Aggregation and diversification effect of dependent random variables

Wüthrich, Mario V.

Winterthur Insurance

Römerstrasse 17

P.O. Box 357

CH-8401 Winterthur, Switzerland

Phone +41 52 261 86 27

E-mail address: mario.wuethrich@winterthur.ch

Abstract

We choose two identically distributed dependent risks X_1 and X_2 with dependence structure modelled by an Archimedean copula. Then we are able to analyze diversification effects in the tails of aggregate dependent risks, i.e. for large u we study $P[X_1 + X_2 \geq u] \approx c \cdot P[X_1 \geq u/2]$, where c describes the diversification effect.

Keywords: Archimedean copula, dependent risks, Value-at-Risk, extreme value theory, diversification effect, multivariate extremes.

1 Introduction and Motivation

The aim of the present work is to give an overview on our results obtained in [11, 16, 1]. These papers study tail dependence from a distributional point of view.

Currently, there is a wide range of discussions going on about solvency requirements for insurance companies and financial institutes. In many countries one is on the way to define (or has already) risk-adjusted solvency requirements. Some of them use a distributional approach, i.e. they define probabilistic models (like dynamic financial analysis models) and determine quantiles; some of them are doing stress testing using different scenarios; and others try to determine risk-adjusted factors, which are applied to certain volume measures (like premium, claims incurred or claims reserves).

In all these different approaches one faces similar difficulties and questions: How should one choose the stochastic model? How should we determine risk-adjusted factors? How should one aggregate random variables? What kind of dependence structure/measure should one use? Etc. In the present article we give asymptotic results for the aggregation of two dependent random variables. We also provide some feeling for the understanding of the diversification effects: We obtain results of the following type

$$P[X_1 + X_2 \geq u] \sim c \cdot P[X_1 \geq u/2], \qquad \text{as } u \to \infty, \tag{1.1}$$

where we assume that X_1 and X_2 have the same marginals and c is a constant measuring the diversification effect (of course it depends on the dependence strength). This expression enables to give e.g. asymptotic Value-at-Risk estimates. It also helps to find risk-adjusted proportionality factors taking into account the dependence strength.

In recent research there is a wide range of literature on this kind of problem. It is understood that correlation is not sufficient to capture the full range of possible outcomes of extremal events (see e.g. Embrechts-McNeil-Straumann [8]). Many authors have tried to find upper und lower bounds for expressions like formula (1.1) (see e.g. Dhaene-Denuit [6], Denuit-Genest-Marceau [5], Bäuerle-Müller [2], Cossette-Denuit-Marceau [4], of course this list is not complete).

We choose a different approach: instead of finding bounds, we rather analyze the asymptotic behaviour. We find certain universalities (weak convergence theorems) which enable us to analyze different classes of models. The dependence structure is described using the copula framework.

Copulas were originally introduced in 1959 in the context of probabilistic metric spaces (see Sklar [15] and Schweizer-Sklar [14]). During the past years they have developped rapidly and they have attracted much interest. Copulas are used to describe scale invariant dependencies between random variables. An overview of recent developments and applications can be found in Joe [10], Nelsen [13], Frees-Valdez [9], Embrechts-McNeil-Straumann [8] and the references therein.

In the present paper we restrict ourselves to the bivariate case (for results in the multivariate case we refer to [16, 1]). In the bivariate case we are able to calculate most of the expressions explicitely.

Organization of this paper: In the next section we define bivariate copulas. In Section 3 we define our bivariate model and our main result is presented in Subsection 3.3. In Section 4 we give an example how our results can be applied for calculating risk-adjusted solvency requirements and how this is applied to premium calculations. Finally, in Section 5 we prove the results (which gives to link to [1]).

2 Bivariate Copulas

The idea behind the concept of copulas is to separate a multivariate distribution function into two parts, one describing the dependence structure and the other one describing marginal behaviours, respectively.

Definition 2.1 (Copula) A 2-dimensional copula is a 2-dimensional distribution function restricted to $[0;1]^2$ with uniform-$(0;1)$ marginals.

Theorem 2.2 (Sklar [15], [14], [13]) For a given joint distribution function F with continuous marginals $F_1; F_2$ there exists a unique copula C satisfying

$$F(x_1; x_2) = C(F_1(x_1); F_2(x_2)): \tag{2.1}$$

Conversely, for a given copula C and marginals $F_1; F_2$ we have that (2.1) defines a distribution with marginals F_i.

