On an interaction risk model with delayed claims

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Abstract

Wu and Yuen (2003) proposed the so-called interaction risk model for dependent classes of business. In the case of two classes of business, the interaction comes from the assumption that each main claim in one class induces a by-claim in the other class with a certain probability. In this paper we study the interaction risk model with delayed claims. Specifically the occurrences of induced by-claims may be delayed. Within the framework of compound binomial process, we develop recursive equations for calculating the finite-time survival probabilities and derive an explicit expression for the ultimate survival probability in a special case.

Keywords: By-claim; Main claim; Generating function; Interaction risk model; Recursive formula; Survival probability

1. Introduction

In risk theory much attention has been paid to the problem of dependent classes of business in recent years. In particular the discrete-time interaction risk model of Wu and Yuen (2003) assumes that there are two kinds of individual claims, namely main claim and by-claim, for each class of business and that each main claim in one class may induce a claim in the other class called by-claim. The continuous version of their model can be found in Yuen and Wang (2002). The purpose of this paper is to study the interaction risk model with delayed claims. It is assumed that the settlement of each by-claim may be delayed with a certain probability to a time after the settlement of the associated main claim. For example, in a severe car accident, the claim of the totally damaged car (main claim) can be dealt with immediately while the medical expenses of the injured driver (by-claim) may need a period of time to settle. The problem of delay in claim settlement was also considered in Waters and Papatriandafylou (1985) and Yuen and Guo (2001).

In this paper we use the compound binomial model to describe the surplus process for each class of business. This discrete-time risk model has been studied by various authors, for example, Gerber (1988), Shiu (1989), Willmot (1993), Dickson (1994) and Yuen and Guo (2001). Under the compound binomial assumption, the interaction risk model with delayed by-claims is introduced in Section 2. Section 3 presents recursive equations for the finite-time survival probabilities with numerical examples. In a special case, the ultimate survival probability for the model is derived in Section 4.

2. The model

For simplicity we consider a book of insurance business containing only two dependent classes of business. Within the framework of the compound binomial model, the probability of having a main claim in the *i*th class $(i=1,2)$ in each time period is p_i and hence the probability of no main claim is $q_i = 1 - p_i$. The occurrences of all main claims are independent. It is assumed that each main claim in the first (second) class will induce a by-claim in the second (first) class with probability p_{12} (p_{21}). Each by-claim and its associated main claim may occur simultaneously with probability θ, or the occurrence of the by-claim may be delayed to the next time period with probability $1-\theta$. That is, a main claim in the first (second) class will cause a by-claim in the second (first) class occurring in the same time period with probability $p_{12}\theta$ $(p_{21}\theta)$, or occurring one period after the occurrence of the main claim with probability $p_{12}(1-\theta)$ $(p_{21}(1-\theta))$. Note that $0 \le p_1, p_2, p_{12}, p_{21}, \theta \le 1$.

Suppose that the book collects a unit amount of premium in each time period with initial surplus *u*. Then the surplus process of the book is defined as

$$U_n = u + n - S_n = u + n - S_n^X - S_n^Y \,, \tag{2.1}$$

for $n = 0, 1, 2, \cdots$, where S_n^X (S_n^Y) is the aggregate claims for the first (second) class and S_n is the sum of correlated aggregate claims for the whole book in the first *n* periods. Assume that all claim sizes are independent and that claim sizes in each class are identically

distributed. Denote the claim-size distribution function of Class 1 by F_X with mean μ_X and that of Class 2 by F_Y with mean μ_Y.

Under the assumed model, claims in the first time period can be classified into two groups, one for main claims and the other for their by-claims. On the other hand, claims in any time period after the first can be divided into three groups, one for main claims, one for the by-claims associated with the main claims occurring in the current period, and the remaining one for the by-claims associated with the main claims occurring in the previous period. So we have

$$E[S_1] = (p_1 + p_2 p_{21}\theta)\mu_X + (p_2 + p_1 p_{12}\theta)\mu_Y ,$$

and for $n = 2, 3 \cdots$,

$$\begin{aligned} E[S_n] &= E[S_{n-1}] + (p_1 + p_2 p_{21}\theta)\mu_X + (p_2 + p_1 p_{12}\theta)\mu_Y + p_2 p_{21}(1-\theta)\mu_X + p_1 p_{12}(1-\theta)\mu_Y \\ &= E[S_{n-1}] + (p_1 + p_2 p_{21})\mu_X + (p_2 + p_1 p_{12})\mu_Y \\ &= (p_1 + p_2 p_{21}\theta)\mu_X + (p_2 + p_1 p_{12}\theta)\mu_Y + n[(p_1 + p_2 p_{21})\mu_X + (p_2 + p_1 p_{12})\mu_Y]. \end{aligned}$$

3. Finite-time survival probability

Define $\phi(u,n) = \Pr(U_k \geq 0; 0 \leq k \leq n)$ as the *n*-period finite-time survival probability for the surplus process of U_n of (2.1) and $\psi(u,n) = 1 - \phi(u,n)$ as the *n*-period finite-time ruin probability. Apparently $\phi(u,0) = 1$ and $\psi(u,0) = 0$ for all $u \geq 0$. In practice, an insurer should establish reserves for delayed by-claims. This leads to a concept of ruin for which ruin at time *n* means that the insurer has negative cash at that time or has insufficient cash to set up reserves required at that time. In this paper we only consider the simplest situation in which ruin occurs when the insurer has negative cash.

To study $\phi(u,n)$, we define the following three supplementary surplus processes

$$\begin{aligned} {}_1U_n &= u + n - S_n^X - S_n^Y - X , \\ {}_2U_n &= u + n - S_n^X - S_n^Y - Y , \\ {}_3U_n &= u + n - S_n^X - S_n^Y - X - Y , \end{aligned}$$

for $n = 0, 1, 2, \cdots$, where *X* and *Y* are independent random variables having distributions F_X and F_Y, respectively. Let $\phi_i(u,n)$ be the *n*-period finite-time survival probability for ${}_iU_n$ $(i = 1, 2, 3)$. Assume that F_X and F_Y respectively have discrete probability functions $\{f_i; i = 1, 2, \cdots\}$ and $\{g_j; j = 1, 2, \cdots\}$.

