-1}(u), Y > F_Y^{-1}(v)$, $u,v \in [0,1]$). Nous étudierons plus particulièrement le cas des copulas Archimédien, qui définissent une famille stable, au sens où le copula tronqué est encore un copula Archimédien, pour tout u,v. Nous étudierons également une application de ce copula conditionel dans le cas du risque de crédit. Nous verrons enfin, comment à partir de cette étude de la queue supérieure du point de vue de la distribution, il est possible de définir des mesures de dépendance fonctionelle, en définissant le rho de Spearman de ce copula conditionel à u et v donnés, et d'étudier sa variation en tant que fonction de u et v, et d'étudier son comportement quand u et v tendent vers 1. Cette mesure de dépendance peut alors être utilisée comme critère de choix de modèle, lors d'une modélisation par un copula paramétrique, de façon à vérifier que la famille retenue ne sous-estime pas cette queue de distribution, qui peut avoir un impact énorme en assurance dommage.

Abstract : Families of copulas could be distinguished according to the allocation of the weight among the dependence structure : some copulas could stress more on upper tails (*large claims*), while other could stress more on lower tails...etc. As mentioned by Gary Venter in *Tails of Copulas*, in property and casualty applications, there could be interest in copulas that emphasize correlation among large losses. More precisely, it might be more interesting to focus on the upper tail, to be sure that the model fits well the dependence among large losses - the main idea being that we should chose a copula which does not underestimate the dependence in the upper tail. In this paper, we will study (*in Part 2*) conditional copula, which is the copula of the distribution (X,Y), given X and Y both higher, or lower, than a given threshold (*defined as a quantile, for a given risk level*). After defining these conditional properties, we will give some properties, satisfied by these families of copulas, focusing on a stable family : the family of Archimedian copula. This conditional copula will be used, then (*in Part 3*), to define a functional dependence measure, based on Spearman's rho, call tail rank correlation, which could be seen as a measure of dependence in the tails of the distribution.

Keywords : Archimedian copula; conditional distribution; copulas; dependence; factor representation; rank correlation; Spearman's rho; tail correlation; tail dependence; truncature;

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1 Introduction

1.1 Motivation

Studying multivariate extremes is an important issues in risk management, and to study into details the relationship among extremes, some tail concentration functions have been introduced, to study, first of all, how much probability is in those regions (X *and* Y *'extreme'*), but also the shape of the dependence structure. Upper and lower tail dependence parameters could be an interesting tool to study tail dependence (*because of its simple analytical expression*), but this coefficient can only quantify the amount of dependence in the tail, and it can not describe the shape of the dependence structure in the tails.

Using untruncated data, it is possible to obtain all the overall dependence structure between X and Y. But one can wonder if dependence properties still hold if focusing only on extremes of the distribution. For example, if the correlation between X and Y is positive, could we assume that the correlation between extreme values of X and extreme values of Y is also positive ? More generally, if X and Y satisfy a positive dependence, can we assume that the same dependence property still holds for (X, Y) given X higher than a given threshold and Y higher than a given threshold ?

The same kind of questions could be asked about stochastic processes. The motivation of some risk managers could be to focus only on rare events : knowing that X_t has been an extreme event at time t, in the sense that X_t was lower (*or higher*) than a low (*high*) quantile, what could we expect for X_{t+1}. In that case, it could be interesting to study the dependence structure of (X_t, X_{t+1}) knowing that $X_t \leq F_{X_t}^{-1}(\alpha)$ for some α (*or* $X_t > F_{X_t}^{-1}(\alpha)$). Malevergne and Sornette (2002) focus, for instance, on the correlation conditioned on exceedance of one variable, for financial time series.

Conversely, for some practical issues, only truncated data could be available. More specifically, those data could be truncated data with a deterministic truncature, in the sense that only data above a given threshold where taken into account. These data could give some information about the dependence structure of (X, Y) given X higher than a given threshold and Y higher than a given threshold, but this is only a partial information about the overall structure between untruncated variables X and Y.

In the first case, the motivation is to link the dependence structure of (X, Y) and (X, Y) given X and Y are higher (*or lower*) than given thresholds And the second case could be seen as the dual problem : knowing the dependence structure of (X, Y) given X and Y are higher (*or lower*) than some thresholds, what could be said about the overall dependence structure of (X, Y) ? Because one of the main concept used to capture the dependence structure of a multivariate distribution is the copula distribution function, we will focus, in this paper, on properties of the conditional copula of (X, Y) given X and Y lower than given thresholds.

1.2 Copula

As mentioned above, one of the main concept used to capture the dependence structure of a multivariate distribution is the copula distribution function. Whenever copula can be defined for any multivariate distributions in $\mathbb{R}^d$, we focus on bivariate continuous random vectors for expository purpose. Let us denote $F_{X,Y}(x, y)$ the bivariate cumulative distribution of the pair (X, Y) of random variables X and Y, $F_X(x)$ and $F_Y(y)$ the marginal c.d.f. of X and Y, respectively. As shown in Sklar (1959), the joint c.d.f. of (X, Y) can be written as

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = C(F_X(x), F_Y(y)),$$

where C is the c.d.f. of a distribution on $[0, 1]^2$, with uniform margins. When variables are continuous, C is unique, and is called the copula of (X, Y). Sklar theorem allows to separate the marginal feature and the dependence structure which is represented by the copula. The function C is the c.d.f. of the pair (U, V) where $U = F_X(X)$ and $V = F_Y(Y)$, and

$$c(u, v) = \frac{\partial^2 C}{\partial u \partial v}(u, v),$$

is the associated p.d.f. Sklar's theorem proves the existence and the uniqueness of the copula. It also explains how to construct it from the initial distribution. Indeed, for any $0 \leq u, v \leq 1$, the copula is given by

$$C(u, v) = F_{X,Y}\left(F_X^{-1}(u), F_Y^{-1}(v)\right),$$

where F_X^{-1} and F_Y^{-1} are the marginal quantile functions. The copula characterizes any nonlinear dependence which is invariant by increasing transformation of either X and Y. More precisely we have the following : if ϕ and ψ are strictly increasing functions, then (X, Y) and $(\phi(X), \psi(Y))$ have the same copula.For a given copula C, let $\mathcal{D}_C$ denote the subset

$$\mathcal{D}_C = \{(u, v) \in [0, 1] \times [0, 1] \, | C(u, v) > 0\}.$$

However, the joint survivor function can be more appropriate in problems involving duration variables, or exceedance over a threshold. The survival distribution of (X,Y) is given by

$$\overline{F}_{X,Y}(x,y)=\mathbb{P}(X>x,Y>y)=C^{*}\left(\overline{F}_X(x),\overline{F}_Y(y)\right), \tag{1}$$

where C^* is the c.d.f. of $(1-U,1-V)$. C^* is called survival copula of (X,Y), or dual copula, and is related to copula C by

$$C^{*}(u,v)=1-u-v+C(1-u,1-v).$$

An as well as $\overline{F}_{X,Y}$ denote the survival distribution of (X,Y), $\overline{C}$ will denote the survival distribution of $(F_X(X),F_Y(Y))$, where (X,Y) has copula C.

Example 1 ***Gaussian copula*** - *This copula is symmetric, and is useful for its easy simulation method. Furthermore, it can be easily generalized to higher dimension than 2. It could be defined by its density,*

$$c(x,y)=\frac{1}{\sqrt{1-\theta^2}}\exp\left[-\frac{1}{2}\frac{x^2+y^2-2\rho xy}{1-\rho^2}\right]\exp\left[\frac{1}{2}\left(x^2+y^2\right)\right],\ \textit{where } \theta\in[-1,1],\in$$

so that

$$C(x,y)=\int_0^x\int_0^y c(u,v)\,dudv\ \textit{ on }\ [0,1]\times[0,1].$$

This copula is such that $C(x,y)$ is equal to $C^{-}(x,y)$, $C^{\perp}(x,y)$ and $C^{+}(x,y)$ respectively, when $\theta=-1$, 0 and $+1$. It could also be written $C(x,y)=\Phi_\theta\left(\Phi^{-1}(x),\Phi^{-1}(y)\right)$ where Φ denotes the $N(0,1)$ cdf and Φ_θ is the bivariate standard normal cdf, with correlation θ.

Example 2 ***Gumbel copula*** - *This copula is asymmetric, with more weight in the right tail, and is given by*

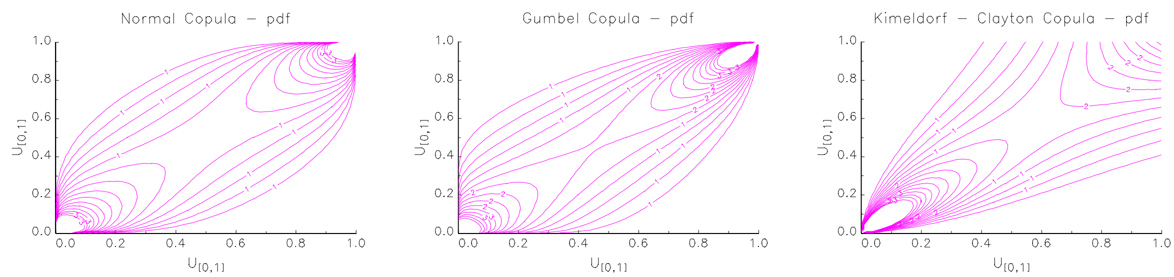
$$C(x,y)=\exp\left(-\left[(-\log x)^{\theta}+(-\log y)^{\theta}\right]^{1/\theta}\right)\ \textit{where } \theta\geq 1,$$

Moreover, it is an extreme value copula, i.e. $C(x^z,y^z)=C(x,y)^z$ for all $z>0$, and it is an Archimedian copula. This copula is also called logistic copula.

Example 3 ***Clayton copula*** - *This copulas is also asymmetric, but with more weight in the left tail, and is given by*

$$C(x,y)=\left[x^{-\theta}+y^{-\theta}-1\right]^{-1/\theta}\ \textit{where } \theta\geq 0,$$

It is also an Archimedian copula. The level curves of the densities of these copulas are given below



1.3 Outline of the paper

As mentioned in Deheuvels (1979), copulas are the 'dependence function' of couple (X,Y). An heuristic interpretation of positive dependence, is that a pair of random variables are positively dependent if large values of one tend to be associated with large values of the other, and small values of one with small values of the other.If we are interested mainly in the dependence among small values, it might be interesting to study the joint distribution of (X,Y) given X and Y "small". In the case the notions of large and small values is based on the quantiles of the distributions, then X and Y are small if they both do not exceed some respective quantiles, and we might be interested in studying the distribution of the joint distribution (X,Y) given $X\leq F_X^{-1}(u),Y\leq F_Y^{-1}(v)$, for some u,v in $[0,1]$. An as well as copulas have been introduced to study the dependence of X and Y, it is possible to introduce the copula of the conditional distribution of (X,Y) given $X\leq F_X^{-1}(u),Y\leq F_Y^{-1}(v)$. Using the rank transformation of X and Y, $U=F_X(X)$ and $V=F_Y(Y)$, we can introduce the copula of (X,Y) given $U\leq u,V\leq v$, which is also the copula of the conditional couple (U,V) given $U\leq u,V\leq v$.

When studying small values among bivariate pairs of random variables, we will introduce the conditional copula, based on the lower orthant, defined as the copula of $(U,V)\,|U\leq u, V\leq v$. Similarly, when studying large values among bivariate pairs of random variables, we can introduce the conditional copula, based on the lower orthant, defined as the copula of (U,V) given $U>u, V>v$. In that case, it could be more interesting to study the survival distribution of the copula associated with (U,V) given $U>u, V>v$. In the first part of this paper, we will focus on the expression of this lower orthant conditional copula. We will see how dependence orderings are transferred to the extremes, and how positive (*or negative*) dependence of (X,Y) imply weaker positive (*or negative*) dependence of (X,Y) given $X\leq F_X^{-1}(u), Y\leq F_Y^{-1}(v)$, or $X>F_X^{-1}(u), Y>F_Y^{-1}(v)$. PQD or NQD properties of extremes are obtained in the case where F_{XY} is either $TP2$ or $RR2$. This part will summarize some theoretical results obtained in Charpentier (2003). We will then study into details the case of Archimedian copulas in the second part, showing, first, that for any u and v, the conditional copula of (U,V) given $U\leq u, V\leq v$ is still an Archimedian copula. Furthermore, if the Archimedian copula of (X,Y) has a factor representation, also has the conditional copula. Furthermore, the only 'invariant' copula (*in the sense that the conditional copula does not depend on u and v*) necessarily belongs to the family of Clayton copulas. An explicit applications we be detailed, where the distribution of the distribution in the tail is needed , in Credit Risk. In that case, we study the dependence between two non-independent default times. In that case, if the copula at time 0, between the two default times was C, it might be interesting to study, at time $t>0$, the dependence structure given the fact that there has been no default between 0 and t, i.e. the conditional copula of τ_X and τ_Y given $\tau_X>t$ and $\tau_Y>t$.

The last part will deal with an application of this conditional copula, defining a functional dependence measure. As done in Venter (2001) and Malevergne and Sornette (2002), it could be interesting to define a correlation measure which describe the behavior in the tails. Venter introduced a cumulative tau (*from Kendall's tau*), normalized, so that it is equal to 0 in 0 and to 1 in 1 (*as Lorentz curve*). But from this measure, it is rather difficult to quantify the dependence in the tails. Malevergne and Sornette (2002) introduced a conditional correlation, but the correlation is the one introduced by Pearson, which is very sensitive to the marginal distribution. The conditional correlation we introduce in this paper could be seen as an extension of the correlation introduced in Malevergne and Sornette (2001), using Spearman's correlation instead of Pearson's. We will see, for example, that this measure of dependence could be used for calibration issues.

2 Conditional copula : distribution in the tails

2.1 Definition

Let (X,Y) be a random pair with copula C, and continuous marginal c.d.f.'s F_X and F_Y, respectively. For any $(u,v)\in\mathcal{D}_C$, the conditional distribution of (U,V) given $U\leq u, V\leq v$ is

$$F(C,u,v)(x,y)=\mathbb{P}(U\leq x, V\leq y|U\leq u, V\leq v)=\frac{C(x,y)}{C(u,v)},$$

where $0\leq x\leq u$ and $0\leq y\leq v$. Since marginal distributions of U and V given $U\leq u, V\leq v$ are not uniforms, $F(C,u,v)$ is not a copula.