Sklar's theorem is a motivation for calling a copula a dependence structure. In fact, (2.1) means that C couples the marginals F_i to the joint distribution function F. One important property of copulas is that copula of a random vector $(X_1; X_2)$ is invariant under strictly increasing transformations.

There are several special copulas, e.g. the so called comonotonic copula $C_U(x_1; x_2) = \min\{x_1; x_2\}$, which corresponds to total positive dependence or the independent copula which is the copula of independent random variables: $C_I(x_1; x_2) = x_1 \cdot x_2$ (for more background information we refer to [8, 10, 13]). In this article we focus on Archimedean copulas.

Definition 2.3 (Strict Archimedean copula) Let $\psi : [0,1] \to [0, \infty]$ be strictly decreasing, convex and such that $\psi(0) = \infty$ and $\psi(1) = 0$. Define for $x_i \in [0,1]$, $i = 1,2$,

$$C^{\psi}(x_1, x_2) = \psi^{-1}(\psi(x_1) + \psi(x_2)) . \tag{2.2}$$

The function ψ is called generator of the copula C^{ψ}.

Archimedean copulas are interesting in practice because they are easy to construct, but still we obtain a rich family of dependence structures. Usually they have only one parameter which is a great advantage when one needs to estimate parameters from data.
Examples of Archimedean copulas:

- The Clayton copula with $\alpha > 0$ is generated by $\psi(t) = t^{-\alpha} - 1$ and takes the form

$$C^{Cl,\alpha}(x_1, x_2) = (x_1^{-\alpha} + x_2^{-\alpha} - 1)^{-1/\alpha}. \tag{2.3}$$

- Another example is the Gumbel copula

$$C^{Gu,\alpha}(x_1, x_2) = \exp\left\{ -[(-\log x_1)^{\alpha} + (-\log x_2)^{\alpha}]^{1/\alpha} \right\} , \tag{2.4}$$

  whose generator is $\psi(t) = (-\log(t))^{\alpha}$ with $\alpha \geq 1$.

3 Main results

In the first subsection we define our model. It is a bivariate stochastic model with two exchangeable random variables. The dependence structure of their survival function is given by an Archimedean copula. In Subsection 2 we recall the results about marginal behaviours (see also Embrechts-Klüppelberg-Mikosch [7], Chapter 3), and finally in the remaining subsections we provide our main results.

3.1 Dependence structure

Assumption 3.1 Assume that the random vector (X_1, X_2) satisfies:

1) Both coordinates X_i have the same continuous marginal $F(x) = P[X_i \leq x]$.
2) (X_1, X_2) has copula $C(u, v) = u + v - 1 + C^{\phi}(1 - u, 1 - v)$, where C^{ϕ} is a strict Archimedean copula with generator ϕ.

Lemma 3.2 The function $C(u, v)$ given in Assumption 3.1 is a copula.

Remarks:

- At the first sight, the copula definition in Assumption 3.1 looks rather complicated. In fact it describes the copula which is related to the survival function (see also Juri-Wüthrich [12], Definition 1.2). The reason for this transformation is that the calculations are much easier for extremes below low thresholds than for excesses above high thresholds. This comes from the fact that $(X, Y) \mapsto (-X, -Y)$ is not a "symmetric" transformation, from a distributional point of view. Unfortunately the formulas get much more complicated for excesses above high thresholds (see [12]).

- From a practical point of view, this definition (going over the survival copula $C(u, v) = u + v - 1 + C^{\phi}(1 - u, 1 - v)$) is not a problem. Our results capture the whole range of possible asymptotic behaviours.

- Often the marginal behaviours of X_1 and X_2 are not the same (i.e. we have no exchangeability). In that case one sees, that simply one coordinate is dominant (in Theorem 3.4). Of course one can always make the other coordinate "more dangerous", such that both live on the same scale (and we can apply our theory).

One crucial assumption for the asymptotic behaviour of aggregate random variables is the speed at which the generator ϕ goes to infinity at the origin. We therefore define

Definition 3.3 A function f is regularly varying with index $\alpha \in \mathbb{R}$ at x^- (x^+, resp.) if for all $t > 0$

$$\lim_{y \uparrow x} \frac{f(yt)}{f(y)} = t^{\alpha}, \qquad \lim_{y \downarrow x} \frac{f(yt)}{f(y)} = t^{\alpha}, \quad \text{resp.} \tag{3.1}$$

We write $f \in \mathcal{R}_{\alpha}^{x^-}$ ($f \in \mathcal{R}_{\alpha}^{x^+}$, resp.). For $\alpha = 0$ we say f is slowly varying; for $\alpha = -\infty$ rapidly varying. A standard reference on regular variation is Bingham-Goldie-Teugels [3].