Conditioning on the event occurring in the first time period, one can construct the following equation by the law of total probability

$$
\begin{aligned}
&\phi(u-1,n) \\
&= q_1q_2\phi(u,n-1) + p_1q_2\Bigg\{\sum_{i\le u}[q_{12}\phi + p_{12}(1-\theta)\phi_2]\circ(u-i,n-1)\,f_i \\
&\quad + p_{12}\theta\sum_{i+j\le u}\phi(u-i-j,n-1)\,f_ig_j\Bigg\} + q_1p_2\Bigg\{p_{21}\theta\sum_{i+j\le u}\phi(u-i-j,n-1)\,f_ig_j \\
&\quad + \sum_{j\le u}[q_{21}\phi + p_{21}(1-\theta)\phi_1]\circ(u-j,n-1)g_j\Bigg\} + p_1p_2\Bigg\{\sum_{i+j\le u}[q_{12}q_{21}\phi \\
&\quad + p_{12}p_{21}(1-\theta)\phi_2 + q_{12}p_{21}(1-\theta)\phi_1 + p_{12}p_{21}(1-\theta)^2\phi_3]\circ(u-i-j,n-1)\,f_ig_j \\
&\quad + \sum_{i+j+k\le u}[q_{12}p_{21}\theta\phi + p_{12}p_{21}\theta(1-\theta)\phi_2]\circ(u-i-j-k,n-1)\,f_ig_jf_k \\
&\quad + \sum_{i+j+k\le u}[p_{12}q_{21}\theta\phi + p_{12}p_{21}\theta(1-\theta)\phi_1]\circ(u-i-j-k,n-1)\,f_ig_jg_k \\
&\quad + \sum_{i+j+k+l\le u}p_{12}p_{21}\theta^2\phi(u-i-j-k-l,n-1)\,f_ig_jf_kg_l\Bigg\}, \qquad (3.1)
\end{aligned}
$$

for $n=1,2,\cdots$, where $(A\phi + B\phi_i)\circ(\cdot,n-1)$ means $A\phi(\cdot,n-1)+B\phi_i(\cdot,n-1)$ for any coefficients *A* and *B*. The first term on the right-hand side of (3.1) is for the event that there is no claim in the first period and hence the surplus process is renewed at the beginning of the second time period. The second term takes care of the event that a main claim in Class 1 but no main claim in Class 2 occur in the first period. Note that there are three possible cases in this event: no induced by-claim in Class 2 with probability q_{12}; an induced by-claim in Class 2 occurring in the next time period with probability $p_{12}(1-\theta)$; and an induced by-claim in Class 2 occurring in the first period with probability $p_{12}\theta$. Along the same line, the third term handles the event that a main claim in Class 2 but no main claim in Class 1 occurs while the fourth term deals with the event that a main claim occurs in each of the two classes.

Parallel to (3.1) we obtain the following three equations

$$
\begin{aligned}
&\phi_1(u-1,n) \\
&= q_1q_2\sum_{m_1\le u}\phi(u-m_1,n-1)\,f_{m_1} + p_1q_2\Bigg\{\sum_{i+m_1\le u}[q_{12}\phi + p_{12}(1-\theta)\phi_2] \\
&\quad \circ(u-i-m_1,n-1)\,f_if_{m_1} + p_{12}\theta\sum_{i+j+m_1\le u}\phi(u-i-j-m_1,n-1)\,f_ig_jf_{m_1}\Bigg\} \\
&\quad + q_1p_2\Bigg\{\sum_{j+m_1\le u}[q_{21}\phi + p_{21}(1-\theta)\phi_1]\circ(u-j-m_1,n-1)g_jf_{m_1} \\
&\quad + p_{21}\theta\sum_{i+j+m_1\le u}\phi(u-i-j-m_1,n-1)\,f_ig_jf_{m_1}\Bigg\} + p_1p_2\Bigg\{\sum_{i+j+m_1\le u}[p_{12}q_{21}(1-\theta)\phi_2
\end{aligned}
$$

$$+ q_{12}q_{21}\phi + q_{12}p_{21}(1-\theta)\phi_1 + p_{12}p_{21}(1-\theta)^2\phi_3]\circ(u-i-j-m_1,n-1)\, f_i g_j f_{m_1}$$
$$+ \sum_{i+j+k+m_1\le u} [q_{12}q_{21}\theta\phi + p_{12}p_{21}\theta(1-\theta)\phi_2]\circ(u-i-j-k-m_1,n-1)\, f_i g_j f_k f_{m_1}$$
$$+ \sum_{i+j+k+m_1\le u} [p_{12}q_{21}\theta\phi + p_{12}p_{21}\theta(1-\theta)\phi_1]\circ(u-i-j-k-m_1,n-1)\, f_i g_j g_k f_{m_1}$$
$$+ \sum_{i+j+k+l+m_1+\le u} p_{12}p_{21}\theta^2\phi(u-i-j-k-l-m_1,n-1)\, f_i g_j f_k g_l f_{m_1}\Bigg\}, \tag{3.2}$$

$$\phi_2(u-1,n)$$
$$= q_1q_2\sum_{m_2\le u}\phi(u-m_2,n-1)g_{m2} + p_1q_2\Bigg\{\sum_{i+m_2\le u}[q_{12}\phi + p_{12}(1-\theta)\phi_2]$$
$$\circ(u-i-m_2,n-1)\, f_i g_{m_2} + p_{12}\theta\sum_{i+j+m_2\le u}\phi(u-i-j-m_2,n-1)\, f_i g_j g_{m_2}\Bigg\}$$
$$+ q_1p_2\Bigg\{\sum_{j+m_2\le u}[q_{21}\phi + p_{21}(1-\theta)\phi_1]\circ(u-j-m_2,n-1)g_j g_{m_2}$$
$$+ p_{21}\theta\sum_{i+j+m_2\le u}\phi(u-i-j-m_2,n-1)\, f_i g_j g_{m_2}\Bigg\} + p_1p_2\Bigg\{\sum_{i+j+m_2\le u}[p_{12}q_{21}(1-\theta)\phi_2$$
$$+ q_{12}q_{21}\phi + q_{12}p_{21}(1-\theta)\phi_1 + p_{12}p_{21}(1-\theta)^2\phi_3]\circ(u-i-j-m_2,n-1)\, f_i g_j g_{m_2}$$
$$+ \sum_{i+j+k+m_2\le u}[q_{12}p_{21}\theta\phi + p_{12}p_{21}\theta(1-\theta)\phi_2]\circ(u-i-j-k-m_2,n-1)\, f_i g_j f_k g_{m_2}$$
$$+ \sum_{i+j+k+m_2\le u}[p_{12}q_{21}\theta\phi + p_{12}p_{21}\theta(1-\theta)\phi_1]\circ(u-i-j-k-m_2,n-1)\, f_i g_j g_k g_{m_2}$$
$$+ \sum_{i+j+k+l+m_2\le u} p_{12}p_{21}\theta^2\phi(u-i-j-k-l-m_2,n-1)\, f_i g_j f_k g_l g_{m_2}\Bigg\}, \tag{3.3}$$