The marginal c.d.f.'s of the conditional distributions are given by

$$F_X(C,u,v)(x)=\frac{C(x,v)}{C(u,v)} \text{ and } F_Y(C,u,v)(y)=\frac{C(u,y)}{C(u,v)} \text{ , respectively.}$$

The copula of the conditional distribution is[1]

$$\Phi(C,u,v)(x,y)=\frac{C\left(F_X(C,u,v)^{-1}(x), F_Y(C,u,v)^{-1}(y)\right)}{C(u,v)},$$

where $F_X(C,u,v)^{-1}(t)=x$ if and only if $C(x,v)=tC(u,v)$, and similarly for Let $F_Y(C,u,v)^{-1}$. This copula is called lower orthant conditional copula.

[1] One can notice that $\Phi(C,u,v)(x,0)=0$ because $F_Y(C,u,v)(0)=C(u,0)/C(u,v)=0$ and so, $F_Y(C,u,v)^{-1}(0)=0$. Furthermore $\Phi(C,u,v)(x,1)=x$ because $F_Y(C,u,v)(v)=1$ (*so that,* $F_Y(C,u,v)^{-1}(1)=v$) and then,

$$\Phi(C,u,v)(x,1)=\frac{C\left(F_X(C,u,v)^{-1}(x),v\right)}{C(u,v)}=F_X(C,u,v)\left(F_X(C,u,v)^{-1}(x)\right)=x,$$

because $F_X(C,u,v)(x)=C(x,v)/C(u,v)$.

Similarly, it is possible to study the dependence for large values. In that case, it is more convenient to study the dependence in terms of survival copulas. Using expression (1), the survival copula could be obtained from joint and marginal survival distributions.

For any $u, v \in]0,1]$, the survival distribution of (U,V) given $U > u, V > v$ is

$$\overline{G}(C,u,v)(x,y) = \mathbb{P}(U > x, V > y | U > u, V > v) = \frac{C^*(1-x,1-y)}{C^*(1-u,1-v)} \text{ on } [u,1] \times [v,1].$$

The marginal survival distributions of the conditional distribution are given by

$$\overline{G}_X(C,u,v)(x) = \frac{C^*(1-x,1-v)}{C^*(1-u,1-v)} \text{ and } \overline{G}_Y(C,u,v)(y) = \frac{C^*(1-u,1-y)}{C^*(1-u,1-v)}, \text{ respectively.}$$

The survival copula of the conditional distribution is[2]

$$\Psi(C^*,u,v)(x,y) = \frac{C^*\left(1-\overline{G}_X(C,u,v)^{-1}(x), 1-\overline{G}_Y(C,u,v)^{-1}(y)\right)}{C^*(1-u,1-v)},$$

where $\overline{G}_X(C,u,v)^{-1}(t) = x$ if and only if $C^*(1-x,1-v) = tC^*(1-u,1-v)$, and similarly for Let $\overline{G}_Y(C,u,v)^{-1}$. This copula is called upper orthant conditional copula.

Proposition 1 *Let (U,V) be a pair of uniform variables, with c.d.f. C, and $(u,v) \in \mathcal{D}_C$. The conditional copula of (U,V) given $U \leq u$ and $V \leq v$ is*

$$\Phi(C,u,v)(x,y) = \frac{C\left(F_X(C,u,v)^{-1}(x), F_Y(C,u,v)^{-1}(y)\right)}{C(u,v)}.$$

Let C^* be the survival copula of (U,V), then, the survival copula of (U,V) given $U > u$ and $V > v$ is

$$\Psi(C^*,u,v)(x,y) = \frac{C^*\left(1-\overline{G}_X(C,u,v)^{-1}(x), 1-\overline{G}_Y(C,u,v)^{-1}(y)\right)}{C^*(1-u,1-v)}.$$

Example 4 ***Mixture of Fréchet upper bond C^+ and independent copula $C^\perp$*** *- For all θ in $[0,1]$, let $C_\theta(x,y) = \theta C^+(x,y) + (1-\theta)C^\perp(x,y)$, i.e. $C_\theta(x,y) = \theta \min(x,y) + (1-\theta)xy$, on $[0,1]\times[0,1]$. Let (U,V) be a pair with distribution C_θ for some θ, which could be seen as the following factor model, with factor Z, with values 0 and 1,*

$$\begin{cases} \mathbb{P}(U \leq x, V \leq y | Z = 1) = C^+(x,y) & \text{, i.e. } V = U \quad \text{with probability } \theta \\ \mathbb{P}(U \leq x, V \leq y | Z = 0) = C^\perp(x,y) & \text{, i.e. } V \perp U \quad \text{with probability } 1-\theta \end{cases}$$

Let $\Phi(C_\theta, t)$ denote the copula of the conditional distribution of (U,V) given $U \leq t$ and $V \leq t$, for all $t \in]0,1]$. Then,

$$\mathbb{P}(Z=1|U \leq t, V \leq t) = \frac{\mathbb{P}(Z=1)}{\mathbb{P}(U \leq t, V \leq t)}\mathbb{P}(U \leq t, V \leq t | Z=1) = \frac{\theta}{\theta + t - \theta t} = \theta(t),$$

The joint cumulative distribution of (U,V) given $U \leq t$ and $V \leq t$ is given by

$$F(x,y) = \mathbb{P}(U \leq x, V \leq y | U \leq t, V \leq t) = \frac{C_\theta(x,y)}{C_\theta(t,t)} = \frac{\theta \min(x,y) + (1-\theta)xy}{\theta t + (1-\theta)t^2},$$

so that, the marginal cumulative distributions are

$$\begin{aligned} F_{X,t}(x,y) &= \mathbb{P}(U \leq x | U \leq t, V \leq t) = \frac{C_\theta(x,t)}{C_\theta(t,t)} \\ &= \frac{\theta x + (1-\theta)xt}{\theta t + (1-\theta)t^2} = \frac{x}{t} \text{ where } x \leq t. \end{aligned}$$

[2] Similarly, one can notice that $\Psi(C^*,u,v)(x,0) = 0$ because $\overline{G}_Y(C^*,u,v)(1) = C^*(1-u,0)/C^*(1-u,1-v) = 0$ and so, $1-\overline{G}_Y(C^*,u,v)^{-1}(0) = 1-1 = 0$. Furthermore $\Phi(C^*,u,v)(x,1) = x$ because $\overline{G}_Y(C^*,u,v)(v) = 1$ (*so that,* $\overline{G}_Y(C^*,u,v)^{-1}(1) = v$) and then,

$$\Psi(C^*,u,v)(x,1) = \frac{C^*\left(1-\overline{G}_X(C,u,v)^{-1}(x), 1-v\right)}{C^*(1-u,1-v)} = \overline{G}_X(C,u,v)\left(\overline{G}_X(C,u,v)^{-1}(x)\right) = x,$$

because $\overline{G}_X(C^*,u,v)(x) = C^*(1-x,1-v)/C^*(1-u,1-v)$.

It comes that the distribution of U given $U \leq t, V \leq t$ is uniform on $[0,t]$. So, F is the copula of the joint distribution, i.e. $\Phi(C_\theta,t)(x,y) = F(x,y)$ and could be written

$$\Phi(C_\theta,t)(x,y) = \theta(t)\, C^+(x,y) + [1-\theta(t)]\, C^\perp(x,y) = C_{\theta(t)}(x,y) \text{ where } \theta(t) = \frac{\theta}{\theta + t - \theta t},$$

which is still in the same family of mixtures. And similarly, one can obtain easily that $\Psi(C^*_\theta,t)(x,y) = \Phi(C_\theta,t)(x,y)$ for all x,y in $[0,1]$.

From the results above, we can write similarly, for the lower tail copula,

$$\frac{C(x,y)}{C(u,v)} = \Phi(C,u,v)\left(\frac{C(x,v)}{C(u,v)}, \frac{C(u,y)}{C(u,v)}\right), \text{ where } x,y \in [0,u]\times[0,v],$$

and for the upper tail conditional survival copula

$$\frac{C^*(1-x,1-y)}{C^*(1-u,1-v)} = \Psi(C^*,u,v)\left(\frac{C^*(1-x,1-v)}{C^*(1-u,1-v)}, \frac{C^*(1-u,1-y)}{C^*(1-u,1-v)}\right), \text{ where } x,y \in [u,1]\times[v,1].$$

One can get easily the following result : let $0 < u,v \leq 1$, and let Γ be a continuous copula, then there is a copula C such that $\Gamma = \Phi(C,u,v)$. It comes from this proposition that, for a given couple (X,Y), with given threshold u and v, the dependence structure of (X,Y) given $X \leq F_X^{-1}(u)$ and $Y \leq F_Y^{-1}(v)$ has no constraint : it could be any copula from the lower to the upper Fréchet bound.

Remark 1 *For some simple transformation, the conditional copula could be obtained easily. For example, let C be a copula, and α in $]0,1]$, then $\Gamma(x,y) \mapsto C(x^\alpha,y^\alpha)^{1/\alpha}$ is a copula, and furthermore, $\Phi(\Gamma,u,v) = [\Phi(C,u^\alpha,v^\alpha)]^{1/\alpha}$.*

Remark 2 *One can notice that these definitions could be extended from the bivariate case to the multivariate case, where $d \geq 2$.*

2.2 Conditional copula on the diagonal : $\Phi(C,t)$

This copula has been introduced recently in copulas literature, such as in Juri and Würthrich (2002). This conditional copula could be used to define some functional dependence measures. For example, it is possible to define the 'lower quadrant correlation' using Spearman's rho :

$$\rho(C,t) = 12 \int_{[0,1]\times[0,1]} \Phi(C,t,t)(x,y)\, dx dy - 4$$

which is the rank correlation of (X,Y) given $X \leq F_X^{-1}(t)$ and $Y \leq F_Y^{-1}(t)$. But is this conditional copula can lead to simplified calculations, one can wonder, for practical issues, why X and Y should be lower (*or higher*) than the same percentile.

An alternative could be to study $\Phi(C,t,\phi(t))$ and $\Psi(C^*,t,\psi(t))$ for some functionals ϕ and ψ.

Example 5 *For example, let X and Y be two default times, with the joint survival distribution $\overline{F}_{X,Y}(x,y) = C^*\left(\overline{F}_X(x), \overline{F}_Y(y)\right)$, at time $t=0$, where C^* is the survival copula of (X,Y). If no defaults occur between time 0 and time t, then, the conditional survival distribution at time t is*

$$\mathbb{P}(X > x, Y > y | X > t, Y > t) = \frac{\overline{C}\left(\overline{F}_X(x), \overline{F}_Y(y)\right)}{\overline{C}\left(\overline{F}_X(t), \overline{F}_Y(t)\right)} = \Psi\left(C^*, \overline{F}_X(t), \overline{F}_Y(t)\right)\left(\overline{F}_X(x), \overline{F}_Y(y)\right) \text{ where } x,y > t,$$

where $\Psi\left(C^*, \overline{F}_X(t), \overline{F}_Y(t)\right)$ is the survival copula of the conditional distribution. Studying the temporal dynamic evolution of the conditional copula, given that at time t no defaults occur, is similar as studying $\Psi(C^*,t,\psi(t))$ where $\psi = \overline{F}_Y \circ \overline{F}_X^{-1}$. This will developed in Section(2.7).

2.3 Conditional copula on the border : $\Phi(C,u,0)$ or $\Phi(C,0,v)$

This case is interesting while conditions depends on only one of the two parameter, while studying $(X,Y)|X \leq F_X^{-1}(u)$, or $(X,Y)|Y \leq F_Y^{-1}(v)$. This conditional copula in interesting when studying time series : given some information on the variable of interest at time t (*for example, X_t was a 'large' value, in the sense that X_t was* *higher than* $F_t^{-1}(p)$ *where* p *is close to* 1), it might be interesting to study the dependence between the variable at time t and the variable at time $t+1$.

For example, let (X_t) be a stochastic process, with marginal cdf at time t F_t. Let $U_t = F_t(X_t)$ be the associated rank at time t. Let C_t be the copula of (U_{t-1}, U_t), which represents the dependence structure between the rank at time $t-1$ and the rank at time t. $\Phi(C, u, 0)$ represents the dependence structure knowing that, at time $t-1$, the rank was low (*at least in the u-th lower percentile*), and $\Psi(C^*, u, 0)$ represents the dependence structure knowing that, at time $t-1$, the rank was high (*at least in the u-th upper percentile*). This could lead to several application to credibility models (*if* X_t *denotes the cost at time* t) or financial modeling (*if* X_t *denotes a financial time series, such as an index price variation*).

Example 6 *Studying this conditional dependence could be interesting in regime-switching models. For example, let* $0 = p_1 < p_2 < ... < p_n = 1$, *and* (X_t) *be a self-exciting threshold autoregressive* ($SETAR$) *process, of order* 1, *i.e.*

$$X_{t+1} = \alpha_i + \phi_i X_t + \varepsilon_t^i \text{ if } F_t^{-1}(p_i) \leq X_t < F_t^{-1}(p_{i+1})$$

where (ε_t^i) *is a sequence of i.i.d. random variables with mean* 0, *such that* (ε_t^i) *and* $\left(\varepsilon_t^j\right)$ *are independent for* $i \neq j$. *In that case, it could be interesting to study the dependence between* X_t *and* X_{t+1} *given some information on* X_t.

This kind of conditional copulas could be used in the case variables X and Y could be linked by a causal relationship. This kind of conditional copula would be used in Section (3.3.1), while studying the dependence between the cost of a claim (*the loss*) and the allocated expenses. Given some information on the cost, it might be interesting to get a better understanding of the dependence between the loss and the expenses.

2.4 Conditional invariant copulas on $[0,1] \times [0,1]$

A copula C is said to be have invariant conditional copulas if $\Phi(C, u, v)$ does not depend on u and v. This implies that $\Phi(C, u, v) = C$ for all u, v in $[0,1] \times [0,1]$.