Remark: The generator of the Clayton copula belongs to $R^{0+}_{-\alpha}$, whereas the Gumbel copula belongs to R^{0+}_{0}. This highlights one important difference between the Clayton and the Gumbel copula. In fact, the index $-\alpha$ of regular variation at 0 is a coefficient measuring the dependence strength in the tails. It also determines the so-called tail-dependence coefficient $\lambda = 2^{-1/\alpha}$ (see Embrechts-McNeil-Straumann [8], and Juri-Wüthrich [11], Theorem 3.9). Independent random variables have $\lambda = 0$, so we say that $\lambda > 0$ corresponds to positive tail dependence. (Be careful: $\lambda = 0$ does not necessarily mean that in the tails the random variables are independent (see Fallacy 5.1 in [12]), but for $\lambda = 0$ we have a rather weak tail dependence which can not be seen on that scale.)

3.2 Marginal behaviours

For marginals there are exactly three different types of non-degenerated extreme value behaviours: In the Fisher-Tippett Theorem and the Pickands-Balkema-de Haan Theorem (see e.g. Theorems 3.2.2 and 3.4.5 in [7]) one sees that the one-dimensional distributions (marginals) can be classified into three groups for analyzing extremal events. The crucial conditition is the rate of decay at ∞, i.e. the fatness of the tails: We define for $X \sim F$

$$\overline{F}(x) = 1 - F(x) = P[X > x]. \tag{3.2}$$

Let x_F denote the right-endpoint of the distribution F. We obtain the following three classes of marginal behaviours (excesses over high thresholds, see Embrechts-Klüppelberg-Mikosch [7] Theorems 3.4.5, 3.3.7, 3.3.12 and 3.3.27):

1. Fréchet case: F belongs to the Fréchet case iff for some $\beta > 0$: $\overline{F}(\cdot) \in R^{\infty}_{-\beta}$. Examples are Pareto, Cauchy, Loggamma distribution.

2. Weibull case: F belongs to the Weibull case iff for some $\beta > 0$: $x_F < \infty$ and $\overline{F}(x_F - 1/\cdot) \in R^{\infty}_{-\beta}$. Examples are Uniform, Beta distribution.

3. Gumbel case: F belongs to the Gumbel case iff $x_F \leq \infty$ and there exists a positive measurable function $a(\cdot)$ such that for $t \in \mathbb{R}$

$$\lim_{u \uparrow x_F} \frac{\overline{F}(u + ta(u))}{\overline{F}(u)} = e^{-t}. \tag{3.3}$$

   Examples are Gamma, Normal, Lognormal distribution.

3.3 Main theorem

Theorem 3.4 Choose $\alpha, \beta > 0$. There exist constants $q^F(\alpha;\beta)$, $q^W(\alpha;\beta)$, $q^G(\alpha) \in (0;\infty)$ such that for all $(X_1; X_2)$ satisfying Assumption 3.1 with $\psi \in R^{0+}_{-\alpha}$ and

a) Fréchet case: marginals F with $\overline{F}(\cdot) \in R_{-1/\beta}$:

$$\lim_{u\to\infty} \frac{1}{\overline{F}(u)} P[X_1 + X_2 \geq 2u] = 2^{-\beta} \cdot q^F(\alpha;\beta): \tag{3.4}$$

b) Weibull case: marginals F with $x_F < \infty$ and $\overline{F}(x_F - 1/\cdot) \in R_{-1/\beta}$:

$$\lim_{u\to\infty} \frac{1}{\overline{F}(x_F - 1/u)} P[X_1 + X_2 \geq 2x_F - 1/u] = q^W(\alpha;\beta): \tag{3.5}$$

c) Gumbel case: marginals F satisfying (3.3) for some $a(\cdot)$, we have

$$\lim_{u\uparrow x_F} \frac{1}{\overline{F}(u)} P[X_1 + X_2 \geq 2u] = q^G(\alpha): \tag{3.6}$$

The main idea in the proof is to show a weak convergence result. It tells that if we know that one coordinate is large, then with high probability also the other coordinate is large. This comes from the fact that we have positive tail dependence which favors joint extremes.