and

$$
\begin{aligned}
&\phi_3(u-1,n) \\
&= q_1 q_2 \sum_{m_1+m_2\le u} \phi(u-m_1-m_2,n-1) f_{m_1} g_{m_2} \\
&\quad + p_1 q_2 \Bigg\{ \sum_{i+m_1+m_2\le u} [q_{12}\phi + p_{12}(1-\theta)\phi_2] \circ (u-i-m_1-m_2,n-1) f_i f_{m_1} g_{m_2} \\
&\quad + p_{12}\theta \sum_{i+j+m_1+m_2\le u} \phi(u-i-j-m_1-m_2,n-1) f_i g_j f_{m_1} g_{m_2} \Bigg\} \\
&\quad + q_1 p_2 \Bigg\{ \sum_{i+m_1+m_2\le u} [q_{21}\phi + p_{21}(1-\theta)\phi_1] \circ (u-j-m_1-m_2,n-1) g_j f_{m_1} g_{m_2} \\
&\quad + p_{21}\theta \sum_{i+j+m_1+m_2\le u} \phi(u-i-j-m_1-m_2,n-1) f_i g_j f_{m_1} g_{m_2} \Bigg\} \\
&\quad + p_1 p_2 \Bigg\{ \sum_{i+j+m_1+m_2\le u} [p_{12}q_{21}(1-\theta)\phi_2 + q_{12}q_{21}\phi + q_{12}p_{21}(1-\theta)\phi_1 \\
&\quad + p_{12}p_{21}(1-\theta)^2\phi_3] \circ (u-i-j-m_1-m_2,n-1) f_i g_j f_{m_1} g_{m_2} \\
&\quad + \sum_{i+j+k+m_1+m_2\le u} [q_{12}p_{21}\theta\phi(u-i-j-k-m_1-m_2,n-1) \\
&\quad + p_{12}p_{21}\theta(1-\theta)\phi_2(u-i-j-k-m_1-m_2,n-1)] f_i g_j f_k f_{m_1} g_{m_2} \\
&\quad + \sum_{i+j+k+m_1+m_2\le u} [p_{12}q_{21}\theta\phi(u-i-j-k-m_1-m_2,n-1) \\
&\quad + p_{12}p_{21}\theta(1-\theta)\phi_1(u-i-j-k-m_1-m_2,n-1)] f_i g_j g_k f_{m_1} g_{m_2} \\
&\quad + \sum_{i+j+k+l+m_1+m_2+\le u} p_{12}p_{21}\theta^2\phi(u-i-j-k-l-m_1-m_2,n-1) f_i g_j f_k g_l f_{m_1} g_{m_2} \Bigg\}. \qquad (3.4)
\end{aligned}
$$

Following the work of Willmot (1993) and Yuen and Guo (2001), we apply the technique of generating function to (3.1)-(3.4) and hence develop a recursive method for computing $\phi(u,n)$. Let the probability generating functions of f and g be $\tilde{f}(z)=\sum_{i=0}^{\infty} f_i z^i$ and $\tilde{g}(z)=\sum_{j=0}^{\infty} g_i z^j$. Define the generating function of ϕ and ϕ_i $(i=1,2,3)$ as

$$
\tilde{\phi}(z,n)=\sum_{u=0}^{\infty}\phi(u,n)z^u \quad \text{and} \quad \tilde{\phi}_i(z,n)=\sum_{u=0}^{\infty}\phi_i(u,n)z^u, \qquad i=1,2,3.
$$

Multiplying both sides of (3.1)-(3.4) by z^u and summing over u from 1 to ∞, we have