Proposition 2 (i) *C is an invariant copula if and only if it satisfies the functional equation*

$$\frac{C(x,y)}{C(u,v)} = C\left(\frac{C(x,v)}{C(u,v)}, \frac{C(u,y)}{C(u,v)}\right) \text{ for all } x \in [0,u], y \in [0,v], \; u, v \in [0,1],$$

(ii) *C is an invariant copula if and only if it satisfies*

$$\frac{C_2(x,1)}{C_2(1,1)} C_1(x,y) = \frac{C_1(1,y)}{C_1(1,1)} C_2(x,y), \text{ where } C_1(x,y) = \partial C(x,y)/\partial x \text{ and } C_2(x,y) = \partial C(x,y)/\partial y.$$

Proof. Charpentier (2003). ■

Example 7 *Let C belong to the family of Clayton copulas, i.e.* $C(x,y) = \left(x^{-\theta} + y^{-\theta} - 1\right)^{-1/\theta}$, *so that* $C_1(x,y) = x^{-\theta-1}\left(x^{-\theta} + y^{-\theta} - 1\right)^{-1/\theta-1}$. *It comes that*

$$\frac{C_2(x,1)}{C_2(1,1)} C_1(x,y) = \left(x^{-\theta}\right)^{-1/\theta-1} .x^{-\theta-1}\left(x^{-\theta} + y^{-\theta} - 1\right)^{-1/\theta-1} = \left(x^{-\theta} + y^{-\theta} - 1\right)^{-1/\theta-1} = \frac{C_1(1,y)}{C_1(1,1)} C_2(x,y)$$

and so, C is invariant on $[0,1] \times [0,1]$.

2.4.1 Conditional invariant copulas on the first diagonal

A weaker condition of invariance could be defined : a copula C is said to be have invariant conditional copulas on the diagonal if $\Phi(C, t, t)$ does not depend on t and t,i.e. $\Phi(C, t, t) = C$ for all t in $[0,1]$.

$$\frac{C(x,y)}{C(t,t)} = C\left(\frac{C(x,t)}{C(t,t)}, \frac{C(t,y)}{C(t,t)}\right) \text{ for all } x, y \in [0,t], \, t \in [0,1]$$

Proposition 3 (i) *C is an invariant copula on the first diagonal if and only if it satisfies the functional equation*

$$\frac{C(x,y)}{C(t,t)} = C\left(\frac{C(x,t)}{C(t,t)}, \frac{C(t,y)}{C(t,t)}\right) \text{ for all } x, y \in [0,t], \, t \in [0,1]$$

(ii) *C is an invariant copula on the first diagonal if and only if it satisfies*

$$C(x,y) = \left[x - \frac{C_2(x,1)}{C_1(1,1) + C_2(1,1)}\right] C_1(x,y) + \left[y - \frac{C_1(1,y)}{C_1(1,1) + C_2(1,1)}\right] C_2(x,y)$$

Proof. Gouriéroux and Monfort (2003). ■

2.5 Ordering and conditional copulas

In this part, we will study the evolution of the dependence, while working on the conditional copula. For example, if (X,Y) is 'positively ordered' we can expect that (X,Y) given $X \leq F_X^{-1}(u)$ and $Y \leq F_Y^{-1}(v)$, or $X > F_X^{-1}(u)$ and $Y > F_Y^{-1}(v)$, to be also 'positively ordered'. The two following paragraph will define some positive dependence concepts, the first one being the most popular : the PQD concept. But we will see in the third part that this concept is not sufficient : if (X,Y) is PQD, it does not necessarily imply that (X,Y) given $X \leq F_X^{-1}(u)$ and $Y \leq F_Y^{-1}(v)$, or $X > F_X^{-1}(u)$ and $Y > F_Y^{-1}(v)$ is PQD. This is the reason why we will introduce a second concept of positive dependence, based on $TP2$ functions.

The positive quadrant dependence (PQD), introduced in Lehmann (1966), is a pointwise comparison between the joint c.d.f. and the product of marginal c.d.f's, or equivalently, between C and $C^{\perp}$: a pair (X,Y), with joint c.d.f. F_{XY}, and marginal c.d.f.'s F_X and F_Y, is said to be PQD if, for all x,y, $F_{XY}(x,y) \geq F_X(x) F_Y(y)$, or similarly, that for all $u,v \in [0,1]$, $C(u,v) \geq C^{\perp}(u,v)$ where C is the copula of (X,Y), and $C^{\perp}$ denotes the independent copula.

One can obtain easily that this definition remains unchanged using survival functions instead of c.d.f. : (X,Y) is PQD if and only if $\overline{F}_{XY}(x,y) \geq \overline{F}_X(x)\overline{F}_Y(y)$.

From this concept of positive dependence, it is possible to define the following stochastic ordering : (X_1,Y_1), with joint c.d.f. F_1, is said to be smaller than (X_2,Y_2), with joint c.d.f. F_2, denoted $(X_1,Y_1) \preceq_{PQD} (X_2,Y_2)$, or similarly $F_1 \preceq_{PQD} F_2$ if and only if $F_1(x,y) \geq F_2(x,y)$ for all x,y.

In the case of bivariate vectors, one can see easily that this condition could be replace equivalently by $\overline{F_1}(x,y) \geq \overline{F_2}(x,y)$ for all x,y. In the case where the two pairs have the same marginal distributions, i.e. $X_1 \overset{\mathcal{L}}{=} X_2$ and $Y_1 \overset{\mathcal{L}}{=} Y_2$, $(X_1,Y_1) \preceq_{LCSD} (X_2,Y_2)$ if and only if $C_1(x,y) \geq C_2(x,y)$ for all x,y where C_1 and C_2 denote the copulas of (X_1,Y_1) and (X_2,Y_2) respectively.

Analogously, it is possible to define a negative dependence concept as follows, a pair (X,Y), with joint c.d.f. F_{XY}, and marginal c.d.f.'s F_X and F_Y, is said to be NQD if, for all x,y, $F_{XY}(x,y) \leq F_X(x) F_Y(y)$, or similarly, that for all $u,v \in [0,1]$, $C(u,v) \leq C^{\perp}(u,v)$ where C is the copula of (X,Y), and $C^{\perp}$ denotes the independent copula.

One can obtain easily that this definition remains unchanged using survival functions instead of c.d.f. : (X,Y) is NQD if and only if $\overline{F}_{XY}(x,y) \leq \overline{F}_X(x)\overline{F}_Y(y)$.

The $TP2$ concept was introduced in Karlin (1968), and a nonnegative function h is totally positive of order 2 ($TP2$) if, for all $x_1 < x_2$, $y_1 < y_2$,

$$h(x_1,y_1).h(x_2,y_2) \geq h(x_1,y_2).h(x_2,y_1).$$

This condition could also be written as the determinant of a square matrix of order 2[3].Nelsen (1999) defines from this concept, two notions of dependence : a pair (X,Y) of continuous random variables with joint c.d.f. F_{XY} is $LCSD$ (*left corner set decreasing*) if F_{XY} is $TP2$. And (X,Y) is $RCSI$ (*right corner set increasing*) if $\overline{F}_{XY}$ is $TP2$.

One can notice that these notion could be defined equivalently using copula functions instead of c.d.f. Furthermore, X and Y are $LCSD$ if and only if $\mathbb{P}(X \leq x, Y \leq y | X \leq x', Y \leq y')$ is non-increasing in x' and y' for all x and y. And similarly, X and Y are $RCSI$ if and only if $\mathbb{P}(X > x, Y > y | X > x', Y > y')$ is non-decreasing in x' and y' for all x and y. These notions of dependence are stronger than the PQD condition, i.e. if (X,Y) is $LCSD$ or $RCSI$, then (X,Y) is PDQ.

From these definition, it is possible to define an ordering relationship, as done in Szehli (1995) : (X_1,Y_1), with joint c.d.f. F_1, is said to be smaller, in the $LSCD$ order, than (X_2,Y_2), with joint c.d.f. F_2, denoted $(X_1,Y_1) \preceq_{LCSD} (X_2,Y_2)$, or similarly $F_1 \preceq_{LSCD} F_2$ if and only if

$$F_1(\min\{x_1,x_2\},\min\{y_1,y_2\}).F_2(\max\{x_1,x_2\},\max\{y_1,y_2\}) \geq F_1(x_1,y_2).F_2(x_2,y_1),$$

for allx_1,x_2,y_1,y_2, And similarly, (X_1,Y_1), with joint c.d.f. F_1, is said to be smaller, in the $RCSI$ order, than (X_2,Y_2), with joint c.d.f. F_2, denoted $(X_1,Y_1) \preceq_{RSCI} (X_2,Y_2)$, or similarly $F_1 \preceq_{RSCI} F_2$ if and only if

$$\begin{cases} \overline{F}_1(x_1,y_1).\overline{F}_2(x_2,y_2) \geq \overline{F}_1(x_1,y_2).\overline{F}_2(x_2,y_1) \\ \overline{F}_1(x_1,y_1).\overline{F}_2(x_2,y_2) \geq \overline{F}_1(x_2,y_1).\overline{F}_2(x_1,y_2) \end{cases} \text{for all } x_1 \leq x_2, y \leq y_2.$$

In the case where the two pairs have the same marginal distributions, i.e. $X_1 \overset{\mathcal{L}}{=} X_2$ and $Y_1 \overset{\mathcal{L}}{=} Y_2$,

[3] i.e.

$$\det\begin{pmatrix} h(x_1,y_1) & h(x_2,y_1) \\ h(x_1,y_2) & h(x_2,y_2) \end{pmatrix} \geq 0, \text{ for all } x_1 < x_2, y_1 < y_2$$

$(X_1,Y_1) \preceq_{LCSD} (X_2,Y_2)$ if and only if

$$\begin{cases} C_1(x_1,y_1).C_2(x_2,y_2) \geq C_1(x_1,y_2).C_2(x_2,y_1) \\ C_1(x_1,y_1).C_2(x_2,y_2) \geq C_1(x_2,y_1).C_2(x_1,y_2) \end{cases} \text{for all } x_1 \leq x_2, y \leq y_2 \text{ in } [0,1],$$

where C_1 and C_2 denote the copulas of (X_1,Y_1) and (X_2,Y_2) respectively, and $(X_1,Y_1) \preceq_{LCSD} (X_2,Y_2)$ if and only if

$$\begin{cases} \overline{C}_1(x_1,y_1).\overline{C}_2(x_2,y_2) \geq \overline{C}_1(x_1,y_2).\overline{C}_2(x_2,y_1) \\ \overline{C}_1(x_1,y_1).\overline{C}_2(x_2,y_2) \geq \overline{C}_1(x_2,y_1).\overline{C}_2(x_1,y_2) \end{cases} \text{for all } x_1 \leq x_2, y \leq y_2 \text{ in } [0,1].$$

Remark 3 *The fact that, if (X,Y) is LCSD or RCSI, then (X,Y) is PDQ, could be extended to the stochastic orderings defined above : if $(X_1,Y_1) \preceq_{LCSD} (X_2,Y_2)$ or $(X_1,Y_1) \preceq_{RSCI} (X_2,Y_2)$, then $(X_1,Y_1) \preceq_{PQD} (X_2,Y_2)$.*

Analogously, it is possible to define a negative dependence concept as follows : a nonnegative function h is reverse regular of order 2 ($RR2$) if, for all $x_1 < x_2$, $y_1 < y_2$,

$$h(x_1,y_1).h(x_2,y_2) \leq h(x_1,y_2).h(x_2,y_1).$$

And similarly, it is possible to extend the notions of $RSCI$ and $LSCD$ orderings as follows, for negative dependence : a pair (X,Y) of continuous random variables with joint c.d.f. F_{XY} is $LCSI$ (*left corner set increasing*) if F_{XY} is $RR2$; and (X,Y) is $RCSD$ (*right corner set decreasing*) if $\overline{F}_{XY}$ is $RR2$.

Proposition 4 *(i) If (X,Y) is LSCD, then, for any $(u,v) \in \mathcal{D}_C$, $\Phi(C,u,v)(x,y) \geq C^{\perp}(x,y)$ for all $x,y \in [0,1]$.*

(ii) If (X,Y) is $RCSI$, then, for any $(u,v) \in \mathcal{D}_{\overline{C}}$, $\Psi(C^*,u,v)(x,y) \geq C^{\perp}(x,y)$ for all $x,y \in [0,1]$

(iii) If (X,Y) is $LSCI$, then, for any $(u,v) \in \mathcal{D}_C$, $\Phi(C,u,v)(x,y) \leq C^{\perp}(x,y)$ for all $x,y \in [0,1]$.

(iv) If (X,Y) is $RCSD$, then, for any $(u,v) \in \mathcal{D}_{\overline{C}}$, $\Psi(C^*,u,v)(x,y) \leq C^{\perp}(x,y)$ for all $x,y \in [0,1]$.

Proof. Charpentier (2003). ■

2.6 Case of Archimedian copulas

An Archimedian copula is defined by

$$C(u,v) = \phi^{-1}(\phi(u) + \phi(v))$$

where ϕ is a convex, decreasing function on $]0,1]$ such that $\phi(1) = 0$. ϕ is called the generator of the copula. The generator of an Archimedian copula is defined up to a scale function. More precisely, two Archimedian copulas with generators ϕ and $\widetilde{\phi}$ are equal if and only if there exists a constant $c > 0$, such that $\phi = c\widetilde{\phi}$ (*Schweizer and Sklar* (1983)[4]).

Example 8 ***Gumbel copula*** *(Gumbel (1960))- It is defined by*

$$C(u,v) = \exp\left[-\left[(-\log u)^{\theta} + (-\log v)^{\theta}\right]^{1/\theta}\right],$$

and corresponds to the generator $\phi(t) = [-\ln t]^{\theta}$, where $\theta \geq 1$.

Example 9 ***Clayton copula*** *(Kimeldorf, Sampson (1975) and Clayton (1978)) - It is defined by*

$$C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta},$$

and correspond to the generator $\phi(t) = t^{-\theta} - 1$, where $\theta \geq 0$.

[4] This result has been obtained for Archimedian binary operation in probabilistic metric spaces (Theorem 5.4.8). The proof remains unchanged for Archimedian copulas.