3.4 Limiting constants

In this subsection we identify the limiting constants $q^F(\alpha;\beta)$, $q^W(\alpha;\beta)$ and $q^G(\alpha)$. We define for $\alpha > 0$ and $y \geq 0$

$$f_\alpha(y) = (1 + y^\alpha)^{-1/\alpha - 1}: \tag{3.7}$$

Theorem 3.5 $f_\alpha(y)$ is a probability density on $[0;\infty)$. And for $Y_\alpha \sim f_\alpha$ we have

1. Fréchet case:
$$q^F(\alpha;\beta) = 1 + E\left[\left(1 + Y_\alpha^{-1/\beta}\right)^{\beta - 1}\right]: \tag{3.8}$$

2. Weibull case:
$$q^W(\alpha;\beta) = E\left[\left(1 + Y_\alpha^{1/\beta}\right)^{-\beta - 1}\right]: \tag{3.9}$$

3. Gumbel case:
$$q^G(\alpha) = \frac{1}{2} E\left[Y_\alpha^{-1/2}\right] = \frac{\Gamma^2(1 + 1/(2\alpha))}{\Gamma(1 + 1/\alpha)} = q^F(\alpha;2)/2 - 1: \tag{3.10}$$

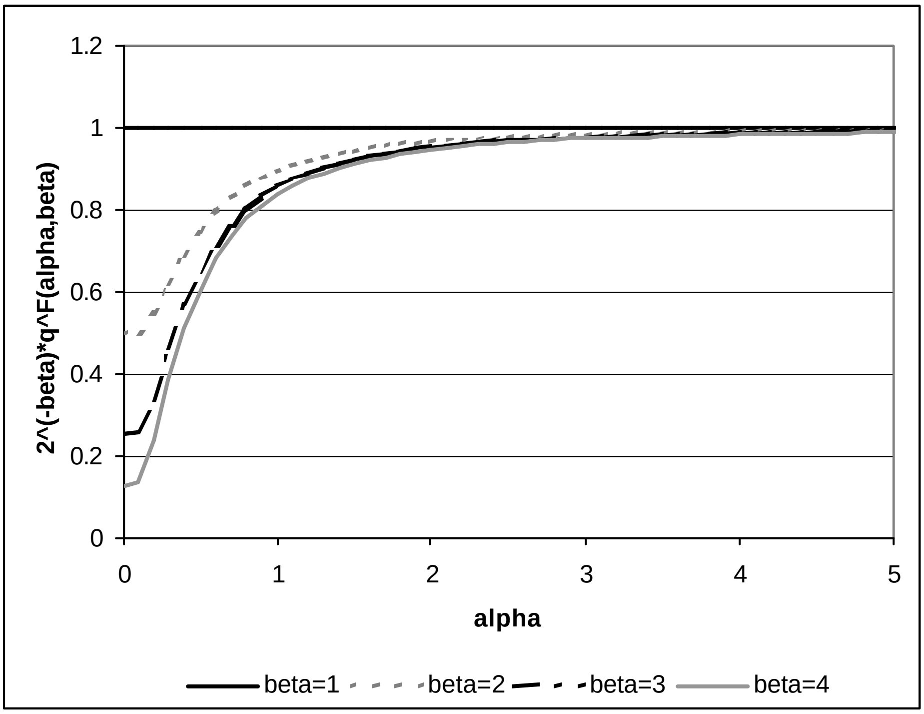

Figure 1: Fréchet case: limiting constant $2^{-\beta} \cdot q^F(\alpha;\beta)$ as a function of α.

We see that in two dimensions the limiting constants have an easy form. They can be explicitely calculate in many cases, and in the other cases we can at least determine them numerically, e.g. Fréchet case with $\beta \in \mathbb{N} \setminus \{0\}$ we have

$$q^F(\alpha;\beta) = \sum_{k=0}^{\beta} \binom{\beta}{k} \frac{\Gamma\left(\frac{\beta-k}{\alpha}+1\right)\Gamma\left(\frac{k}{\alpha}+1\right)}{\Gamma(1+1/\alpha)}. \tag{3.11}$$

Proposition 3.6 (Diversification effect)

1. Fréchet case: The constant $q^F(\alpha;\beta)$ is strictly increasing in β.

   For $\beta > 1$, the constant $q^F(\alpha;\beta)$ is strictly increasing in α.

   For $\beta = 1$, we have $q^F(\alpha;1) = 2$ for all $\alpha > 0$.

   For $\beta < 1$, we have $q^F(\alpha;\beta)$ is strictly decreasing in α.

   Moreover, $\lim_{\alpha\to 0} 2^{-\beta} \cdot q^F(\alpha;\beta) = 2^{-\beta}$ and $\lim_{\alpha\to\infty} 2^{-\beta} \cdot q^F(\alpha;\beta) = 1$.