$$
\begin{aligned}
z\tilde{\phi}(z,n) = {} & q_1 q_2 (\tilde{\phi}(z,n-1) - \phi(0,n-1)) \\
& + p_1 q_2 \tilde{f}(z)[q_{12}\tilde{\phi}(z,n-1) + p_{12}(1-\theta)\tilde{\phi}_2(z,n-1) + p_{12}\theta\tilde{\phi}(z,n-1)\tilde{g}(z)] \\
& + q_1 p_2 \tilde{g}(z)[q_{21}\tilde{\phi}(z,n-1) + p_{21}(1-\theta)\tilde{\phi}_1(z,n-1) + p_{21}\theta\tilde{\phi}(z,n-1)\tilde{f}(z)] \\
& + p_1 p_2 \tilde{f}(z)\tilde{g}(z)\{ q_{12}q_{21}\tilde{\phi}(z,n-1) + p_{12}q_{21}(1-\theta)\tilde{\phi}_2(z,n-1) \\
& + q_{12}p_{21}(1-\theta)\tilde{\phi}_1(z,n-1) + p_{12}p_{21}(1-\theta)^2\tilde{\phi}_3(z,n-1) \\
& + [q_{12}p_{21}\theta\tilde{\phi}(z,n-1) + p_{12}p_{21}\theta(1-\theta)\tilde{\phi}_2(z,n-1)]\tilde{f}(z) \\
& + [p_{12}q_{21}\theta\tilde{\phi}(z,n-1) + p_{12}p_{21}\theta(1-\theta)\tilde{\phi}_1(z,n-1)]\tilde{g}(z) \\
& + p_{12}p_{21}\theta^2\tilde{\phi}(z,n-1)\tilde{f}(z)\tilde{g}(z)\Big\},
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
z\tilde{\phi}_1(z,n) = {} & q_1 q_2 \tilde{\phi}(z,n-1)\tilde{f}(z) + p_1 q_2 \tilde{f}^2(z)[p_{12}\theta\tilde{\phi}(z,n-1)\tilde{g}(z) \\
& + q_{12}\tilde{\phi}(z,n-1) + p_{12}(1-\theta)\tilde{\phi}_2(z,n-1)] + q_1 p_2 \tilde{f}(z)\tilde{g}(z) \\
& \times [q_{21}\tilde{\phi}(z,n-1) + p_{21}(1-\theta)\tilde{\phi}_1(z,n-1) + p_{21}\theta\tilde{\phi}(z,n-1)\tilde{f}(z)] \\
& + p_1 p_2 \tilde{f}^2(z)\tilde{g}(z)\{q_{12}q_{21}\tilde{\phi}(z,n-1) + p_{12}q_{21}(1-\theta)\tilde{\phi}_2(z,n-1) \\
& + q_{12}p_{21}(1-\theta)\tilde{\phi}_1(z,n-1) + p_{12}p_{21}(1-\theta)^2\tilde{\phi}_3(z,n-1) \\
& + [q_{12}p_{21}\theta\tilde{\phi}(z,n-1) + p_{12}p_{21}\theta(1-\theta)\tilde{\phi}_2(z,n-1)]\tilde{f}(z) \\
& + [p_{12}q_{21}\theta\tilde{\phi}(z,n-1) + p_{12}p_{21}\theta(1-\theta)\tilde{\phi}_1(z,n-1)]\tilde{g}(z) \\
& + p_{12}p_{21}\theta^2\tilde{\phi}(z,n-1)\tilde{f}(z)\tilde{g}(z)\},
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
z\tilde{\phi}_2(z,n) = {} & q_1 q_2 \tilde{\phi}(z,n-1)\tilde{g}(z) + p_1 q_2 \tilde{f}(z)\tilde{g}(z)[p_{12}\theta\tilde{\phi}(z,n-1)\tilde{g}(z) \\
& + q_{12}\tilde{\phi}(z,n-1) + p_{12}(1-\theta)\tilde{\phi}_2(z,n-1)] + q_1 p_2 \tilde{g}^2(z) \\
& \times [q_{21}\tilde{\phi}(z,n-1) + p_{21}(1-\theta)\tilde{\phi}_1(z,n-1) + p_{21}\theta\tilde{\phi}(z,n-1)\tilde{f}(z)] \\
& + p_1 p_2 \tilde{f}(z)\tilde{g}^2(z)\{q_{12}q_{21}\tilde{\phi}(z,n-1) + p_{12}q_{21}(1-\theta)\tilde{\phi}_2(z,n-1) \\
& + q_{12}p_{21}(1-\theta)\tilde{\phi}_1(z,n-1) + p_{12}p_{21}(1-\theta)^2\tilde{\phi}_3(z,n-1) \\
& + [q_{12}p_{21}\theta\tilde{\phi}(z,n-1) + p_{12}p_{21}\theta(1-\theta)\tilde{\phi}_2(z,n-1)]\tilde{f}(z) \\
& + [p_{12}q_{21}\theta\tilde{\phi}(z,n-1) + p_{12}p_{21}\theta(1-\theta)\tilde{\phi}_1(z,n-1)]\tilde{g}(z) \\
& + p_{12}p_{21}\theta^2\tilde{\phi}(z,n-1)\tilde{f}(z)\tilde{g}(z)\},
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
z\tilde{\phi}_3(z,n) = {} & q_1q_2\tilde{\phi}(z,n-1)\tilde{f}(z)\tilde{g}(z) + p_1q_2\tilde{f}^2(z)\tilde{g}(z)[p_{12}\theta\tilde{\phi}(z,n-1)\tilde{g}(z) \\
& + q_{12}\tilde{\phi}(z,n-1) + p_{12}(1-\theta)\tilde{\phi}_2(z,n-1)] + q_1p_2\tilde{f}(z)\tilde{g}^2(z) \\
& \times[q_{21}\tilde{\phi}(z,n-1) + p_{21}(1-\theta)\tilde{\phi}_1(z,n-1) + p_{21}\theta\tilde{\phi}(z,n-1)\tilde{f}(z)] \\
& + p_1p_2\tilde{f}^2(z)\tilde{g}^2(z)\{q_{12}q_{21}\tilde{\phi}(z,n-1) + p_{12}q_{21}(1-\theta)\tilde{\phi}_2(z,n-1) \\
& + q_{12}p_{21}(1-\theta)\tilde{\phi}_1(z,n-1) + p_{12}p_{21}(1-\theta)^2\tilde{\phi}_3(z,n-1) \\
& + [q_{12}p_{21}\theta\tilde{\phi}(z,n-1) + p_{12}p_{21}\theta(1-\theta)\tilde{\phi}_2(z,n-1)]\tilde{f}(z) \\
& + [p_{12}q_{21}\theta\tilde{\phi}(z,n-1) + p_{12}p_{21}\theta(1-\theta)\tilde{\phi}_1(z,n-1)]\tilde{g}(z) \\
& + p_{12}p_{21}\theta^2\tilde{\phi}(z,n-1)\tilde{f}(z)\tilde{g}(z)\}. \qquad (3.8)
\end{aligned}
$$

Furthermore we define the bivariate generating functions

$$
\bar{\phi}(z,t) = \sum_{n=0}^{\infty}\tilde{\phi}(z,n)t^n, \quad \text{and} \quad \bar{\phi}_i(z,t) = \sum_{n=0}^{\infty}\tilde{\phi}_i(z,n)t^n, \qquad i = 1, 2, 3.
$$

By analogy with the derivations of (3.5)-(3.8), we get

$$
z(\bar{\phi}(z,t) - \tilde{\phi}(z,0)) = A - q_1q_2t\bar{\phi}_0(t), \qquad (3.9)
$$

$$
z(\bar{\phi}_1(z,t) - \tilde{\phi}_1(z,0)) = A\tilde{f}(z), \qquad (3.10)
$$

$$
z(\bar{\phi}_2(z,t) - \tilde{\phi}_2(z,0)) = A\tilde{g}(z), \qquad (3.11)
$$

$$
z(\bar{\phi}_3(z,t) - \tilde{\phi}_3(z,0)) = A\tilde{f}(z)\tilde{g}(z), \qquad (3.12)
$$

where $\bar{\phi}_0(t) = \sum_{n=0}^{\infty}\phi(0,n)t^n$ and

$$
\begin{aligned}
A = {} & q_1q_2t\bar{\phi}(z,t) + p_1q_2t\tilde{f}(z)[p_{12}\theta\bar{\phi}(z,t)\tilde{g}(z) + q_{12}\bar{\phi}(z,t) + p_{12}(1-\theta)\bar{\phi}_2(z,t)] \\
& + q_1p_2t\tilde{g}(z)[q_{21}\bar{\phi}(z,t) + p_{21}\theta\bar{\phi}(z,t)\tilde{f}(z) + p_{21}(1-\theta)\bar{\phi}_1(z,t)] \\
& + p_1p_2t\tilde{f}(z)\tilde{g}(z)[q_{12}q_{21}\bar{\phi}(z,t) + p_{12}q_{21}(1-\theta)\bar{\phi}_2(z,t) + q_{12}p_{21}(1-\theta)\bar{\phi}_1(z,t) \\
& + p_{12}p_{21}(1-\theta)^2\bar{\phi}_3(z,t) + q_{12}p_{21}\theta\bar{\phi}(z,t)\tilde{f}(z) + p_{12}p_{21}\theta(1-\theta)\bar{\phi}_2(z,t)\tilde{f}(z) \\
& + p_{12}q_{21}\theta\bar{\phi}(z,t)\tilde{g}(z) + p_{12}p_{21}\theta(1-\theta)\bar{\phi}_1(z,t)\tilde{g}(z) + p_{12}p_{21}\theta^2\bar{\phi}(z,t)\tilde{f}(z)\tilde{g}(z)].
\end{aligned}
$$