2.6.1 Conditional copula

Proposition 5 *The conditional copula of and Archimedian copula with generator ϕ is also an Archimedian copula, with generator*

$$\phi_{u,v}(t) = \phi(tC(u,v)) - \phi(C(u,v)).$$

Proof. Charpentier (2003). ■

Example 10 ***Gumbel copula*** *- Gumbel copulas C have generator $\phi(t) = [-\ln t]^\theta$ where $\theta \geq 0$. For any $0 < u, v < 1$, the corresponding conditional copula has generator*

$$\phi_{u,v}(t) = \left[M^{1/\theta} - \ln t\right]^\theta - M \text{ where } M = [-\ln u]^\theta + [-\ln v]^\theta.$$

Example 11 ***Frank copula*** *- Frank copulas C have generator $\phi(t) = -\ln[(\exp(-\theta t) - 1) / ((\exp(-\theta) - 1))]$ where $\theta \in \mathbb{R}\setminus\{0\}$. For any $0 < u, v < 1$, the corresponding conditional copula has generator*

$$\phi_{u,v}(t) = \ln \frac{M}{(1+M)^t - 1} \text{ where } M = \frac{(\exp(-\theta u) - 1)(\exp(-\theta v) - 1)}{\exp(-\theta) - 1}$$

Example 12 ***Clayton copula*** *- Clayton copulas C have generator $\phi(t) = t^{-\theta} - 1$ where $\theta \in \mathbb{R}\setminus\{0\}$. For any $0 < u, v < 1$, the corresponding conditional copula is $\Phi(C, u, v)(x, y) = C(x, y)$.*

As noticed in the Example above, conditional copulas of Clayton copulas do not depend on u, v.

2.6.2 Factor representation - frailty models

A wide class of Archimedian copulas admit a factor representation. Let us assume that X and Y are independent, conditionally on Z, a positive random variable, such that

$$\mathbb{P}(X \leq x|Z) = G_X(x)^Z \text{ and } \mathbb{P}(Y \leq y|Z) = G_Y(y)^Z,$$

where G_X and G_Y are *cdf*'s. The joint *cdf* of couple (X, Y) is given by

$$\begin{aligned} F_{X,Y}(x,y) &= \mathbb{E}(\mathbb{P}(X \leq x, Y \leq y|Z)) = \mathbb{E}(\mathbb{P}(X \leq x|Z)\,\mathbb{P}(Y \leq y|Z)) \\ &= \mathbb{E}\left(G_X(x)^Z G_Y(y)^Z\right) = \mathbb{E}(\exp[-Z(-\log G_X(x))]\exp[-Z(-\log G_Y(y))]) \\ &= \psi(-\log G_X(x) - \log G_Y(y)), \end{aligned}$$

where ψ is the Laplace transform of the distribution of Z, i.e. $\psi(t) = \mathbb{E}(\exp(-tZ))$.

Because the marginal *cdf* of X and Y are given respectively by

$$F_X(x) = F_{X,Y}(x, +\infty) = \psi(-\log G_X(x)) \text{ and } F_Y(y) = \psi(-\log G_Y(y)),$$

the copula of (X, Y) is

$$C(u,v) = F_{XY}\left(F_X^{-1}(u), F_Y^{-1}(v)\right) = \psi\left(\psi^{-1}(u) + \psi^{-1}(v)\right)$$

This copula is an Archimedian copula with generator $\phi = \psi^{-1}$.

Example 13 ***Gumbel copula*** *- Gumbel copulas could be obtained when factor Z has its Laplace transform equal to $\psi(t) = \exp\left[-t^{1/\theta}\right]$.*

Example 14 ***Clayton copula*** *- Clayton copulas are obtained when the heterogeneity factor Z has a Laplace transform equal to $\psi(t) = [1-t]^{-1/\theta}$. The heterogeneity distribution is a Gamma distribution with degrees of freedom $1/\theta$.*

Proposition 6 *Let us consider (X, Y) with Archimedian copula, with f, and let ψ denote the Laplace transform of the heterogeneity factor. Let $0 < u, v \leq 1$, and (X, Y) given $X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(v)$ is an Archimedian copula with a factor representation, where the factor has Laplace transform*

$$\psi_{uv}(t) = \frac{\psi\left(t + \psi^{-1}(u) + \psi^{-1}(v)\right)}{\psi\left(\psi^{-1}(u) + \psi^{-1}(v)\right)}.$$

Proof. Charpentier (2003). ■

2.6.3 Invariant copulas

Proposition 7 *ϕ is the generator of an invariant copula if and only if, for all c, t in $[0,1] \times]0,1]$, $\phi(t) = \gamma\left[t^\theta - 1\right]$, i.e. an Archimedian copula C is invariant if and only if C is a Clayton copula.*

Proof. Charpentier (2003) . ■

A weaker condition of invariance could be defined : a copula C is said to be have invariant conditional copulas on the diagonal if $\Phi(C,t,t)$ does not depend on t and t,i.e. $\Phi(C,t,t) = C$ for all t in $[0,1]$.

Similarly, this notion could be extended to upper orthant dependence, and we can study copulas C such that $\Psi(C^*,t,t) = C^*$ for all t in $[0,1]$.

Proposition 8 *An Archimedian copula C is invariant on the diagonal if and only if C is a Clayton copula*

Proof. Charpentier (2003) . ■

2.6.4 Dynamic evolution and dependence orderings

In the case we consider the evolution if $\Phi(C,t)$ - that is the evolution of the conditional copula on the diagonal - one can expect a dependence structure which is all the more positively dependent as t decreases, or similarly, all the less dependent.In the first case, if $0 \le t_2 \le t_1 \le 1$, $\Phi(C,t_1) \preceq \Phi(C,t_2)$, in the sense that $\Phi(C,t_1)(x,y) \le \Phi(C,t_2)(x,y)$ for all x,y in $[0,1]\times[0,1]$. In the general case, it is difficult to obtain analytical interpretation of that result, but some properties could be obtained in the case of Archimedian copula.

Proposition 9 *Let t_1 and t_2 such that $0 \le t_2 \le t_1 \le 1$, and let C be an Archimedian copula with generator ϕ. Let*

$$f_{12}(x) = \phi\left(\frac{C_1}{C_2}\phi^{-1}(x+\phi(C_2))\right) - \phi(C_1) \text{ and } f_{21}(x) = \phi\left(\frac{C_2}{C_1}\phi^{-1}(x+\phi(C_1))\right) - \phi(C_2),$$

where $C_1 = C(t_1,t_1)$ and $C_2 = C(t_2,t_2)$. Then

(i) $\Phi(C,t_2)(x,y) \le \Phi(C,t_1)(x,y)$ for all x,y in $[0,1]\times[0,1]$ if and only if $f_{21}(x)$is sudadditive,

(ii) $\Phi(C,t_2)(x,y) \ge \Phi(C,t_1)(x,y)$ for all x,y in $[0,1]\times[0,1]$ if and only if $f_{12}(x)$is sudadditive.

Proof. Charpentier (2003). ■

Example 15 *The case of **Clayton** copulas could be seen as a limiting case, in the sense that $\phi(t) = t^{-\theta} - 1$ and so,*

$$f_{12}(x) = ax + b \text{ where } a = C_1^\theta / C_2^\theta.$$

In the case were ϕ is twice differentiable, a sufficient condition for uniform ordering of conditional copula is the following.

Lemma 10 *If ϕ is twice differentiable, let $\psi(x) = \log -\phi'(t)$,*

(i) If ψ is concave on $]0,1]$, then $\Phi(C,t_2)(x,y) \le \Phi(C,t_1)(x,y)$ for all x,y in $[0,1]\times[0,1]$, for all $0 \le t_2 \le t_1 \le 1$.

(ii) Similarly, if $\psi(x)$ is convex on $]0,1]$, then $\Phi(C,t_2)(x,y) \ge \Phi(C,t_1)(x,y)$ for all x,y in $[0,1]\times[0,1]$, for all $0 \le t_2 \le t_1 \le 1$.

Proof. Charpentier (2003) . ■

Example 16 *Let C be a **Ali-Mikhail-Haq** copula, with generator $\phi(x) = \log(1-\theta(1-x)) - \log x$. Then*

$$\phi'(x) = \frac{\theta}{1-\theta(1-x)} - \frac{1}{x} \text{ and } \psi(x) = \log\left(\frac{1}{x} - \frac{\theta}{1-\theta(1-x)}\right)$$

One gets that

$$\begin{aligned}
\psi''(x) &= \frac{\phi'''(x)\phi'(x) - \phi''(x)^2}{\phi'(x)^2} = \frac{2}{\phi'(x)^2}\left[\frac{\theta^3x^3 - (1-\theta(1-x))^3}{x^3(1-\theta(1-x))^3}\right] \\
&= \frac{-2(1-\theta)}{\phi'(x)^2}\left[\frac{3\theta^2x^2 + 3\theta(1-\theta)x + (1-\theta)^2}{x^3(1-\theta(1-x))^3}\right]
\end{aligned}$$

which has the opposite sign of $3\theta^2x^2 + 3\theta(1-\theta)x + (1-\theta)^2$ ($\theta \le 1$) which is positive. So, finally, ψ is a concave function on $[0,1]$, and so $\Phi(C,t_2)(x,y) \le \Phi(C,t_1)(x,y)$ for all x,y in $[0,1]\times[0,1]$, for all $0 \le t_2 \le t_1 \le 1$: (X,Y) given $X \le F_X(t)$ and $Y \le F_Y(t)$ is less and less positively dependent, as t decreases towards 0.

Example 17 *Let C be the copula given by **4.2.19** in Nelsen (1999), that is with generator $\phi(x) = \exp(\theta/x) - \exp(\theta)$. Then, for all t_1 and t_2 such that $0 \leq t_2 \leq t_1 \leq 1$, and let $C_i = \theta/\log[2\exp(\theta/t_i) - \exp(\theta)]$ where $i = 1, 2$. One gets*

$$f_{12}(x) = \exp\left(\frac{\log[2\exp(\theta/t_1) - \exp(\theta)]}{\log[2\exp(\theta/t_2) - \exp(\theta)]}\log(x + 2\exp(\theta/t_2) - \exp(\theta))\right) - 2\exp(\theta/t_1) + \exp(\theta)$$

Let α, β and γ such that

$$f_{12}(x) = \exp(\alpha \log(x + \beta)) - \gamma$$

which gives, derivating two times with respect to x,

$$\frac{d^2}{dx^2} f_{12}(x) = \frac{\alpha(\alpha - 1)}{(x + \beta)^2}\exp(\alpha \log(x + \beta))$$

Because $t_2 \leq t_1$, $C_1(t_1, t_1)/C_2(t_2, t_2) \geq 1$, $\alpha(\alpha - 1) \geq 0$, and then $d^2 f_{12}(x)/dx^2 \geq 0$ and $f_{12}(x)$ is concave. Hence, because $f_{12}(0) = 0$ and $f_{12}(x)$ is convex, then $f_{12}(x)$ is subadditive. For all t_1 and t_2 such that $0 \leq t_2 \leq t_1 \leq 1$, $f_{12}(x)$ is subadditive :(X, Y) given $X \leq F_X(t)$ and $Y \leq F_Y(t)$ is more and more positively dependent, as t decreases towards 0.

One can notice that this case is an application of Lemma (10) :

$$\phi'(x) = -\frac{\theta}{x^2}\exp\left(\frac{\theta}{x}\right) \text{ and } \psi(x) = \log -\phi'(t) = \frac{\theta}{x} + \log\theta - 2\log x$$

which is a convex function on $[0, 1]$, and so $\Phi(C, t_2)(x, y) \geq \Phi(C, t_1)(x, y)$ for all x, y in $[0, 1] \times [0, 1]$, for all $0 \leq t_2 \leq t_1 \leq 1$.

Example 18 *Let C be a copula in the **Gumbel-Barnett** family, that is $\phi(x) = \log(1 - \theta \log x)$. Then*

$$\phi'(x) = \frac{-\theta}{x(1 - \theta \log x)} \text{ and } \psi(x) = \log\theta - \log x - \log(1 - \theta \log x)$$

which is a convex function on $[0, 1]$, and so $\Phi(C, t_2)(x, y) \geq \Phi(C, t_1)(x, y)$ for all x, y in $[0, 1] \times [0, 1]$, for all $0 \leq t_2 \leq t_1 \leq 1$. In that case (X, Y) given $X \leq F_X(t)$ and $Y \leq F_Y(t)$ is more and more positively dependent as t decreases towards 0 should be understood as (X, Y) given $X \leq F_X(t)$ and $Y \leq F_Y(t)$ is less and less negatively dependent as t decreases towards 0. This is a direct implication of the fact that the conditional copula of a Gumbel-Barnett copula remains in this family, with a smaller parameter.

Example 19 *Let C be a **Frank** copula, with generator $\phi(x) = -\log[(\exp(-\theta x) - 1)/(\exp(-\theta) - 1)]$, then*

$$\phi'(t) = \frac{\theta \exp(-\theta x)}{\exp(-\theta x) - 1} \text{ and } \psi(t) = \log\theta - \theta x - \log(1 - \exp(-\theta x))$$

which satisfies $\psi''(x) = -\theta^2 \exp(-\theta x)/[\exp(-\theta x) - 1]^2 \leq 0$: ψ is concave, and so $\Phi(C, t_2)(x, y) \leq \Phi(C, t_1)(x, y)$ for all x, y in $[0, 1] \times [0, 1]$, for all $0 \leq t_2 \leq t_1 \leq 1$.

Example 20 *Let C be a **Clayton** copula, with generator $\phi(x) = x^{-\theta} - 1$, then*

$$\phi'(x) = -\theta x^{-\theta - 1} \text{ and } \psi(x) = \log\theta - (1 + \theta)\log x$$

which is convex and so $\Phi(C, t_2)(x, y) \leq \Phi(C, t_1)(x, y)$ for all x, y in $[0, 1] \times [0, 1]$, for all $0 \leq t_2 \leq t_1 \leq 1$. In that particular case, because Clayton copulas are invariant, one could have expect a linear functional for ψ, that is both convex and concave.