2. Weibull case: The constant $q^W(\alpha;\beta) \le 1$ is strictly decreasing in β and strictly increasing in α.

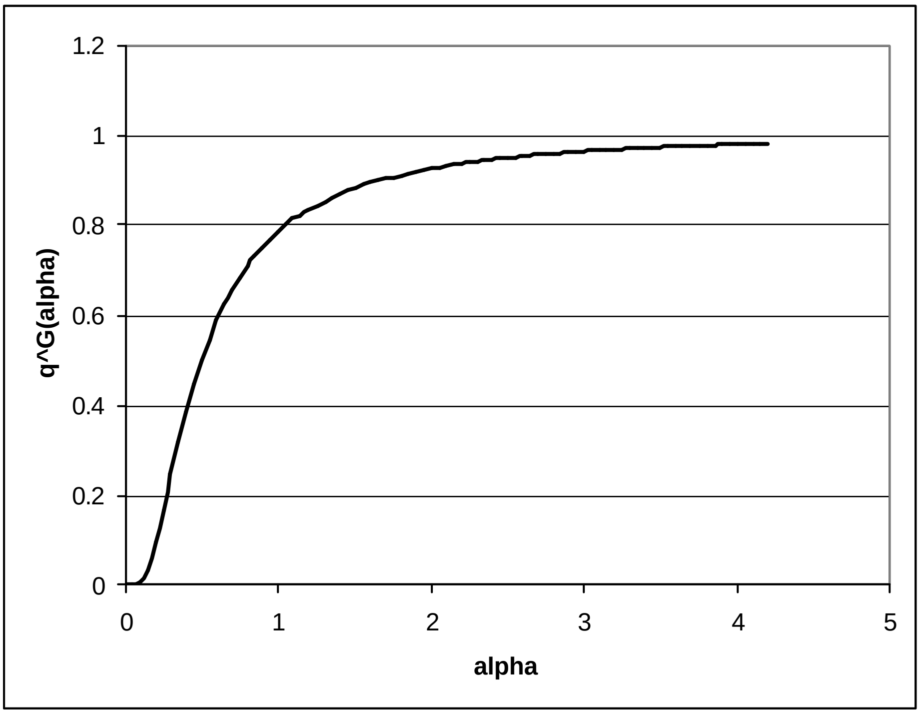

Figure 2: Gumbel case: limiting constant $q^G(\alpha)$ as a funtion of α.

Moreover $\lim_{\alpha \to 0} q^W(\alpha;\beta) = 0$ and $\lim_{\alpha \to \infty} q^W(\alpha;\beta) = 2^{-\beta}$.

3. Gumbel case: The constant $q^G(\alpha) \le 1$ is strictly increasing in α with $\lim_{\alpha \to 0} q^G(\alpha) = 0$ and $\lim_{\alpha \to \infty} q^G(\alpha) = 1$.

Remark, diversification effect: In the Weibull case, in the Gumbel case and in the Fréchet case for $\beta > 1$ we have that the limiting constants are increasing in α. This means: the stronger the dependence strength α the smaller the diversification effect.
In the Fréchet case this does not hold true for $\beta \le 1$. At the first sight this is surprising, since we would not expect a "negative" diversification. The Fréchet case undergoes a phase transition at $\beta = 1$. This comes from the fact that the expectation of the Fréchet distribution does not exists for $\beta \le 1$, i.e. in practice we would not take such risks into our insurance portfolio, since there is no finite pure risk premium for such risks. Our proposition says that we should rather have only one such risk instead of two because there is no diversification.

4 Example: Premium calculation with risk-adjusted loadings in the Gumbel case

4.1 Premium calculation

For the calculation of the premium we have the following elements:

$$\begin{aligned} \text{premium} &= \text{pure risk premium} + \text{risk loading} + \text{other loadings} \\ &\quad - \text{interest} + \text{admininstration expenses} + \text{state contributions.} \end{aligned}$$

Other loadings may e.g. contain the loading for operational risks. In this work we concentrate on the pure risk premium and on the risk loading. We introduce the following notation: The total claim incurred is denoted by the random variable X, hence the pure risk premium is the mean of X, $E[X]$. As risk measure we choose the Value-at-Risk (which fits into our theory of determining quantiles): Choose $p \in (0; 1)$

$$\mathrm{VaR}_X(p) = \inf\{x;\ P[X \le x] \ge p\} - E[X]; \tag{4.1}$$

i.e. if we hold the risk capital $\mathrm{VaR}_X(p)$, we know that we won't have ruin with probability p; therefore the required risk capital is $\mathrm{VaR}_X(p)$ (typically for p close to 1). We need to pick up this capital from investors to run our business. Since it is exposed to risk (they may loose it if the business runs badly), they expect a return r on that capital which is larger than the risk free rate r_0. At the same time the insurance company invests the risk capital at the risk free rate r_0.