By the definition of the generating functions, we have $\tilde{\phi}(z,0) = \tilde{\phi}_i(z,0) = (1-z)^{-1}$, for $i = 1, 2, 3$. Hence comparing the right-hand sides of (3.9)-(3.12) yields

$$
\begin{aligned}
z\bar{\phi}(z,t)-\frac{z}{1-z} \;=\; & t(q_1q_2+p_1q_2q_{12}\tilde{f}(z)+p_1q_2p_{12}\theta\tilde{f}(z)\tilde{g}(z)+q_1p_2q_{21}\tilde{g}(z) \\
& +q_1p_2p_{21}\theta\tilde{f}(z)\tilde{g}(z)+p_1p_2q_{12}q_{21}\tilde{f}(z)\tilde{g}(z)+p_1p_2q_{12}p_{21}\theta\tilde{f}^2(z)\tilde{g}(z) \\
& +p_1p_2p_{12}q_{21}\theta\tilde{f}(z)\tilde{g}^2(z)+p_1p_2p_{12}p_{21}\theta^2\tilde{f}^2(z)\tilde{g}^2(z))\bar{\phi}(z,t) \\
& +tp_2p_{21}(1-\theta)(q_1+p_1q_{12}\tilde{f}(z)+p_1p_{12}\theta\tilde{f}(z)\tilde{g}(z))\tilde{g}(z)\bar{\phi}_1(z,t) \\
& +tp_1p_{12}(1-\theta)(q_2+p_2q_{21}\tilde{g}(z)+p_2p_{21}\theta\tilde{f}(z)\tilde{g}(z))\tilde{f}(z)\bar{\phi}_2(z,t) \\
& +tp_1p_2p_{12}p_{21}(1-\theta)^2\tilde{f}(z)\tilde{g}(z)\bar{\phi}_3(z,t)-tq_1q_2\bar{\phi}_0(t), \qquad (3.13)
\end{aligned}
$$

$$
z\bar{\phi}_1(z,t)-\frac{z}{1-z}=\left(z\bar{\phi}(z,t)-\frac{z}{1-z}+q_1q_2t\bar{\phi}_0(t)\right)\tilde{f}(z), \qquad (3.14)
$$

$$
z\bar{\phi}_2(z,t)-\frac{z}{1-z}=\left(z\bar{\phi}(z,t)-\frac{z}{1-z}+q_1q_2t\bar{\phi}_0(t)\right)\tilde{g}(z), \qquad (3.15)
$$

$$
z\bar{\phi}_3(z,t)-\frac{z}{1-z}=\left(z\bar{\phi}(z,t)-\frac{z}{1-z}+q_1q_2t\bar{\phi}_0(t)\right)\tilde{f}(z)\tilde{g}(z). \qquad (3.16)
$$

Substituting (3.14)-(3.16) into (3.13), we have

$$
\begin{aligned}
& \Big\{z-t[q_1q_2+p_1q_2q_{12}\tilde{f}(z)+q_1p_2q_{21}\tilde{g}(z)+(p_1p_2q_{12}q_{21}+q_1p_2p_{21} \\
& +p_1q_2p_{12})\tilde{f}(z)\tilde{g}(z)+p_1p_2q_{12}p_{21}\tilde{f}^2(z)\tilde{g}(z)+p_1p_2p_{12}q_{21}\tilde{f}(z)\tilde{g}^2(z) \\
& +p_1p_2p_{12}p_{21}\tilde{f}^2(z)\tilde{g}^2(z)]\Big\}\bar{\phi}(z,t) \\
= \; & \frac{z}{1-z}+tp_2p_{21}(1-\theta)(1-\tilde{f}(z))\tilde{g}(z)\left(\frac{q_1+p_1q_{12}\tilde{f}(z)+p_1p_{12}\theta\tilde{f}(z)\tilde{g}(z)}{1-z}\right) \\
& +tp_1p_{12}(1-\theta)\tilde{f}(z)(1-\tilde{g}(z))\left(\frac{q_2+p_2q_{21}\tilde{g}(z)+p_2p_{21}\theta\tilde{f}(z)\tilde{g}(z)}{1-z}\right) \\
& +tp_1p_2p_{12}p_{21}(1-\theta)^2\tilde{f}(z)\tilde{g}(z)\left(\frac{1-\tilde{f}(z)\tilde{g}(z)}{1-z}\right)-tq_1q_2\bar{\phi}_0(t) \\
& \times\Bigg[1-tp_2p_{21}(1-\theta)\tilde{f}(z)\tilde{g}(z)\left(\frac{q_1+p_1q_{12}\tilde{f}(z)+p_1p_{12}\theta\tilde{f}(z)\tilde{g}(z)}{z}\right) \\
& -tp_1p_{12}(1-\theta)\tilde{f}(z)\tilde{g}(z)\left(\frac{q_2+p_2q_{21}\tilde{g}(z)+p_2p_{21}\theta\tilde{f}(z)\tilde{g}(z)}{z}\right) \\
& -tp_1p_2p_{12}p_{21}(1-\theta)^2\frac{\tilde{f}^2(z)\tilde{g}^2(z)}{z}\Bigg]. \qquad (3.17)
\end{aligned}
$$

Assume that delay of by-claim is not allowed. Under this assumption let $S_n^{'}$ be the aggregate claims in the first n periods for the book of business. Denote the probability function, distribution function and probability generating function of $S_n^{'}$ respectively by $h(z,n)$, $H(z,n)$ and $\tilde{h}(z,n)$. It is clear that

$$\tilde{h}(z,1) = q_1q_2 + p_1q_2q_{12}\tilde{f}(z) + q_1p_2q_{21}\tilde{g}(z) + (p_1p_2q_{12}q_{21} + q_1p_2p_{21} + p_1q_2p_{12})\tilde{f}(z)\tilde{g}(z)$$
$$+ p_1p_2q_{12}p_{21}\tilde{f}^2(z)\tilde{g}(z) + p_1p_2p_{12}q_{21}\tilde{f}(z)\tilde{g}^2(z) + p_1p_2p_{12}p_{21}\tilde{f}^2(z)\tilde{g}^2(z), \qquad (3.18)$$

with $\tilde{h}(z,n) = (\tilde{h}(z,1))^n$. In addition we need the following functions to get recursive equations for $\phi(u,n)$

$$\tilde{h}_1(z,1) = q_1q_2 + p_1q_2\tilde{f}(z) + q_1p_2\tilde{g}(z) + p_1p_2\tilde{f}(z)\tilde{g}(z)$$
$$- p_1p_2p_{12}p_{21}\theta\,\tilde{f}(z)\tilde{g}(z)(1-\tilde{f}(z))(1-\tilde{g}(z)),$$

$$\tilde{h}_2(z,1) = q_1q_2 + p_1q_2q_{12}\tilde{f}(z) + q_1p_2q_{21}\tilde{g}(z) + p_1p_2q_{12}q_{21}\tilde{f}(z)\tilde{g}(z)$$
$$- p_1p_2p_{12}p_{21}\,\theta\,\tilde{f}^2(z)\tilde{g}^2(z),$$

with $\tilde{h}_1(z,n) = \tilde{h}_1(z,1)\tilde{h}(z,n-1)$ and $\tilde{h}_2(z,n) = \tilde{h}_2(z,1)\tilde{h}(z,n-1)$.