Example 21 *Let C be a **Gumbel** copula, with generator $\phi(x) = (-\log x)^{\theta}$, $\theta \geq 1$, then*

$$\phi'(x) = -\theta(-\log x)^{\theta - 1}/x, \text{ and } \psi(x) = \log\theta - \log x + (\theta - 1)\log(\log[-x])$$

This function being twice differentiable, one gets

$$\psi''(x) = \frac{(\log x)^2 - [\theta - 1]\log x - [\theta - 1]}{x^2[\log x]^2} = \frac{h(\log x)}{x^2[\log x]^2}$$

where $h(y) = y^2 - [\theta - 1] y - [\theta - 1]$ *this polynomial has two (real) roots, and one is negative*[5]. *So finally,* $\psi''(x) \leq 0$ *on* $]0, x_0]$ *and* $\psi''(x) \geq 0$ *on* $[x_0, 1]$ *for some* x_0 : ψ *is neither concave nor convex. In that particular case,* $\phi^{-1}(x) = \exp\left(-x^{1/\theta}\right)$, *so that*

$$\phi_{t_1}(x) = (-\log C_1 x)^\theta - (-\log C_1)^\theta \text{ and } \phi_{t_1}^{-1}(x) = \frac{1}{C_1} \exp\left(-\left[x + (-\log C_1)^\theta\right]^{1/\theta}\right)$$

Then, for all $0 < t_2 \leq t_1 \leq 1$,

$$f_{21}(x) = \left(-\log\left[\frac{C_2}{C_1} \exp\left(-\left[x + (-\log C_1)^\theta\right]^{1/\theta}\right)\right]\right)^\theta - (-\log C_2)^\theta$$

In the case where $t_1 = 1$, *then* $C_1 = 1$, *and*

$$f_{.2}(x) = (x - \log C_2)^\theta - (-\log C_2)^\theta = (x - \alpha)^\theta - \alpha^\theta \text{ with } \alpha \geq 0,$$

which satisfies

$$f''_{.2}(x) = \theta(\theta - 1)(x + \alpha)^{\theta - 2} \text{ with } x, \alpha \geq 0$$

Because $\theta \geq 1$, *this function is positive, that is* $f_{.2}$ *is concave. Furthermore* $f_{.1}(0) = 0$, *and so,* $f_{.1}$ *is subadditive* : $\Phi(C, t)(x, y) \leq C(x, y)$ *for all* x, y *in* $[0, 1] \times [0, 1]$, *for all* $0 \leq t \leq 1$. (X, Y) *given* $X \leq F_X(t)$ *and* $Y \leq F_Y(t)$ *is less positively dependent than* (X, Y).

2.6.5 Asymptotic behavior of $\Phi(C, u, v)$ when $u, v \to 0$

In extreme value theory, the rate of decay at infinity (*or at zero*) of a function, or the fatness of the tails is usually expressed through an index of regular variation at infinity (*or at zero*).

Definition 1 *A function* $f :]0, +[\to]0, +[$ *is called regularly varying at* 0 *with index* $\rho \in \mathbb{R}$, *if for any* $x > 0$,

$$\lim_{t \to 0} \frac{f(tx)}{f(t)} = x^\rho,$$

and f *belongs to* $\mathcal{R}_\rho^0$. *In the case where* $\rho = 0$, *the function is said to be slow varying at* 0.

Nelsen (1999) studied some Archimedean copulas whose generator are regularly varying.

Example 22 ***Gumbel copula*** - *Gumbel copulas have generator* $\phi(t) = [-\ln t]^\theta$ *where* $\theta \geq 0$, *which belong to* $\mathcal{R}_0^0$..

Example 23 ***Clayton copula*** - *Clayton copulas* C *have generator* $\phi(t) = t^{-\theta} - 1$ *where* $\theta > 0$ *which belong to* $\mathcal{R}_{-\theta}^0$.

Proposition 11 *Let* C *be an Archimedean copula with differentiable generator* $\phi \in \mathcal{R}_{-\alpha}^0$, *where* $0 \leq \alpha \leq +\infty$. *Then, for all* $0 \leq x, y \leq 1$,

$$\lim_{u, v \to 0} \Phi(C, u, v)(x, y) = \Gamma_\alpha(x, y),$$

where Γ_α *is Clayton copula with parameter* α. *The limit above is obtain when either* u *or* v *tend to* 0.

In the case where $\alpha = 0$, *then* $\Phi(C, u, v)$ *converges to* $C^\perp$ *(limit case of Clayton copula when* $\alpha \to 0$*).*

In the case where $\alpha = +\infty$, *then* $\Phi(C, u, v)$ *converges to* C^+ *(limit case of Clayton copula when* $\alpha \to -\infty$*).*

Proof. Charpentier (2003). ■

Remark 4 *This result is an extension of the result obtained by Juri and Würthrich* (2002).

Proposition 12 *Let* C *be an Archimedean copula with generator* ϕ. *The limiting case* $\lim_{u, v \to 0} \Phi(C, u, v)(x, y) = C^+(x, y)$ *is obtained if and only if* ϕ *satisfies*

$$\lim_{x \to 0} \frac{\phi(xt) - \phi(x)}{x\phi'(xt)} = 0 \text{ for all } t \text{ in } [0, 1].$$

Proof. This proposition is obtained using a proposition due to Genest and MacKay (1986), stating that if ϕ_n is a sequence of generator, and C_n the sequence of associated copulas, then $\lim_{u, v \to 0} C_n(x, y) = C^+(x, y)$ if and only if $\phi_n(t) / \phi'_n(t) \to 0$ when $n \to \infty$ for all t in $[0, 1]$. ■

[5] Because $\Delta = (\theta - 1)(\theta + 3) \geq 0$ and the product of the roots is $-(\theta + 1) \leq 0$.

2.6.6 Generalized Archimedian copulas with factor representation

In that case, there is a factor Z such that the copula of (X,Y) given $Z=z$ does not depend on z, and is an extreme value copula C, whereas the conditional marginal distribution depend on Z, i.e. $\mathbb{P}(X\leq x|Z)=G_X(x)^Z$, with a similar expression for Y. In that case, the copula of (X,Y) is a function of ψ, the Laplace transform of Z, and C, given by

$$C_{XY}(x,y)=\psi\left(-\log\left(C\left(\exp\left[-\psi^{-1}(x)\right],\exp\left[-\psi^{-1}(y)\right]\right)\right)\right) \tag{2}$$

In the case where $C=C^{\perp}$, then C_{XY} is an Archimedian copula with generator ψ^{-1}. We will see with the Theorem below that, in the case where the copula of (X,Y) is a generalized Archimedian copula, with a factor representation, then, for all u,v in $[0,1]$, the copula of (X,Y) given $X\leq F_X^{-1}(u), Y\leq F_Y^{-1}(v)$ is also function of a Laplace transform ψ^* and a copula C^*

$$C_{XY}^*(x,y)=\psi^*\left(-\log\left(C^*\left(\exp\left[-\psi^{*-1}(x)\right],\exp\left[-\psi^{*-1}(y)\right]\right)\right)\right) \tag{3}$$

And furthermore, the copula of (X,Y) given $X\leq F_X^{-1}(u), Y\leq F_Y^{-1}(v)$ is a generalized Archimedian copula, with a factor representation if and only if the copula C^* is an extreme value copula.

In the case of Archimedian copula with a factor representation, then the Proposition below proves that the copula of (X,Y) given $X\leq F_X^{-1}(u), Y\leq F_Y^{-1}(v)$ is a also an Archimedian copula with a factor representation.

Proposition 13 *Let us consider (X,Y) with a generalized Archimedian copula, with a factor representation, and let ψ denote the Laplace transform of the heterogeneity factor, C denote the underlying copula, and G_X and G_Y the 'marginal parameters'.*

(1) *Let $0<u,v\leq 1$, then, the copula of (X,Y) given $X\leq F_X^{-1}(u), Y\leq F_Y^{-1}(v)$ is*

$$C_{XY}^*(x,y)=\psi^*\left(-\log\left(C^*\left(\exp\left[-\psi^{*-1}(x)\right],\exp\left[-\psi^{*-1}(y)\right]\right)\right)\right)$$

where

- *ψ^* is the following Laplace transform $\psi^*(t)=\psi(t+\alpha)/\psi(\alpha)$ where $\alpha=-\log(C(u^*,v^*))$ and $u^*=\exp\left[-\psi^{-1}(u)\right]$, $v^*=\exp\left[-\psi^{-1}(v)\right]$: ψ^* is the Laplace transform of Z given $X\leq F_X^{-1}(u), Y\leq F_Y^{-1}(v)$*
- *$\mathbb{P}\left(X\leq x|X\leq F_X^{-1}(u), Y\leq F_Y^{-1}(v), Z=z\right)=G_X^*(x)^z$ and $\mathbb{P}\left(Y\leq y|X\leq F_X^{-1}(u), Y\leq F_Y^{-1}(v), Z=z\right)=G_Y^*(y)^z$ where*

$$G_X^*(x)=\frac{C(G_X(x),v^*)}{C(u^*,v^*)} \text{ and } G_Y^*(y)=\frac{C(u^*,G_Y(y))}{C(u^*,v^*)}$$

- *C^* is the following copula*

$$C^*(x,y)=\frac{C\left(G_X\left(G_X^{*-1}(x)\right),G_Y\left(G_Y^{*-1}(y)\right)\right)}{C\left(G_X\left(F_X^{-1}(u)\right),G_Y\left(F_Y^{-1}(v)\right)\right)}=\frac{C\left(G_X\left(G_X^{*-1}(x)\right),G_Y\left(G_Y^{*-1}(y)\right)\right)}{C(u^*,v^*)}$$

(2) *Furthermore, (X,Y) given $X\leq F_X^{-1}(u), Y\leq F_Y^{-1}(v)$ has a generalized Archimedian copula with a factor representation if and only if C^* is an extreme value copula. That is, C^* is the copula of (X,Y) given $X\leq F_X^{-1}(u), Y\leq F_Y^{-1}(v)$ and $Z=z$ for all z if and only if $C^*\left(x^k,y^k\right)=C^*(x,y)^k$ for all $k>0$ and x,y in $[0,1]$.*

Proof. Charpentier (2003). ■

Example 24 *Let C be a Gumbel copula, such that $C(x,y)=\exp\left(-\left((-\log x)^\theta+(-\log y)^\theta\right)^{1/\theta}\right)$.*

$$C^*(x,y)=\exp\left(\alpha^{1/\theta}-\left(\left[-\log x+\alpha^{1/\theta}\right]^\theta+\left[-\log y+\alpha^{1/\theta}\right]^\theta-\alpha\right)^{1/\theta}\right)$$

where $\alpha=(-\log u^*)^\theta+(-\log v^*)^\theta$. Then, one can notice that,

$$C^*(x^z,y^z)=\exp\left(\alpha^{1/\theta}-\left(\left[-z\log x+\alpha^{1/\theta}\right]^\theta+\left[-z\log y+\alpha^{1/\theta}\right]^\theta-\alpha\right)^{1/\theta}\right)$$

while

$$C^*(x,y)^z=\exp\left(z\left[\alpha^{1/\theta}-\left(\left[-\log x+\alpha^{1/\theta}\right]^\theta+\left[-\log y+\alpha^{1/\theta}\right]^\theta-\alpha\right)^{1/\theta}\right]\right)$$

and, given x, y and $\alpha>0$,, $f(z)=C^*(x,y)^z-C^*(x^z,y^z)=0$ only when $z=1$: in that case, C^* is not an extreme copula.

As well as Archimedian copulas are stable (*in the sense that conditional copulas are still Archimedian copula*), those copulas define a stable family of copulas.

Example 25 *Let us consider some insurance claims, where X_t denotes the cost for year t, and X_{t+1} the cost for year $t+1$, for one contract. Let Θ be a random variable, which denotes the risk variable, such that, for, given $\Theta = \theta$, the survival copula of (X_t, X_{t+1}) is C^*, and such that $\mathbb{P}(X_t > x|\Theta = \theta) = \overline{G}_t(x)^\theta$ and $\mathbb{P}(X_{t+1} > x|\Theta = \theta) = \overline{G}_{t+1}(x)^\theta$. Let ψ denote the Laplace transform of the risk variable Θ, then the unconditional copula of $(X_t; X_{t+1})$ is given by* (2)*. Furthermore, it is possible to focus on 'bad' contracts of year t, and study the dependence, for those contract, between X_t and X_{t+1}, which the copula of (X_t, X_{t+1}), given $X_t > F_t^{-1}(p)$ where $p \in [0,1]$, and is obtained as was obtained* (3)*.*

2.7 Applications of conditional copulas

Let us define the default time of a firm by, for any $i = 1, .., n$ (*for a portfolio of n firms*) $\tau_i = \inf\{t, \gamma_i(t) \leq U_i\}$, where the default trigger variables U_i are defined on $[0,1]$, and $\gamma_i(t)$ are the default countdown processes, defined as

$$\gamma_i(t) = \exp\left(-\int_0^t \lambda_i(s)\,ds\right)$$

where λ_i are non-negative continuous processes, called default intensity processes. For example, as shown in Lando (1998), the time of default of a Cox process with intensity $\lambda_i(t)$ can be written

$$\tau_i = \inf\left\{t, \int_0^t \lambda_i(s)\,ds \geq \theta_i\right\}$$

where θ_i is a unit exponential random variable, independent of the default intensity process $\lambda_i(t)$ (*this setup is equivalent to taking U_i uniform on* $[0,1]$).

If the survival copula, between two default times τ_X and τ_Y at time $t = 0$ is C^*, and if no defaults occur between time 0 and time t, the conditional survival copula at time t is not necessarily C^*, as noticed in Giesecke (2001) and Jouanin (2003). In fact, the joint survival distribution is

$$\mathbb{P}(X > h + x, Y > h + y|X, Y > h) = \frac{C^*\left(\overline{F}_X(h+x), \overline{F}_Y(h+y)\right)}{C^*\left(\overline{F}_X(h), \overline{F}_Y(h)\right)} \text{ where } x, y \geq 0.$$

so that the copula of (X, Y) given $X, Y > h$ is then $\Psi\left(C^*, \overline{F}_X(h), \overline{F}_Y(h)\right)$.