Hence the premium paid by the policyholder is

$$\begin{aligned} pr(X) &\stackrel{\mathrm{def}}{=} \text{pure risk premium} + \text{risk loading} \\ &= E[X] + (r - r_0) \cdot \mathrm{VaR}_X(p): \end{aligned} \tag{4.2}$$

4.2 Numerical example

The total claim amount incurred for two insurance companies is denoted by X_i ($i = 1; 2$). A model which is used quite commonly is the so-called translated lognormal model, i.e. $\log(X_i - V_i) \sim N(\mu; \sigma)$ for some constant translation V_i. Define $Y_i = X_i - V_i$ which is lognormally distributed. The lognormal distribution belongs to the maximum domain of attraction of the Gumbel distribution. A possible choice of the function $a(\cdot)$ can be found via the mean excess function (see [7], Theorem 3.3.27, formulas (3.34)-(3.36) and Table 3.4.7). We choose now the following numerical example: Quantil $p = 99.5\%$, cost-of-capital rate $r = 15\%$, risk free rate $r_0 = 3\%$,

	company 1	company 2
translation V_i	700	750
μ	6.19	6.19
σ	0.20	0.20
pure risk premium $E[X_i]$	1'000	1'050
variational coefficient	8.3%	8.0%
$\mathrm{VaR}_{X_i}(p)$	316.58	316.58
Premium $pr(X_i)$	1'037.99	1'087.99
Loading in % $E[X_i]$	3.17%	3.04%

Table 4.1 Premium pr for portfolios X_1 and X_2.

Of course the effective risk-capital held by the insurance companies is much larger because these considerations do not take into account other risks, like investment risk, the risk coming from the runoff of old accident years, credit risk, operational risk, etc.

Example 4.2 (Joint portfolio of X_1 and X_2)

Now we merge the two portfolios X_1 and X_2 to one portfolio $X = X_1 + X_2$, e.g. company 1 buys company 2. Furthermore we assume that the two portfolios are not independent: (Y_1, Y_2) satisfies Assumption 3.1 with a generator $\psi \in R^{0+}_{-\alpha}$. All we need to specify explicitely is the dependence strength $\alpha > 0$, which corresponds to the tail dependence coefficient $\lambda = 2^{-1/\alpha}$ (see Embrechts-McNeil-Straumann [8], and Juri-Wüthrich [11], Theorem 3.9). Hence from Theorem 3.4 we obtain for $u \to \infty$

$$P[Y_1 + Y_2 \geq 2u] \sim q^G(\alpha)\overline{F}(u); \qquad (4.3)$$

i.e. $q^G(\alpha) < 1$ can be viewed as the diversification effect (see Proposition 3.6). If we consider the two portfolios separately (which is equivalent to comonotonic random variables) we get $q^G(\infty) = 1$.