Using (3.18) to rewrite (3.17), we get

$$(z - t\tilde{h}(z,1))\bar{\phi}(z,t) = \frac{z}{1-z} + \frac{t(1-\theta)}{1-z}[q_1q_2 + p_1q_2\tilde{f}(z) + q_1p_2\tilde{g}(z) + p_1p_2\tilde{f}(z)\tilde{g}(z)$$
$$- p_1p_2p_{12}p_{21}\theta(1-\tilde{f}(z))(1-\tilde{g}(z))\tilde{f}(z)\tilde{g}(z) - \tilde{h}(z,1)]$$
$$- tq_1q_2\bar{\phi}_0(t)\left\{1 - \frac{t(1-\theta)}{z}[\tilde{h}(z,1) - (q_1q_2 + p_1q_2q_{12}\tilde{f}(z) + q_1p_2q_{21}\tilde{g}(z)\right.$$
$$\left. + p_1p_2q_{12}q_{21}\tilde{f}(z)\tilde{g}(z) - p_1p_2p_{12}p_{21}\tilde{f}^2(z)\tilde{g}^2(z)]\right\}.$$

This together with the definitions of $\tilde{h}_1$ and $\tilde{h}_2$ give

$$(z - t\tilde{h}(z,1))\bar{\phi}(z,t) = \frac{z}{1-z} + (1-\theta)\frac{t}{1-z}(\tilde{h}_1(z,1) - \tilde{h}(z,1))$$
$$- tq_1q_2\bar{\phi}_0(t)\left[1 - \frac{t(1-\theta)}{z}(\tilde{h}(z,1) - \tilde{h}_2(z,1))\right]. \qquad (3.19)$$

Dividing both sides of (3.19) by $z - t\tilde{h}(z,1)$ and expressing $(z - t\tilde{h}(z,1))^{-1}$ in terms of a power series in *t*, we have

$$
\begin{aligned}
z^n\tilde{\phi}(z,n) &= \frac{\tilde{h}(z,n)}{1-z} + \frac{(1-\theta)\tilde{h}(z,n-1)}{1-z}(\tilde{h}_1(z,1)-\tilde{h}(z,1)) \\
&\quad + q_1q_2(1-\theta)(\tilde{h}(z,1)-\tilde{h}_2(z,1))\sum_{i=0}^{n-2}\tilde{h}(z,i)\phi(0,n-2-i)z^{n-2-i} \\
&\quad - q_1q_2\sum_{i=0}^{n-1}\tilde{h}(z,i)\phi(0,n-1-i)z^{n-1-i} \\
&= \frac{\theta\tilde{h}(z,n)+(1-\theta)\tilde{h}_1(z,n)}{1-z} - q_1q_2\theta\sum_{i=1}^{n-1}\tilde{h}(z,i)\phi(0,n-1-i)z^{n-1-i} \\
&\quad - q_1q_2(1-\theta)\sum_{i=1}^{n-1}\tilde{h}_2(z,i)\phi(0,n-1-i)z^{n-1-i}, \qquad (3.20)
\end{aligned}
$$

for $n=1,2,3,\cdots$. Then rewriting both sides of (3.20) in terms of the power series in z yields

$$
\begin{aligned}
\sum_{u=n}^{\infty}\phi(u-n,n)z^u &= \theta\sum_{u=0}^{\infty}H(u,n)z^u + (1-\theta)\sum_{u=0}^{\infty}H_1(u,n)z^u \\
&\quad - q_1q_2\theta\Bigg(\sum_{u=0}^{n-2}\sum_{i=0}^{u}h(u-i,n-1-i)\phi(0,i)z^u \\
&\quad - \sum_{u=n-1}^{\infty}\sum_{i=0}^{n-2}h(u-i,n-1-i)\phi(0,i)z^u\Bigg) \\
&\quad - q_1q_2(1-\theta)\Bigg(\sum_{u=0}^{n-2}\sum_{i=0}^{u}h_2(u-i,n-1-i)\phi(0,i)z^u \\
&\quad - \sum_{u=n-1}^{\infty}\sum_{i=0}^{n-2}h_2(u-i,n-1-i)\phi(0,i)z^u\Bigg),
\end{aligned}
$$

where H_1 is the distribution function of and $\tilde{h}_1$ and h_2 is the probability function of $\tilde{h}_2$. Comparing the coefficients of z^u, we get the recursive equations for the finite-time survival probabilities $\phi(u,n)$

$$
\begin{aligned}
\phi(u-n,n) &= \theta H(u,n) + (1-\theta)H_1(u,n) - q_1q_2\theta\sum_{i=0}^{n-2}h(u-i,n-1-i)\phi(0,i) \\
&\quad - q_1q_2(1-\theta)\sum_{i=0}^{n-2}h_2(u-i,n-1-i)\phi(0,i), \qquad (3.21)
\end{aligned}
$$

for $n=1,2,\cdots$ and $u \ge n$ where

$$
\begin{aligned}
\phi(0,n) &= \theta H(n,n) + (1-\theta)H_1(n,n) - q_1q_2\theta\sum_{i=0}^{n-2}h(n-i,n-1-i)\phi(0,i) \\
&\quad - q_1q_2(1-\theta)\sum_{i=0}^{n-2}h_2(n-i,n-1-i)\phi(0,i). \qquad (3.22)
\end{aligned}
$$

It is apparent from (3.21) and (3.22) that $\phi(u,0)=1$ for any $u \geq 0$. Equations (3.21) and (3.22) can be used recursively to calculate $\phi(u,n)$ for any $u \geq 0$ and $n \geq 1$. Note that h, h_2, H and H_1 appearing in (3.21) and (3.22) can be obtained using $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$.

To demonstrate the use of the recursive equations, we present two numerical examples with $p_1 = 0.25$, $p_2 = 0.2$, $p_{12} = 0.4$ and $p_{21} = 0.25$. In the first example we let $\tilde{f}(z) = z$ and $\tilde{g}(z) = z^2$ which implies that $f_1 = g_2 = 1$. For various values of u and θ, the numerical results are summarized in Table 1. As expected Table 1 shows that the bigger the initial surplus the larger the survival probability. Furthermore the book of business is more likely to survive if the probability of delay gets higher. The second example assumes that $\tilde{f}(z) = z$ and $\tilde{g}(z) = z(2-z)^{-1}$, or equivalently, $f_1 = 1$ and $g_j \sim \text{Geometric}(0.5)$. The corresponding survival probabilities are given in Table 2 and exhibit more or less the same pattern as what we observe in the first example.