Remark 5 *As shown in Charpentier* (2003)*, the choice of the starting time is arbitrary, and does not influence.... More precisely, Charpentier* (2003) *proved that, if C is a copula, and u, v in* $]0,1]$*, then*

$$\{\Phi(C, u', v'), 0 \leq u' \leq u, 0 \leq v' \leq v\} = \{\Phi(\Phi(C, u, v), u', v'), 0 \leq u' \leq u, 0 \leq v' \leq v\}$$

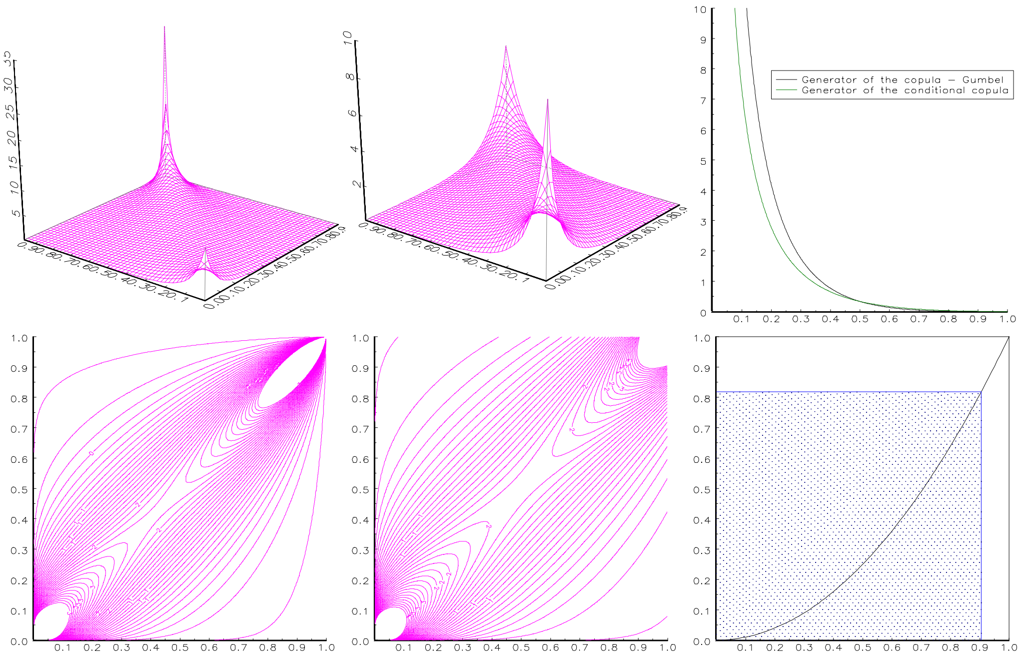
which could be written, in terms of credit risk models as follows : if no defaults occur between time T and time $T + h$, the copula at time $T + h$ is

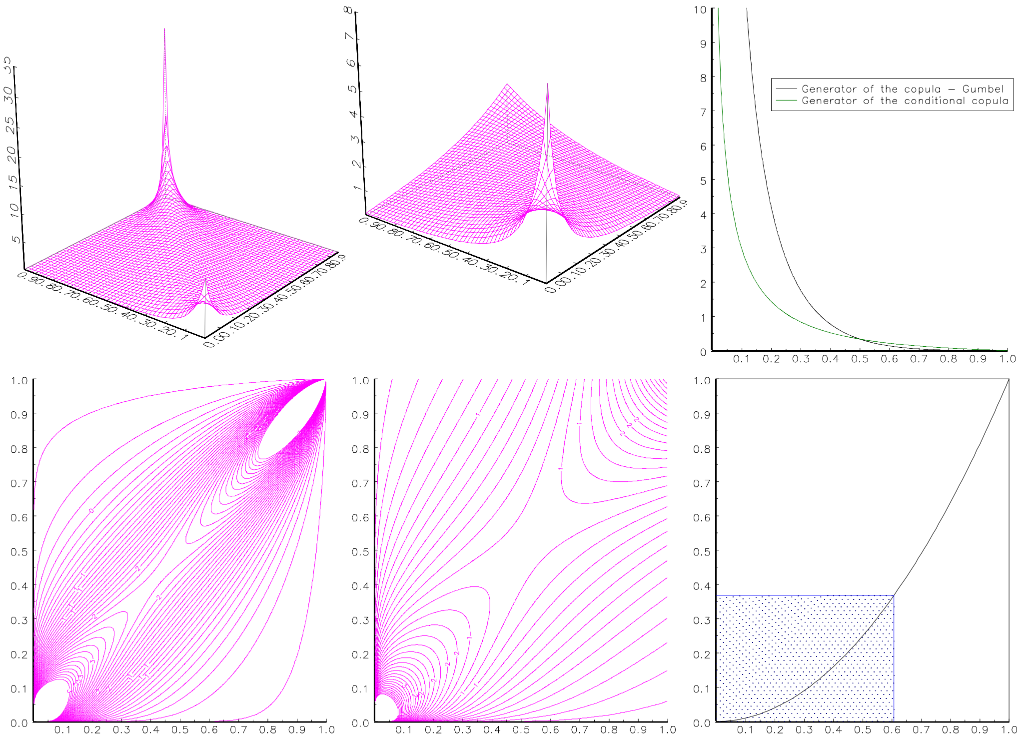
$$\Psi\left(C^*, \overline{F}_X(T+h), \overline{F}_Y(T+h)\right) = \Psi\left(\Psi\left(C^*, \overline{F}_X(T), \overline{F}_Y(T)\right), \overline{F}_{X|X,Y>T}(T+h), \overline{F}_{Y|Y,Y>T}(T+h)\right)$$

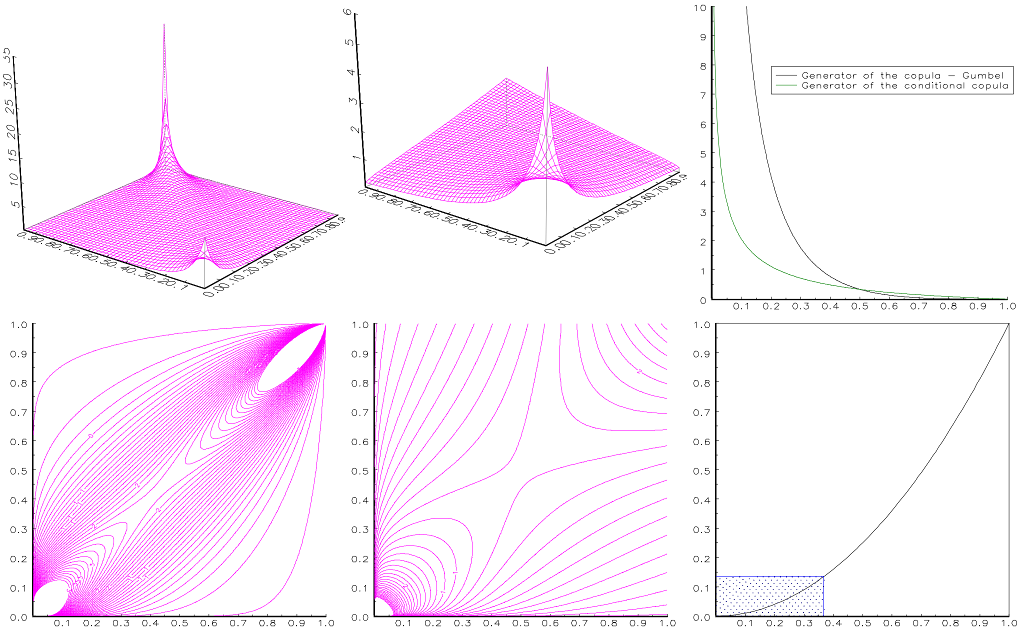
where $\overline{F}_{X|X,Y>T}$ and $\overline{F}_{Y|X,Y>T}$ denote respectively the survival distribution of X and Y given $X, Y > T$. The dynamic evolution of the dependence structure remains unchanged, starting at time 0 *or at time T.*

Example 26 *Let τ_X and τ_Y be exponentially distributed, i.e.* $\mathbb{P}(\tau_X > t) = \exp(-t)$ *while* $\mathbb{P}(\tau_Y > t) = \exp(-2t)$. *Let us assume that at time $t = 0$, the copula of τ_X and τ_Y is a Gumbel copula. The graphs below show the evolution of the joint τ_X and τ_Y at time $t = 1$,* 1 *and* 3*, if no defaults occur between* 0 *and t.*

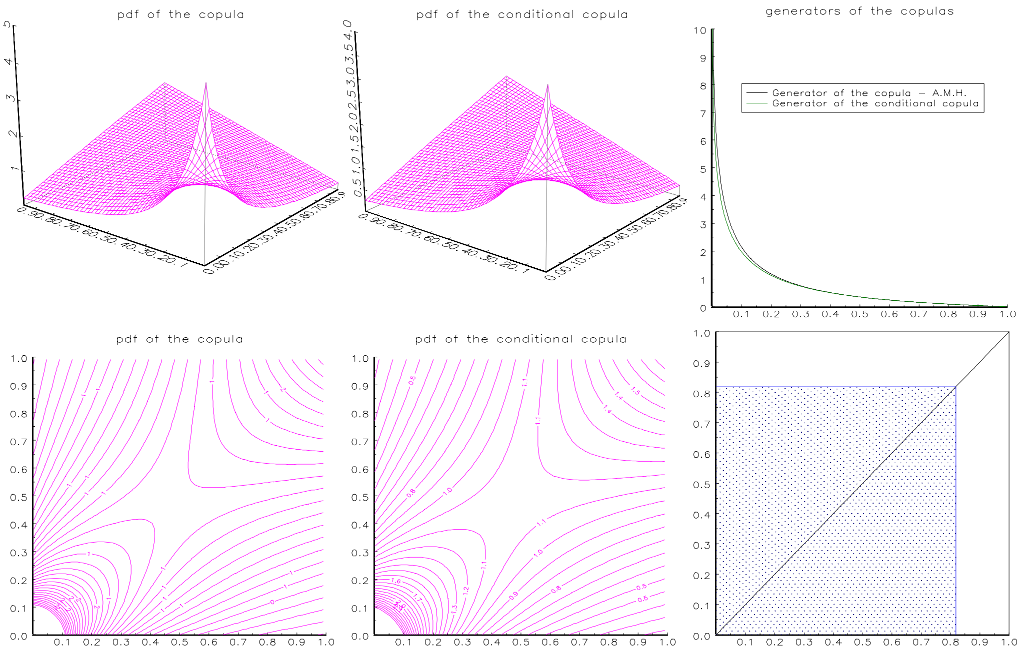
The graphs on the left are the initial shape of the distribution (at time 0),

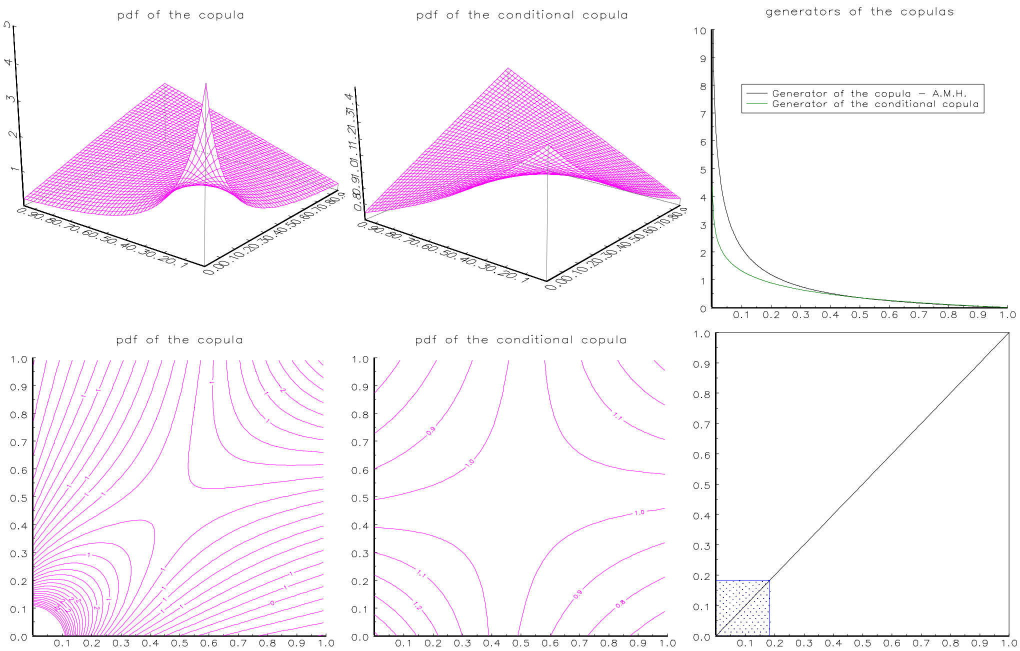






Example 27 *Following the example above, we assume now that,initially, the copula of* (τ_X, τ_Y) *was a Ali-Mikhail-Haq copula.*





3 Conditional rank correlations : tail dependence measure

3.1 Definition of the conditional rank correlations

Some classical measures of dependence, such as Spearman's rho or Kendall's tau, have initially been defined using the notion of '*concordance*'. An heuristic interpretation of concordance is that a pair of random variables are concordant if large values of one tend to be associated with large values of the other, and small values of one with small values of the other : (X_i, Y_i) and (X_j, Y_j) are concordant if $(X_i - X_j)(Y_i - Y_j) > 0$. Similarly, (X_i, Y_i) and (X_j, Y_j) are said discordant if $(X_i - X_j)(Y_i - Y_j) < 0$. As shown in Nelsen (1999) it is also possible to defined some risk measures using copulas. Kendall's tau of couple (X, Y) is defined as

$$\tau(X,Y) = \int\int_{[0,1]^2} C(u,v)\, dC(u,v) = \tau(C)$$

as in Schweizer and Wolf (1981) . Spearman's rho could be defined using the independent copula $C^{\perp}(u,v) = uv$, as[6]

$$\rho(X,Y) = 12\int\int_{[0,1]^2} uv dC(u,v) - 3 = 12\int\int_{[0,1]^2} C(u,v)\, dudv - 3 = \rho(C)$$

Using these two definitions, one can notice that those two measures of dependence depend only on the copula C, and not on marginal distributions F_X and F_Y. This implies that, if ϕ and ψ are strictly increasing functions on the range of X and Y, then $\rho(X,Y) = \rho(\phi(X), \psi(Y))$ and $\tau(X,Y) = \tau(\phi(X), \psi(Y))$.

Another classical measure of dependence is Pearson's (*linear*) correlation, which is defined as follows,

$$r(X,Y) = \frac{cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

where the variances are defined by

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \int_{\mathbb{R}} x^2 dF_X(x) - \left[\int_{\mathbb{R}} x dF_X(x)\right]^2$$

and, as shown in Hottelling (1940)

$$cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \int\int_{\mathbb{R}^2} [F_{XY}(x,y) - F_X(x)F_Y(y)]\, dxdy = \int\int_{[0,1]^2} [C(u,v) - uv]\, dF_X^{-1}(u)\, dF_Y^{-1}(v)$$

[6] Using this expression, Spearman's rho could be seen as the "scaled" volume under the graph of the copula (*volume under the graph of the copula and over the unit square*). Spearman's rho is also proportional to the signed volume between the graphs of the copula C and the product copula $C^{\perp}$. Thus, as notice in Nelsen (1999) is a measure of "average distance" between the distribution of (X,Y) and independence.