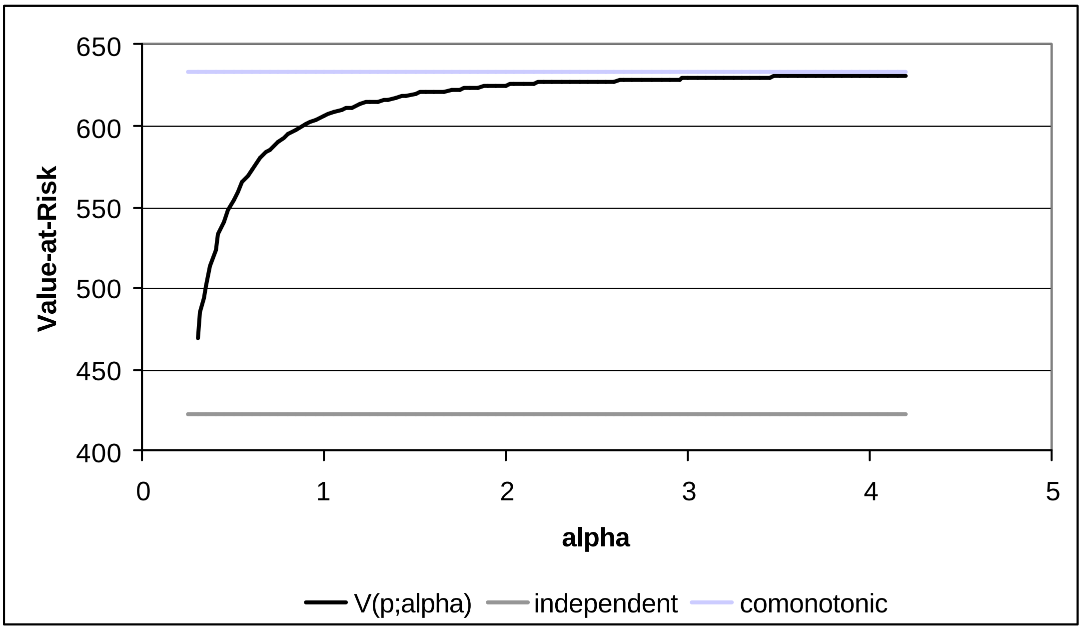

Figure 3: Value-at-Risk of $X_1 + X_2$ as a function of α.

Now we consider the quantiles for small default probabilities $q = 1 - p \in (0, 1)$

$$q = 1 - p = q^G(\alpha)\overline{F}(u). \tag{4.4}$$

The above equality implies

$$u = \overline{F}^{-1}\left(1 - \frac{1-p}{q^G(\alpha)}\right) = \exp\left\{\sigma\Phi^{-1}\left(1 - \frac{1-p}{q^G(\alpha)}\right) + \mu\right\}. \tag{4.5}$$

This implies that asymptotically ($q \to 0$) the Value-at-Risk of $X_1 + X_2$ is given by

$$VaR_{X_1+X_2}(p;\alpha) \approx V(p;\alpha) \stackrel{\text{def.}}{=} 2\left(\exp\left\{\mu + \sigma\Phi^{-1}\left(1 - \frac{1-p}{q^G(\alpha)}\right)\right\} - \exp\left\{\mu + \sigma^2/2\right\}\right). \tag{4.6}$$

The pure risk premium is $E[X_1 + X_2] = 2'450$. Define diversification effect as $\mathrm{def}(\alpha) = 1 - V(p;\alpha)/V(p;\infty)$. In the following table $\alpha = 0$ means independence, $\alpha = \infty$ means

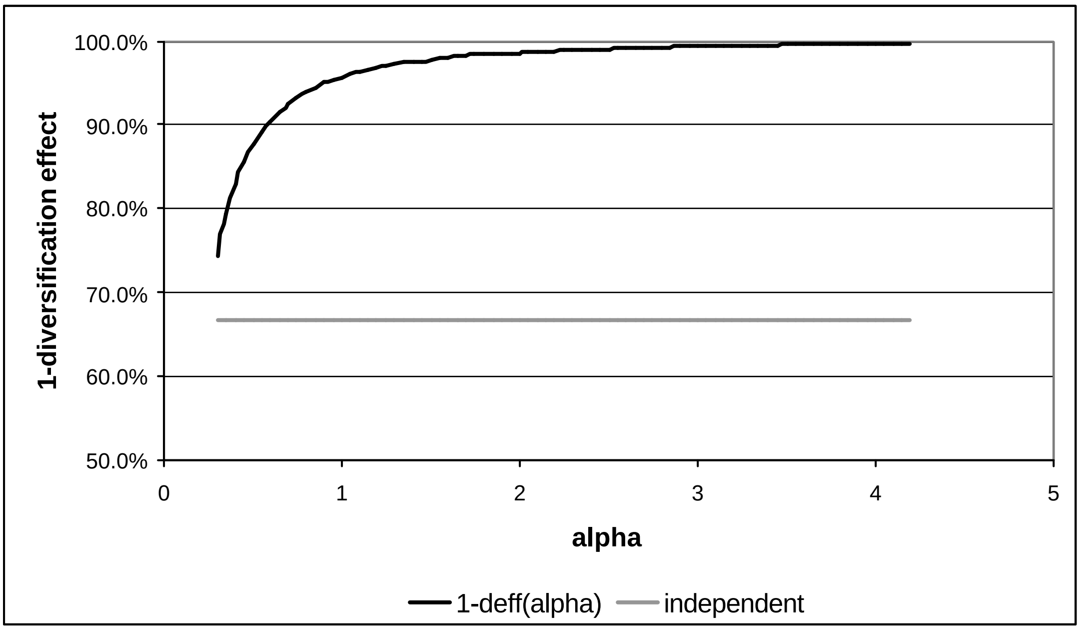

Figure 4: Diversi...cation: V (p; ®)=V (p; 1) = 1-de¤(®) as a function of ®.