4. Ultimate survival probability

In this section we consider the ultimate survival probability for U_n given by $\phi(u) = \lim_{n\to\infty} \phi(u,n)$ with $f_1 = 1$, $g_1 = 1$, $p_{12} = p_{21} = 1$ and $\theta = 0$. In this special case all claims are of unit amount; all main claims certainly induce a by-claim; and each by-claim definitely occurs one period after the occurrence of its associated main claim.

It is easy to see from (3.17) that there exists a unique solution $z(t)$ with $|z(t)|<1$ of the equation

$$z - t(q_1q_2 + q_1p_2\tilde{f}(z)\tilde{g}(z) + q_1q_2\tilde{f}(z)\tilde{g}(z) + p_1p_2\tilde{f}^2(z)\tilde{g}^2(z)) = z - t\tilde{h}(z,1) = 0,$$

for any $|(t)|<1$ and hence (3.17) can be simplified and rewritten as

$$\begin{aligned}\bar{\phi}_0(t) &= \frac{\frac{z(t)}{1-z(t)} + q_1p_2tz(t) + p_1q_2tz(t) + p_1p_2tz^2(t)(1+z(t))}{q_1q_2t(1-q_1p_2tz(t) - p_1q_2tz(t) - p_1p_2tz(t)^3)} \\ &= \frac{z(t)}{q_1q_2t(1-z(t))}\left(1 + \frac{q_1p_2t + p_1q_2t + p_1p_2tz(t)}{1-q_1p_2tz(t) - p_1q_2tz(t) - p_1p_2tz(t)^3}\right) \\ &= \frac{z(t)}{q_1q_2t(1-z(t))}\left(1 + \frac{(q_1p_2 + p_1q_2)z(t)}{q_1q_2} + \frac{p_1p_2z^2(t)}{q_1q_2}\right). \end{aligned} \tag{4.1}$$

For an analytic function r with $r(0) = 0$, Lagrange's expansion yields the following power series in t

$$r(z(t)) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{ds^{n-1}} \left[r'(s)(\tilde{h}(s,1))^n\right]_{s=0}. \tag{4.2}$$

Due to the facts that

$$\tilde{h}(s,n) = (\tilde{h}(s,1))^n,$$
$$h(n-1,n) = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\tilde{h}(s,n)\big|_{s=0},$$
$$h(n-2,n) = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}s\tilde{h}(s,n)\big|_{s=0},$$

we obtain from (4.2) that

$$z(t) = \sum_{n=1}^{\infty}\frac{t^n}{n}h(n-1,n),$$
$$z^2(t) = 2\sum_{n=2}^{\infty}\frac{t^n}{n}h(n-2,n),$$
$$\frac{z(t)}{1-z(t)} = \sum_{n=1}^{\infty}\left(\sum_{i=0}^{n-1}H(i,n)\right)\frac{t^n}{n}.$$

Substituting $z(t)$, $z^2(t)$ and $z(t)(1-z(t))^{-1}$ into (4.1) yields

$$\begin{aligned}\sum_{n=0}^{\infty}\phi(0,n)t^n &= \frac{1}{q_1q_2t}\left[\frac{z(t)}{1-z(t)} + \frac{q_1p_2+p_1q_2}{q_1q_2}\left(\frac{z(t)}{1-z(t)} - z(t)\right)\right.\\ &\quad\left. + \frac{p_1p_2}{q_1q_2}\left(\frac{z(t)}{1-z(t)} - z(t) - z^2(t)\right)\right]\\ &= \frac{1}{q_1^2q_2^2t}\left[\frac{z(t)}{1-z(t)} - (1-q_1q_2)z(t) - p_1p_2z^2(t)\right]\\ &= \frac{1}{q_1^2q_2^2}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}H(i,n+1)\right)\frac{t^n}{n+1} - \frac{1-q_1q_2}{q_1^2q_2^2}\sum_{n=0}^{\infty}\frac{h(n,n+1)}{n+1}t^n\\ &\quad - \frac{2p_1p_2}{q_1^2q_2^2}\sum_{n=1}^{\infty}\frac{h(n-1,n+1)}{n+1}t^n.\end{aligned}$$

By comparing the coefficients of t^n, we get

$$\phi(0,n) = \frac{\sum_{i=0}^{n}H(i,n+1) - (1-q_1q_2)h(n,n+1) - 2p_1p_2h(n-1,n+1)}{q_1^2q_2^2(n+1)}. \tag{4.3}$$

Therefore an explicit expression for the ultimate survival probability $\phi(0)$ can be obtained by letting $n\to\infty$ in (4.3). Making use of the result of Willmot (1993), we have

$$\lim_{n\to\infty}\frac{1}{q_1^2q_2^2(n+1)}\sum_{i=0}^{n}H(i,n+1) = \frac{1-2(p_1+p_2)}{q_1^2q_2^2}.$$

This together with the fact that

$$\lim_{n\to\infty}\frac{1}{(n+1)}\,h(n,n+1)=\lim_{n\to\infty}\frac{1}{n+1}\,h(n-1,n+1)=0\,,$$

lead to

$$\phi(0)=\lim_{n\to\infty}\phi(0,n)=\frac{1-2(p_1+p_2)}{q_1^2q_2^2}. \tag{4.4}$$

Finally we discuss how to deal with $\phi(u)$ for $u>0$. Define $\tilde{\phi}(z)=\sum_{u=0}^{\infty}\phi(u)z^u$ as the generating function of $\phi(u)$. With $f_1=g_1=1$, $p_{12}=p_{21}=1$ and $\theta=0$, we use (3.17), (4.4) and a Tauberian theorem for power series to come up with

$$\begin{aligned}
\phi(z) &= \lim_{t\to1}(t-1)\bar{\phi}(z,t)\\
&= \left(\frac{q_1q_2(1-p_1q_2z-q_1p_2z-p_1p_2z^3)}{q_1q_2+p_1q_2z^2+q_1p_2z^2+p_1p_2z^4-z}\right)\lim_{t\to1}(t-1)\bar{\phi}_0(t)\\
&= \left(\frac{q_1q_2(1-p_1q_2z-q_1p_2z-p_1p_2z^3)}{q_1q_2+p_1q_2z^2+q_1p_2z^2+p_1p_2z^4-z}\right)\left(\frac{1-2(p_1+p_2)}{q_1^2q_2^2}\right)\\
&= \frac{1-2(p_1+p_2)}{(q_1q_2)^2}\sum_{i=0}^{\infty}\left(1-p_1q_2z-q_1p_2z-p_1p_2z^3\right)^{i+1}\frac{1}{(q_1q_2)^i}\,z^i.
\end{aligned} \tag{4.5}$$