This coefficient has not simple expression and depends on the copula C but also on marginal distributions. This means that, if ϕ and ψ are strictly increasing functions on the range of X and Y, then $r(X,Y)$ and $r(\phi(X),\psi(Y))$ have no reason to be equal (*except in some cases where ϕ and ψ are increasing linear function*).

As mentioned before, because variables X and Y are continuous, $U = F_X(X)$ and $V = F_Y(Y)$ are uniform on $[0,1]$ (*so that U and V have mean* $1/2$ *and variance* $1/12$) and then,

$$\rho(X,Y) = 12\int\int_{[0,1]^2} C(u,v)\,dudv - 3 = 12\mathbb{E}(UV) - 3 = \frac{\mathbb{E}(UV) - \mathbb{E}(U)\mathbb{E}(V)}{\sqrt{Var(U)}\sqrt{Var(V)}} = r(U,V) = r(F_X(X), F_Y(Y))$$

Spearman's rho is equal to Pearson's correlation between $F_X(X)$ and $F_Y(Y)$. This relationship gives a simple way to estimate the measure using the ranks of sample : Spearman's correlation is the correlation (*Pearson's linear correlation*) of the ranks, *i.e.* the sample expression of Spearman's rho is the (*linear*) correlation of the pairs (R_i, S_i) where R_j is the rank of X_j among the sample $X_1, ..., X_n$, and S_j is the rank of Y_j among the sample $Y_1, ..., Y_n$.

If $\tau(X,Y) = +1$ or $\rho(X,Y) = +1$ then. Conversely, let F_X and F_Y marginal c.d.f. , then there is a couple (X,Y) with marginal c.d.f. F_X and F_Y such that $\rho(X,Y) = +1$. This property does not stand of Person's correlation : given two marginal c.d.f. there is usually no couple (X,Y) such that $r(X,Y) = +1$.

Example 28 *Let X and Y have respectively log-normal distributions $LN(0,1)$ and $LN(0,\sigma^2)$. As shown in Wang* (1997)*, Pearson's correlation $r(X,Y)$ is bounded as follows*

$$\underline{r}(X,Y) \leq r(X,Y) \leq \overline{r}(X,Y) \text{ where } \underline{r}(X,Y) = \frac{\exp(-\sigma) - 1}{\sqrt{\exp(\sigma^2) - 1}\sqrt{e-1}} \text{ and } \overline{r}(X,Y) = \frac{\exp(\sigma) - 1}{\sqrt{\exp(\sigma^2) - 1}\sqrt{e-1}}$$

For example, if $\sigma^2 = 3$, then the bounds are -0.008 and 0.16, while $\rho(X,Y)$ and $\tau(X,Y)$ can take any value from -1 to $+1$.

Definition 2 *The conditional rank correlation of couple (U,V) with copula C, based on Spearman's rho, is defined on $[0,1]\times[0,1]$ by*

$$\underline{\rho}(C,u,v) = \rho(C,[0,u],[0,v]) : (u,v) \mapsto \rho(U,V|U \leq u, V \leq v).$$

The functional measure of dependence[7] *will also be called "lower tail conditional rank correlation", on $[0,u]\times[0,v]$, or "truncated rank correlation", and could be denoted $\underline{\rho}(C,u,v)$. Similarly, one can define easily the "upper tail conditional rank correlation", on $[u,1]\times[v,1]$, as*

$$\overline{\rho}(C,u,v) = \rho(C,[u,1],[v,1]) : (u,v) \mapsto \rho(U,V|U > u, V > v).$$

As mentioned before, if Spearman's correlation could be written using the copula of (U,V), then, these tail rank correlations could be written using conditional copulas :

$$\underline{\rho}(C,u,v) = 12\int\int_{[0,1]^2} \Phi(C,u,v)(x,y)\,dxdy - 3 \text{ and } \overline{\rho}(C,u,v) = 12\int\int_{[0,1]^2} \Psi(C^*,u,v)(x,y)\,dxdy - 3$$

Remark 6 *Malevergne and Sornette* (2002) *have studied a function that is called "conditional rank correlation", defined as $v \mapsto r(U,V|V > v)$.*

From the properties of the conditional copulas, one could obtain several interesting results on these tail conditional rank correlations. For example,

[7] The function is base on the conditional distribution of $(U,V)\,|\,(U,V) \in [0,u]\times[0,v]$ but is also base on the conditional distribution of (X,Y) :

$$\begin{aligned} \underline{\rho}(C,u,v) &= \rho(U,V|0 \leq U \leq u, 0 \leq V \leq v) \\ &= \rho(X,Y|0 \leq U \leq u, 0 \leq V \leq v) \\ &= \rho\left(U,V|X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(v)\right) \\ &= \rho\left(X,Y|X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(v)\right) \end{aligned}$$

But

$$\begin{aligned} \underline{\rho}(C,u,v) &= \rho(U,V|0 \leq U \leq u, 0 \leq V \leq v) \\ &\neq r(U,V|0 \leq U \leq u, 0 \leq V \leq v) \end{aligned}$$

Proposition 14 *If (X,Y) is PQD then $\underline{\rho}(u,v)=\rho\left([0,u]\times[0,v]\right)\geq 0$ for all $0<u,v\leq 1$.*

A natural estimate of Spearman's rho is to consider the correlation (*Pearson's linear correlation*) of the ranks. Similarly, a natural estimate of the tail conditional correlations is given by the correlation of the ranks of (U,V) given either $U\leq u$ and $V\leq v$, or $U>u$ and $V>v$. Let $(X_1,Y_1),...,(X_n,Y_n)$ be a n-bivariate sample, and let

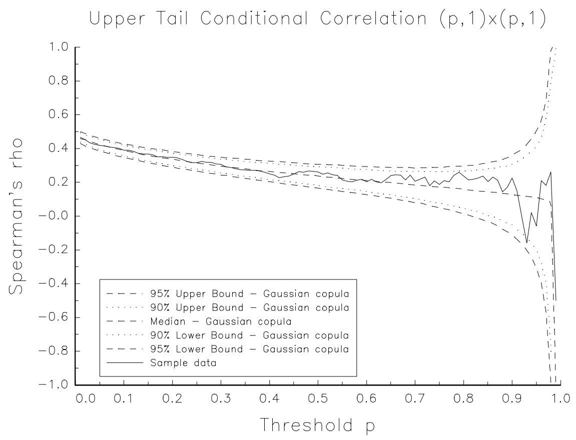
$$U_i=\widehat{F}_{X,n}(X_i) \text{ and } V_i=\widehat{F}_{Y,n}(Y_i) \text{ where } \widehat{F}_{X,n}(x)=\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}(X_i\leq x) \text{ and } \widehat{F}_{Y,n}(x)=\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}(Y_i\leq x),$$

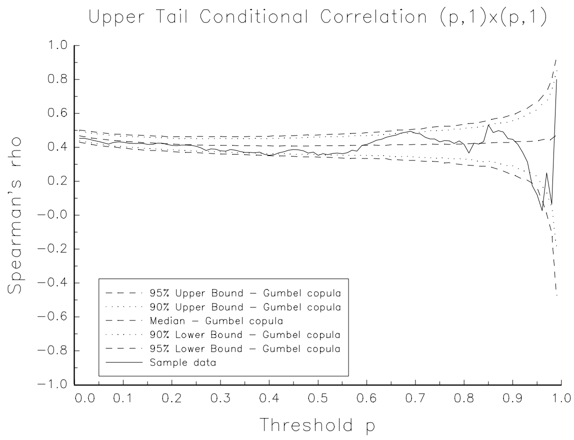
so that nU_i and nV_i are respectively the ranks of X_i within $X_1,...,X_n$ and Y_i.within $Y_1,...,Y_n$. If $\mathcal{I}(u,v,n)=\{i|U_i\leq u,V_i\leq v\}$, then

$$\widehat{\underline{\rho}}(C,u,v)=corr\left(U_i,V_i\,|i\in\mathcal{I}(u,v,n)\right)=\frac{\sum_{i\in\mathcal{I}(u,v,n)}\left(U_i-\overline{U}_{u,v}\right)\left(V_i-\overline{V}_{u,v}\right)}{\sqrt{\sum_{i\in\mathcal{I}(u,v,n)}\left(U_i-\overline{U}_{u,v}\right)^2}\sqrt{\sum_{i\in\mathcal{I}(u,v,n)}\left(V_i-\overline{V}_{u,v}\right)^2}}$$

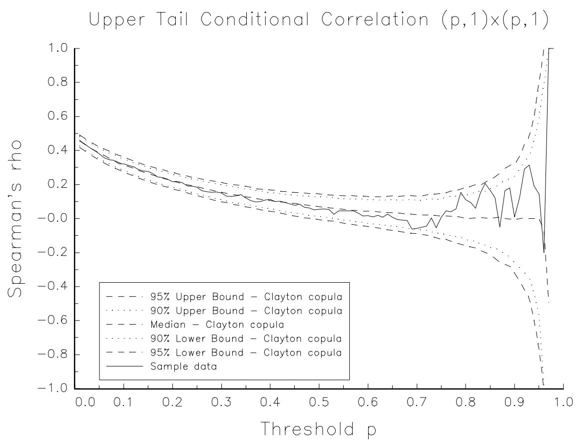
3.2 Examples

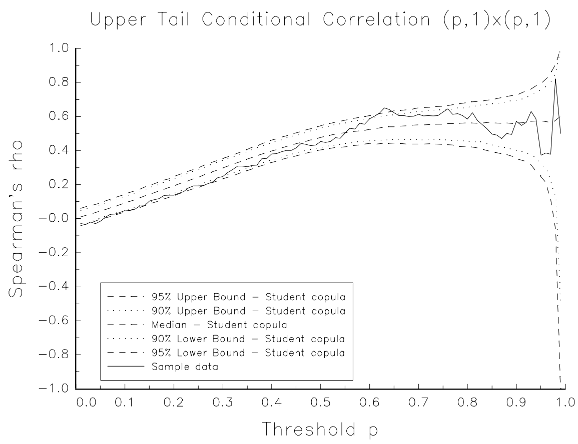
The graphs below show the evolution of the upper conditional rank correlation, for some copulas, for some simulated data, for samples with 1500 observations. Dotted lines are the confidence intervals[8] (95% *and* 90%) for 1500 observations. The two first graphs are the cases of Gaussian and a Gumbel copulas, with an overall rank correlation around 0.5.





The two graphs below are the cases of Clayton and Student copulas, with an overall rank correlation around 0.5.





One can notice that the case of the Student copula is all the more interesting that the overall rank correlation is 0 : fitting a one-parameter copula among usual Archimedian copula should lead to the independent copula.

[8] These confidence bounds have been obtained using Monte Carlo simulations, with 10,000 simulations of 1,500-samples.

And assuming that these data are independent would lead to substantial problems : in the upper-tail, the rank correlation is around 0.5, saying that data are not independent in the tails. These graphs could be summarized in the following table, giving the evolution of the conditional correlation,

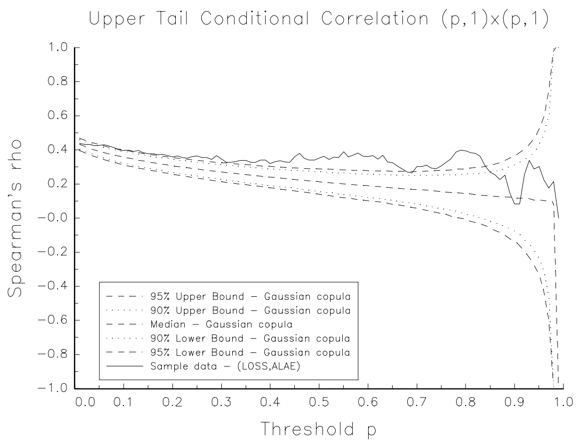
	Gaussian	Gumbel	Survival Gumbel	Clayton	Survival Clayton	Frank
$\overline{\rho}(C,0)=\rho_S$	0.60	0.60	0.60	0.60	0.60	0.60
$\overline{\rho}(C,50\%)$	0.33	0.51	0.26	0.12	0.60	0.27
$\overline{\rho}(C,75\%)$	0.25	0.51	0.17	0.00	0.60	0.11
$\overline{\rho}(C,90\%)$	0.20	0.51	0.12	0.00	0.60	0.03
$\overline{\rho}(C,95\%)$	0.17	0.51	0.10	0.00	0.60	0.01

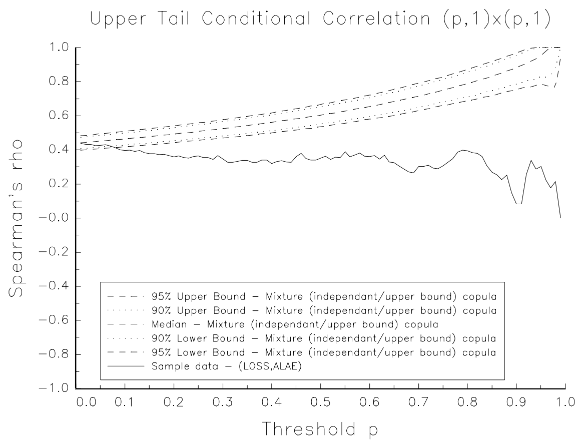
In *Tails of copulas*, Gary Venter said that '*correlation is stronger for large events*'. But we can notice that it is not exactly the case : apart from the case of Student copula, and the survival Clayton copula (*which the only invariant copula, as mentioned in the previous part*), the more we look into details in the upper tail of the distribution ('*large events*'), the smaller the dependence (*theoretical results on that topic were developed in Part* (2.6.4)).