comonotonicity, for p = 99:5% we obtain

®	0	0:5	1:0	1:5	2:0	3:0	4:0	1
¸	0:0%	25:0%	50:0%	63:0%	70:7%	79:4%	84:1%	100%
V (p; ®)	422:7	554:4	606:0	619:3	624:7	629:0	630:7	633:2
1 ¡ de¤(®)	66:8%	87:6%	95:7%	97:8%	98:7%	99:3%	99:6%	100%
Loading	50:72	66:53	72:72	74:31	74:96	75:48	75:69	75:98
pr	$2^0$500:7	$2^0$516:5	$2^0$522:7	$2^0$524:3	$2^0$525:0	$2^0$525:5	$2^0$525:7	$2^0$526:0
Load. in %	2:07%	2:72%	2:97%	3:03%	3:06%	3:08%	3:09%	3:10%

Table 4.3 Premium pr for di¤erent values of ®.

Conclusions: In our example we see that the diversi...cation e¤ect decreases rather quickly to zero, i.e. already for small ® we are close to the comonotonic case. E.g. for ® = 0:6 (which corresponds to a tail dependence coe¢cient of 31.5 %, cf. Juri-Wüthrich [11]) the diversi...cation e¤ect is already less than 10%. For Archimedean copulas we have always this kind of asymptotic behaviour (because asymptotically they depend only on the coe¢cient ® of regular variation at the origin). Henceforth we need to choose non-Archimedean copulas to obtain other asymptotic behaviours.

5 Proofs

Proof of Lemma 3.2. First we prove that C is a distribution. Choose $(U, V) \sim C^{\psi}$,

$$P[1 - U \le u, 1 - V \le v] = P[U \ge 1 - u, V \ge 1 - v] = C(u, v); \tag{5.1}$$

i.e. $(1 - U, 1 - V)$ has distribution C (which immediately implies that C is a distribution, i.e. increasing in both arguments and it satisfies the two-increasing property). For the marginals we obtain $C(u, 1) = u + C^{\psi}(1 - u, 0) = u$, since C^{ψ} is a copula, and equivalently $C(1, v) = v$. This finishes the proof.

$\Box$

Proof of Theorems 3.4 and 3.5 and Proposition 3.6. We apply Theorem 3.2 of [1], or Theorem 3.3 of [16], respectively. Therefore we need to transform our random variables since those theorems only apply to lower tails. Define $Y_i = -X_i$, for $i = 1, 2$. Hence all the statements in Theorem 3.4 can be rewritten as follows (we only treat the Fréchet case since the other cases are similar)

$$P[X_1 + X_2 \ge u] = P[Y_1 + Y_2 \le -u]. \tag{5.2}$$

Our theorem follows from Theorem 3.2 in [1] once we have proven that (Y_1, Y_2) satisfies the necessary conditions. Assume Y_i have marginals $\tilde{F}$ then

$$\tilde{F}(y) = P[Y_i \le y] = P[-X_i \le y] = P[X_i \ge -y] = \overline{F}(-y) = 1 - F(-y); \tag{5.3}$$

Hence $\tilde{F}(\cdot) = \overline{F}(-\cdot)$, and $\tilde{F}^{-1}(\cdot) = -F^{-1}(1 - \cdot)$. Therefore the marginal behaviours of $\overline{F}$ for $u \to \infty$ and of $\tilde{F}$ for $u \to -\infty$ are the same. Next we show that (Y_1, Y_2) has copula C^{ψ} with regularly varying index $-\alpha$ at the origin: Using a similar calculation as in the proof of Lemma 3.2 and Sklar's theorem we obtain

$$\begin{aligned} P\left[Y_1 \le \tilde{F}^{-1}(u), Y_2 \le \tilde{F}^{-1}(v)\right] &= P\left[X_1 \ge -\tilde{F}^{-1}(u), X_2 \ge -\tilde{F}^{-1}(v)\right] \\ &= P\left[X_1 \ge F^{-1}(1 - u), X_2 \ge F^{-1}(1 - v)\right] \\ &= C^{\psi}(u, v); \end{aligned} \tag{5.4}$$

hence (Y_1, Y_2) has Archimedean copula with regularly varying generator $\psi \in R^{0+}_{-\alpha}$. Therefore the claim of the theorem follows from Theorem 3.2 in [1] and Theorem 3.3 of [16]. This finishes the proof of Theorem 3.4.

The proofs of Theorem 3.5 and Proposition 3.6 now immediatly follow from Lemma 3.4 and Theorems 3.5, 3.7 and 3.9 in [1].

2

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