Furthermore the summation on the right-hand side of (4.5) can be expressed as

$$\begin{aligned}
&\sum_{i=0}^{\infty}\left(1-p_1q_2z-q_1p_2z-p_1p_2z^3\right)^{i+1}\frac{1}{(q_1q_2)^i}\,z^i\\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i+1}\sum_{k=0}^{j}\left(\frac{(i+1)!}{(i+1-j)!(j-k)!k!}\right)\left(\frac{(p_1q_2+q_1p_2)^{j-k}(p_1p_2)^k(-1)^j}{(q_1q_2)^i}\right)z^{i+j+2k}\\
&= \sum_{i=0}^{\infty}\sum_{j=i}^{2i+1}\sum_{k=0}^{j-i}\left(\frac{(i+1)!}{(2i+1-j)!(j-i-k)!k!}\right)\left(\frac{(p_1q_2+q_1p_2)^{j-i-k}(p_1p_2)^k(-1)^{j-i}}{(q_1q_2)^i}\right)z^{j+2k}\\
&= \sum_{i=0}^{\infty}\sum_{i=[\frac{j}{2}]}^{j}\sum_{k=0}^{j-i}\left(\frac{(i+1)!}{(2i+1-j)!(j-i-k)!k!}\right)\left(\frac{(p_1q_2+q_1p_2)^{j-i-k}(p_1p_2)^k(-1)^{j-i}}{(q_1q_2)^i}\right)z^{j+2k}\\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{j-[\frac{j}{2}]}a_{jk}z^{j+2k},
\end{aligned} \tag{4.6}$$

where

$$a_{jk}=\sum_{i=[\frac{j}{2}]}^{j-k}\left(\frac{(i+1)!}{(2i+1-j)!(j-i-k)!k!}\right)\left(\frac{(p_1q_2+q_1p_2)^{j-i-k}(p_1p_2)^k(-1)^{j-i}}{(q_1q_2)^i}\right),$$

and $[c]$ means the integral part of c. By letting $l = j + 2k$, (4.6) beomces

$$\sum_{i=0}^{\infty} \left(1 - p_1 q_2 z - q_1 p_2 z - p_1 p_2 z^3\right)^{i+1} \frac{1}{(q_1 q_2)^i} z^i = \sum_{l=0}^{\infty} b_l z^l, \tag{4.7}$$

where $b_l = \sum_{n=0}^{n_l} a_{l-2n,n}$ with $n_l = \left[\frac{l - \left[\frac{l}{2}\right]}{2}\right] = \left[\frac{l+1}{4}\right]$. From the definition of $\tilde{\phi}(z)$, (4.5) and (4.7), it is obvious that the ultimate survival probability takes the form

$$\begin{aligned}
\phi(u) &= \frac{1 - 2(p_1 + p_2)}{(q_1 q_2)^2} \sum_{n=0}^{[\frac{u+1}{4}]} a_{u-2n,n} \\
&= \sum_{n=0}^{[\frac{u+1}{4}]} \sum_{i=[\frac{u}{2}]-n}^{u-3n} \left(\frac{(i+1)!}{(2i+1-u+2n)!(u-i-3n)!n!} \right) (-1)^{u-i} (1 - 2p_1 - 2p_2) \\
&\quad \times (p_1 q_2 + q_1 p_2)^{u-i-3n} (p_1 p_2)^n (q_1 q_2)^{-i-2}.
\end{aligned} \tag{4.8}$$

As an example we consider the case with $p_1 = 0.25$ and $p_2 = 0.2$. Using (4.8) we obtain various values of $\phi(u)$ for $u = 0, 1, \cdots, 25$. The values shown in Table 3 are consistent with the fact that the ultimate survival probability increases as u increases.

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Table 1

Values of $\phi(u, n)$ with $\tilde{f}(z) = z$ and $\tilde{g}(z) = z^2$

		n						
		1	2	3	4	5	10	20
$\theta = 0$	$\phi(0, n)$	0.8000	0.6660	0.6046	0.5539	0.5160	0.4142	0.3342
	$\phi(3, n)$	1.0000	0.9805	0.9552	0.9261	0.9000	0.7993	0.6880
	$\phi(5, n)$	1.0000	0.9983	0.9923	0.9832	0.9720	0.9131	0.8235
$\theta = 0.2$	$\phi(0, n)$	0.7840	0.6500	0.5892	0.5388	0.5016	0.4018	0.3238
	$\phi(3, n)$	0.9960	0.9740	0.9474	0.9175	0.8908	0.7892	0.6781
	$\phi(5, n)$	1.0000	0.9972	0.9904	0.9806	0.9687	0.9080	0.8173
$\theta = 0.8$	$\phi(0, n)$	0.7360	0.6019	0.5438	0.4955	0.4604	0.3673	0.2953
	$\phi(3, n)$	0.9840	0.9533	0.9238	0.8913	0.8632	0.7597	0.6500
	$\phi(5, n)$	1.0000	0.9937	0.9842	0.9723	0.9584	0.8926	0.7993

Table 2

Values of $\phi(u, n)$ with $\tilde{f}(z) = z$ and $\tilde{g}(z) = z(2-z)^{-1}$

		n						
		1	2	3	4	5	10	20
$\theta = 0$	$\phi(0, n)$	0.8750	0.7520	0.6761	0.6216	0.5803	0.4635	0.3702
	$\phi(3, n)$	0.9844	0.9518	0.9196	0.8896	0.8624	0.7599	0.6476
	$\phi(5, n)$	0.9961	0.9846	0.9703	0.9548	0.9390	0.8678	0.7702
$\theta = 0.2$	$\phi(0, n)$	0.8553	0.7336	0.6582	0.6045	0.5639	0.4496	0.3587
	$\phi(3, n)$	0.9801	0.9461	0.9130	0.8824	0.8547	0.7516	0.6396
	$\phi(5, n)$	0.9948	0.9824	0.9673	0.9513	0.9351	0.8626	0.7644
$\theta = 0.8$	$\phi(0, n)$	0.7960	0.6793	0.6065	0.5555	0.5173	0.4108	0.3272
	$\phi(3, n)$	0.9669	0.9286	0.8927	0.8605	0.8318	0.7274	0.6166
	$\phi(5, n)$	0.9905	0.9754	0.9582	0.9405	0.9230	0.8471	0.7475

Table 3

Values of $\phi(u)$ with $f_1 = g_1 = 1$, $p_{12} = p_{21} = 1$ and $\theta = 0$

u	$\phi(u)$	u	$\phi(u)$
0	0.27778	7	0.72178
1	0.36574	8	0.75743
2	0.44753	9	0.78851
3	0.51865	10	0.81560
4	0.58020	15	0.90710
5	0.63398	20	0.95320
6	0.68089	25	0.97642