3.3 Applications

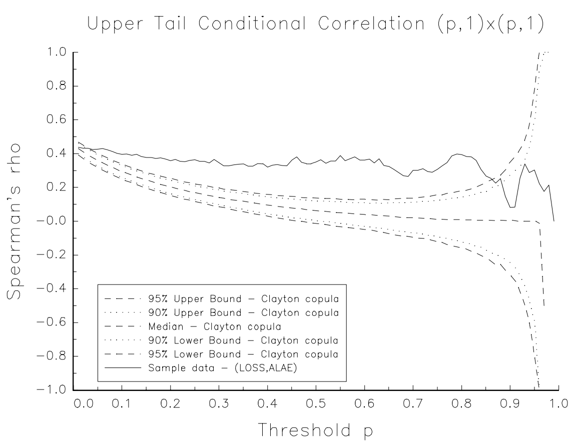
3.3.1 Loss Adjustment Expense (Frees and Valdez (1997))

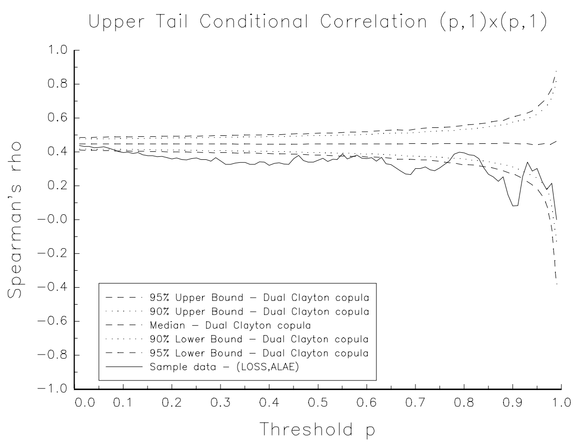
If this rank correlation could be used to get a better understanding of the dependence in the tails of the distribution, it is also possible to use it while fitting copulas.The two first graphs are the cases of Gaussian and a mixture of copulas ($C_\theta(x,y)=\theta C^{+}(x,y)+(1-\theta)C^{\perp}(x,y)$, *as in Charpentier* (2003)), with an overall rank correlation around 0.5.



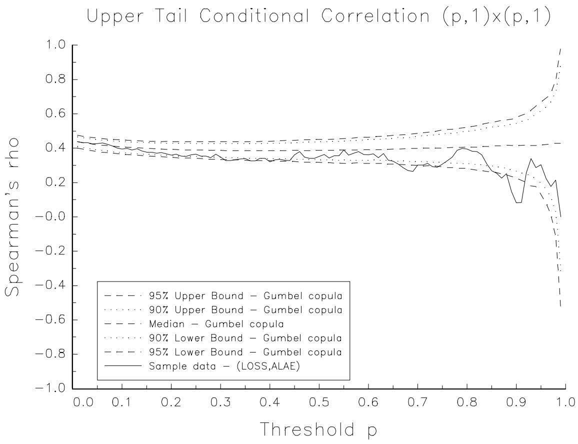


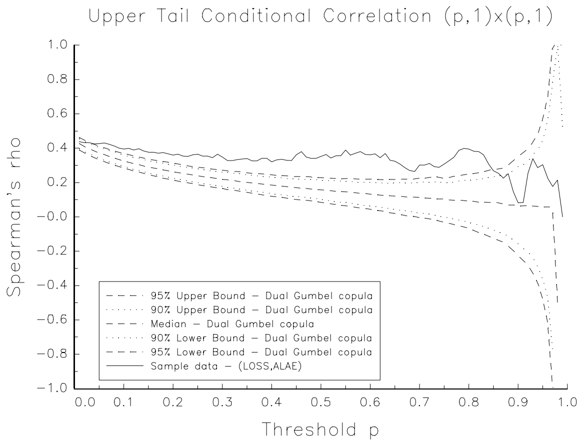
The two graphs below are the cases of Clayton and the dual of Clayton copulas, with an overall rank correlation around 0.5.



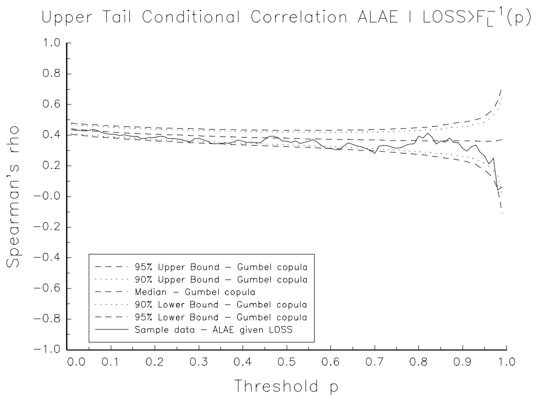


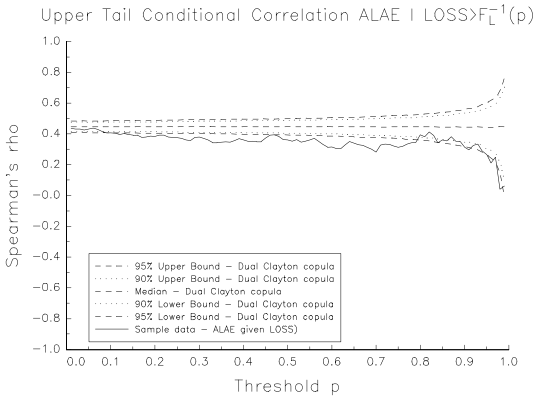
The two graphs below are the cases of Gumbel and the dual of Gumbel copulas, with an overall rank correlation around 0.5





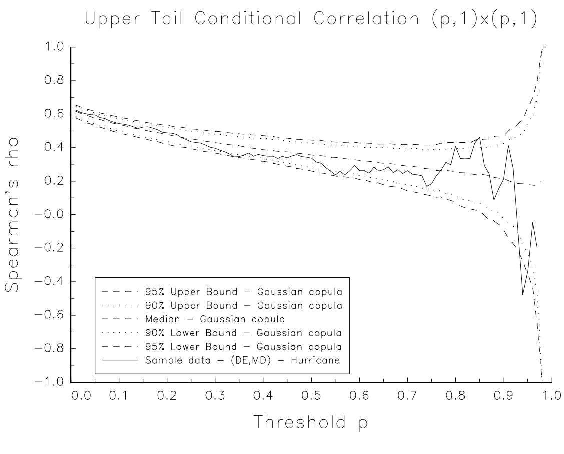
Furthermore, it could be more interesting to study the dependent between the losses and the expenses, given the fact that the loss is important, that is studying the conditional copula of $(LOSS, ALAE)$ given $LOSS > F_L^{-1}(u)$. One can notice that, if the problem is knowing the expenses given the loss amount, then Gumbel copula is the best copula.

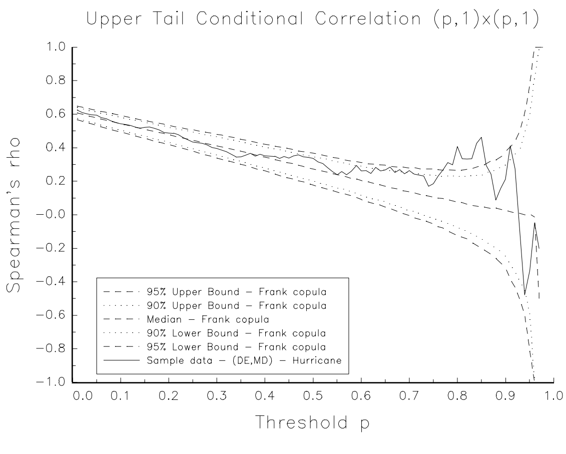




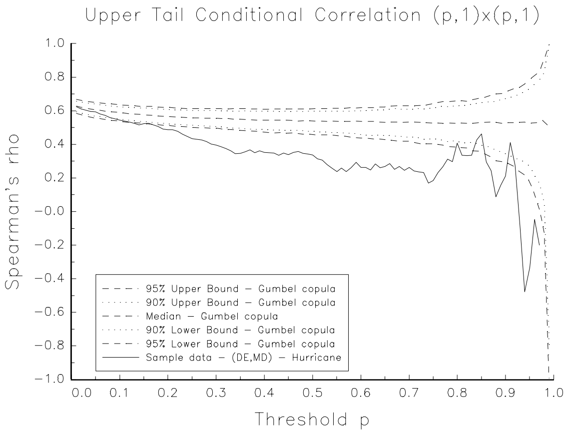
3.3.2 Simulated Hurricane Losses (Venter (2001))

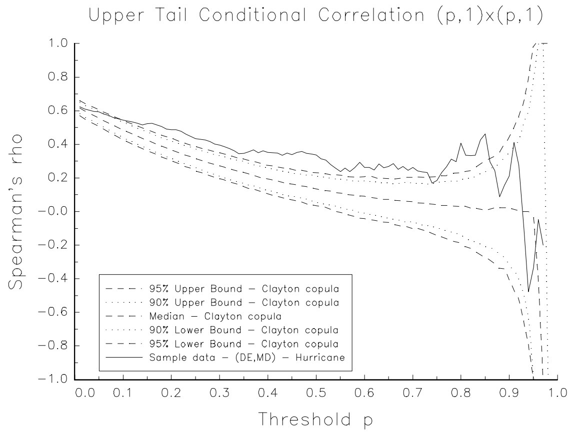
This sample data set, used in Tail of Copulas, is a simulation of 727 losses from a hurricane loss generator, with Maryland and Delaware exposures. Among several copula models (*Gumbel, Frank, Gaussian, and Survival Gumbel*), Gary Venter said that Frank and Gaussian copulas where the best : they had the highest AIC (*or log-likelihood, all the copulas having a single parameter*), and, moreover, the empirical J function (*the cumulative tau*) is close to the ones obtained with Frank or Gaussian copulas. The graphs below show the evolution of the upper conditional rank correlation, compared with the case of Gaussian copulas (*on the left*), and Frank copula (*on the right*),





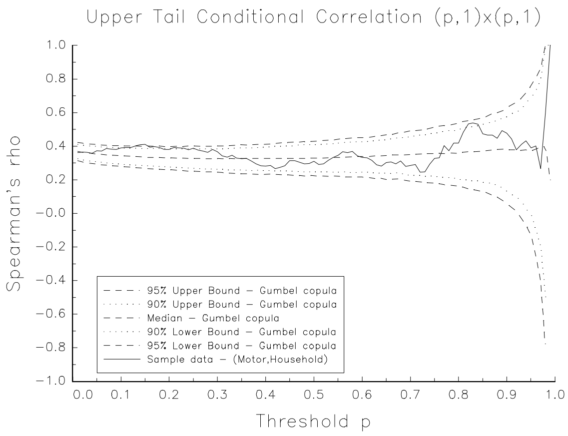
These graphs confirm that those parametric copula provide the best fit. The following graphs represent the case of Gumbel copula (*which overestimates the upper tail*) and Clayton copula (*which underestimates the upper tail*)

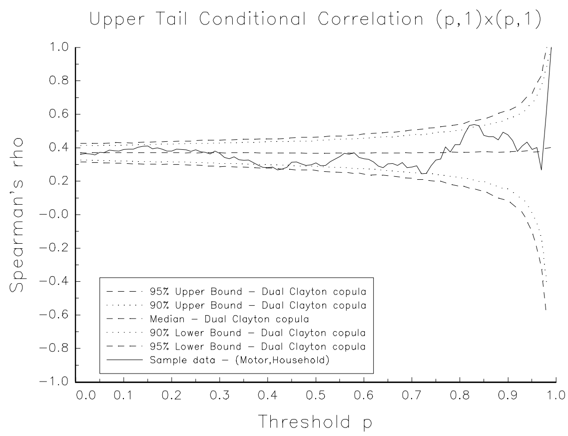




3.3.3 Motor and Household claims (Belguise (2001))

Following the methodology developed by Gary Venter, Belguise fitted a copula using the cumulative tau. The two copulas which provide the best fit are Gumbel copula, and the survival Clayton copula. The graph below compare the empirical upper rank correlation with the cases of Gumbel and the dual Clayton copula,





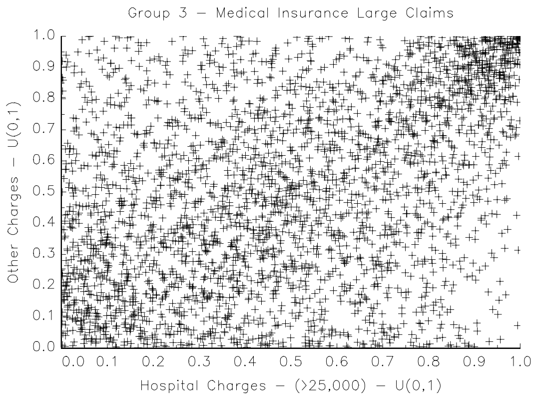
These graphs confirm that those two copulas provide a good fit.

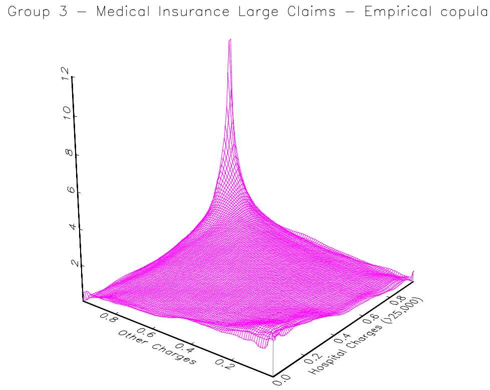
3.3.4 Group Medical Insurance : Hospital vs. other expenses

The dataset used here, from the medical insurance large claims database (*available from the website of the Society of Actuaries*) includes over 3 million claims, over the years 1991 and 1992. In this par, only records relating to Plan type 3 are used (*Cebrian, Denuit and Lambert* (2003) *have focused on Plan type 4 records*). This dataset contains only claims exceeding \$25,000, splitted between the hospital charge ($HOSP$) and the other expenses ($OTHER$). For some reinsurance issue, it might be interesting to study the dependence between these two amounts, and more precisely, to study the dependence given the hospital charge. For example, as in Cebrian, Denuit and Lambert, given the expenses and the hospital charge, the reinsurance indemnity might be defined as

$$g(HOSP, OTHER) = \begin{cases} 0 & \text{if } HOSP \leq R \\ HOSP - R + \frac{HOSP-R}{HOSP}.OTHER & \text{if } HOSP > R \end{cases}.$$

In that case, it might be interesting to get a good understanding of the dependence of $(HOSP, OTHER)$ given $HOSP > x$. The graphs below give an illustration of the relationship between these two variables, with the scatterplot of the copula-type transformation of these variables, i.e. (U_i, V_i) where $U_i = \widehat{F}_H(HOSP_i)$ and $V_i = \widehat{F}_O(OTHER_i)$ (*given* $HOSP$ *exceeds* \$25,000), on the left, and, on the right, the empirical density of the copula.





The graph below show the evolution of the conditional rank correlation of $(HOSP, OTHER)$ given $HOSP > F_H^{-1}(p)$ where $F_H^{-1}(p)$ denotes the quantile of order p, for Plan type 3 claims, with claims over the years 1991 and 1992, with confidence bonds under the assumption of a Gumbel copula.

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